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Gevrey Solvability for Semilinear Partial Differential Equations with Multiple Characteristics.

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Sunto. – *Vengono considerate equazioni alle derivate parziali semilineari con caratteristiche multiple. Si studia in particolare la loro risolubilità locale e la buona positura del problema di Cauchy nell'ambito delle classi di Gevrey.*

1. – Introduction and statement of the main results.

The present paper studies semilinear PDE of the form

$$P(x, D)\, v + F(x, \partial^\alpha v)_{|\alpha| \leq m-1} = f(x) \tag{1.1}$$

with linear part

$$P(x, D) = \sum_{|\alpha| \leq m} c_\alpha(x)\, D^\alpha \tag{1.2}$$

having analytic or Gevrey coefficients, and multiple characteristics. The vectorial notations in (1.1), (1.2) are standard, in particular in (1.2) we write D^α for $(-i)^{|\alpha|}\partial^\alpha$. As it concerns the nonlinear term F, it is a smooth complex-valued $C^\infty(\mathbb{R}^n \times \mathbb{C}^N : \mathbb{C})$ function, where $N = \sum_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \leq m-1} 1$.

Our general results will be for two cases: in the first one we assume that F in (1.1) is an entire function in $v, \ldots, \partial^\alpha v, \ldots, |\alpha| \leq m-1$, analytic with respect to $x = (x_1, \ldots, x_n)$ say in an open neighborhood Ω of the origin in $\mathbb{R}^n$, while in the second one, much more involved from the technical point of view, we only require that F is G^θ Gevrey in all variables, with $F(x, 0) = 0$ in any case.

In fact, one of the main problems under investigation of the present paper is the solvability of (1.1) for a right-hand side $f(x)$ in the Gevrey class G^σ, $1 < \sigma < \infty$ (in the second case $1 < \theta \leq \sigma < \infty$), i.e. we assume for a suitable

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constant $C > 0$

$$\sup_{x \in \Omega} |\partial^\alpha f(x)| \leqslant C^{|\alpha|+1} (\alpha!)^\sigma, \qquad \alpha \in \mathbb{Z}^n_+ . \tag{1.3}$$

Note that the setting $f \in G^\sigma$ represents an intermediate choice between the choice of f in the analytic class, for which solvability is granted by the Cauchy-Kowalewsky theorem, and $f \in C^\infty$, for which solvability heavily depends on the lower order terms in (1.1), (1.2). We address to L. Rodino [44], M. Mascarello, L. Rodino [34] for a general introduction to the study of linear PDE in Gevrey class. Furthermore, if the operator $P(x, D)$ is weakly hyperbolic, we will study the local well-posedness of the corresponding Cauchy problem for (1.1).

The results in both directions will be based on a kind of a nonlinear calculus in the framework of suitable Banach spaces of Gevrey functions. We show a priori estimates for nonlinear superpositions, generalize Moser type estimates and as a consequence we are able to apply fixed point theorems in Gevrey-Banach spaces. Let us recall that in the C^∞ category and more generally, in the framework of the classical Sobolev spaces, the main tools are a priori energy estimates for strictly hyperbolic systems with or without the use of the theory of the paradifferential operators. We cite for example M. Taylor [45]. A typical problem is the following one: given a smooth function F, with $F(0) = 0$, and given two Banach spaces of regular functions $X \subset Y$ (say, Sobolev type), can we find a positive continuous function g defined on $[0, +\infty[$, depending only on F and X, such that the following estimate holds

$$\|F(u)\|_X \leqslant \|u\|_X \, g(\|u\|_Y), \qquad \text{for all } u \in X . \tag{E}$$

We stress that (E) has been proved in such a general form for $F \in C^\infty$, $X = H^s_p(\mathbb{R}^n)$, $s > n/p$, $Y = L^\infty(\mathbb{R}^n)$ by means of paradifferential operators techniques by J.-M. Bony [2] and Y. Meyer [36] (see also J. Rauch and M. Reed [42] for an alternative proof when $p = 2$). The estimate (E) has been proved by H. Chen and L. Rodino [6] for F being analytic and

$$X = H^{s,\tau}_\sigma(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n) \colon \|e^{\tau|D|^{1/\sigma}} u\|_{H^s} < \infty\},$$

$$Y = H^{s_0,\tau}_\sigma(\mathbb{R}^n), \qquad \frac{n}{2} < s_0 < s ,$$

for $\tau > 0$, $\sigma > 1$, as an application of the paradifferential calculus in Gevrey classes developed by the authors. We stress that, in the limit case $\sigma = 1$, similar type of spaces of analytic functions and estimates as (E) have been employeed in the study of the analytic regularity in the x variables for $t > 0$ of the solutions of the Navier-Stokes type equations and more generally, semilinear parabolic equations with the nonlinear term being an entire function, cf. C. Foias and R. Temam [17], K. Promislow [41], A. Ferrari and E. Titi [16]. See also C. D. Levermore and M. Oliver [33], where analyticity for a generalized Euler equation is investigated.

At present we are not aware of any results on the estimate (E) in the framework of the Banach G^σ Gevrey spaces assuming that F is G^θ with $1 < \theta \leqslant \sigma$. We mention that M. Cicognani and L. Zanghirati [8] study the G^σ regularity of the solutions to some weakly hyperbolic equations provided $\theta < \sigma$, while M. Reissig and K. Yagdjian [43] investigate the Gevrey well-posedness for second order weakly hyperbolic equations in one space dimension but without estimates of the type (E).

Before presenting our results, we would like to give three model examples of equations of the type (1.1), which will serve both as an illustration of the main novelties of our paper and as a comparison with the previous results on nonlinear PDE with multiple characteristics.

a) Let P be weakly hyperbolic with respect to x_n of multiplicity m. Then in case F is analytic, as a corollary from results on the local well-posedness of the Cauchy problem for general fully nonlinear weakly hyperbolic systems due to J. Leray and Y. Ohya [32] and K. Kajitani [30], the local solvability in G^σ for $1 < \sigma < m/(m-1)$ is valid.

If $\sigma = m/(m-1)$ or F is G^θ Gevrey, $1 < \theta \leqslant \sigma$, we are not aware of general results of G^σ local well-posedness and even G^σ local solvability.

We are able to show such results. We stress that in the case $\sigma = \theta$ we impose a kind of small norm requirement of the nonlinearity in order to have solvability and local well-posedness. Actually the smallness requirement for the critical index $\sigma = \theta$ comes from the nonlinear superposition estimates.

b) Let $n = 2$ and $P = (D_{x_2} + ic_1 x_2^{2h_1} D_{x_1}) \circ \ldots \circ (D_{x_2} + ic_m x_2^{2h_m} D_{x_1})$, where $c_j \in \mathbb{R}\backslash 0$, $h_j \in \mathbb{N}$, $j = 1, \ldots, m$. If all c_j, h_j are equal and F is linear in $\partial_x^\alpha u$, $|\alpha| \leqslant m-1$, it is well known that if $\sigma > m/(m-1)$ the operators could be not solvable under suitable assumptions on the lower order term, see for example T. Okaji [38], while for $1 < \sigma \leqslant m/(m-1)$ positive results are proved by T. Gramchev [23]. If not all c_j are equal, there are the classical results of V. Grushin [26] on nonsolvability in C^∞ provided suitable discrete conditions on the lower order terms are imposed, see A. Corli and L. Rodino [11] for the Gevrey case. We will show that if all c_j have the same sign, the semilinear equation (1.1) is G^σ solvable for any $1 \leqslant \sigma \leqslant m/(m-1)$ with smallness requirements if $\sigma = m/(m-1)$.

c) If $n = 2$, we can consider P as a product of Mizohata type operators as in *b*) and first order hyperbolic operators. For such operators we do not know any results concerning the equation (1.1). We will prove results on local solvability in G^σ, $1 < \sigma \leqslant m/(m-1)$, with smallness requirements if $\sigma = m/(m-1)$.

We hope that our nonlinear calculus in the Gevrey-Banach spaces will lead in the future to applications for fully nonlinear PDE with multiple characteristics via Nash-Moser type theorem (see T. Gramchev and M. Yoshino [25] for rapidly convergent method in Gevrey classes on the torus for normal forms of Gevrey orientation preserving mappings of the unit circle). As it concerns the Cauchy problem for second order nonlinear weakly hyperbolic equations and the use of Nash-Moser theorem in the C^∞ category we refer to the recent survey paper of P. D'Ancona and M. Reissig [13] and the references therein.

Let us begin by presenting the results in the case of the analytic nonlinearity. We shall also assume here that the coefficients of the linear part $P(x, D)$ are analytic in Ω. In the following we shall only argue on the principal symbol of the linear part

$$p_m(x, \xi) = \sum_{|\alpha| = m} c_\alpha(x)\, \xi^\alpha , \tag{1.4}$$

assuming it has multiple characteristics and satisfies suitable hypotheses which guarantee solvability of the linear equation

$$P(x, D)\, v = f \in G^\sigma . \tag{1.5}$$

Before specifying such hypotheses, we recall that a local solution v (not subjected to any initial or boundary condition) of the semilinear equation (1.1) exists for any $f \in C^\infty(\Omega)$ when $p_m(x, \xi)$ is elliptic, see for example S. Alinhac and P. Gérard [1] § 3.2.4, and also when $p_m(x, \xi)$ is of real principal type, as shown by J. Goodman and D. Yang [20] in the fully nonlinear case by means of the Nash-Moser method, see K. Payne [37] for a systematic presentation. The local solvability result keeps valid for (1.1) with $f \in C^\infty(\Omega)$ when $p_m(x, \xi)$ is complex-valued of principal type, provided it satisfies somewhat stronger conditions than the Nirenberg-Trèves linear solvability (P) condition, as it was proved by B. Dehman [14], T. Gramchev and P. Popivanov [24], J. Hounie [27]; precisely, it was assumed

$$d_\xi \operatorname{Re} p_m(x, \xi) \neq 0 \quad \text{and} \quad \operatorname{Im} p_m(x, \xi) \text{ does not change sign for } (x, \xi) \tag{1.6}$$

$$\text{in a neighborhood of the characteristic manifold} .$$

Observe that (1.6) is satisfied by all the sub-elliptic symbols $p_m(x, \xi)$, cf. F. Trèves [46], Vol. II, § 11.3 and L. Hörmander [29], Vol. IV, § 27.3. More recently, J. Hounie and P. Santiago [28] obtained local solvability for (1.1) under (P) condition in full generality.

Coming now to symbols with multiple characteristics, we shall assume $p_m(x, \xi)$ has a smooth decomposition into factors satisfying (1.6). Precisely, let us write Σ for the characteristic manifold:

$$\Sigma = \{(x, \xi) \in \Omega \times (\mathbb{R}^n \backslash 0), p_m(x, \xi) = 0\}. \tag{1.7}$$

Let $k \geq 1$ be a fixed integer.

We suppose for every $\varrho = (x_0, \xi_0) \in \Sigma$ there exists a conic neighborhood Γ_ϱ of ϱ such that

$$p_m(x, \xi) = e_{m-k}(x, \xi)\, a_1(x, \xi) \ldots a_k(x, \xi) \quad \text{for } (x, \xi) \in \Gamma_\varrho, \tag{1.8}$$

where $e_{m-k}(x, \xi)$ is an analytic elliptic symbol homogeneous of order $m-k$, and the first order homogeneous symbols $a_j(x, \xi)$, $j = 1, \ldots, k$, are analytic of nondegenerate principal type, i.e. $d_\xi a_j(x, \xi) \neq 0$ when $a_j(x, \xi) = 0$ in Γ_ϱ. After a re-setting of the elliptic factor $e_{m-k}(x, \xi)$ and possibly after a linear change of variables and a shrinking of Γ_ϱ, there is then no loss of generality in assuming

$$\partial_{\xi_n} \operatorname{Re} a_j(x, \xi) > 0 \quad \text{for all } j = 1, \ldots, k \text{ and } (x, \xi) \in \Gamma_\varrho. \tag{1.9}$$

Modelling on (1.6) we add

$$\operatorname{Im} a_j(x, \xi) \geq 0 \quad \text{for all } j = 1, \ldots, k \text{ and } (x, \xi) \in \Gamma_\varrho. \tag{1.10}$$

One can obviously replace (1.10) with

$$\operatorname{Im} a_j(x, \xi) \leq 0 \quad \text{for all } j = 1, \ldots, k \text{ and } (x, \xi) \in \Gamma_\varrho, \tag{1.11}$$

since in this case the change of variable $x_n' = -x_n$ allows a new factorization satisfying (1.9), (1.10).

Looking first to the linear equation (1.5), we observe that the assumption in (1.10), (1.11), that all the $\operatorname{Im} a_j(x, \xi)$ have the same sign, is essential. In fact it is known from the above mentioned works of V. Grushin [26] , A. Corli and L. Rodino [11] and also A. Menikoff [35] that sub-elliptic factors with conflicting signs may give rise to non-hypoellipticity and non-solvability results for $f \in C^\infty$ and also $f \in G^\sigma$, $1 < \sigma < \infty$.

It is also well known that, under the assumptions (1.8), (1.9), (1.10), the local solvability of the linear equation (1.5) with $f \in G^\sigma$, $k/(k-1) < \sigma < \infty$, as well as with $f \in C^\infty$, depends on the lower order terms; see A. Corli [9], [10], T. Gramchev [22], P. Popivanov [39], [40]. What we may expect, without any assumption on lower order terms, is G^σ-solvability for $1 < \sigma < k/(k-1)$.

This is in fact our preliminary result concerning the linear equation, which we express in a microlocal form using the Gevrey-Sobolev spaces $H^{s,\psi}_{\tau,\sigma}$, by short H^s_σ, of all the functions f such that

$$\|f\|_{H^s_\sigma} = \|e^{\tau\psi(x, D)} f\|_{H^s} < \infty ,$$

where σ represents the Gevrey order, s the Sobolev index, τ is a positive parameter, $\psi(x, \xi)$ a suitable symbol, homogeneous of degree $1/\sigma$ for large $|\xi|$, so that $\bigcup_{\tau, s} H^{s, \psi}_{\tau, \sigma} \supset G^\sigma$ locally; see the next § 2.

THEOREM 1.1. – *Under the assumptions* (1.8), (1.9), (1.10) *for the principal symbol $p_m(x, \xi)$ in the neighborhood Γ_ϱ of ϱ, the linear equation*

$$P(x, D)v = f \in H^s_\sigma, \qquad 1 < \sigma < k/(k-1), \qquad s \in \mathbb{R},$$

admits a solution $v \in H^{s+m-k(1-1/\sigma)}_\sigma$, microlocally at ϱ.

A more precise statement will be given in § 3, where we shall obtain the theorem following closely the arguments of K. Kajitani and S. Wakabayashi [31] concerning micro-hyperbolic operators. Note that for the solution v we get a loss of $k(1-1/\sigma) < 1$ derivatives in the Gevrey-Sobolev spaces H^s_σ, whereas in the standard Sobolev spaces the loss for multiple characteristics would be at least $k/2$ in any case.

Aiming now at the local solvability of the semilinear equation (1.1), we shall have first to patch together the microlocal results from Theorem 1.1; this requires a global choice of the weight ψ which in the definition of H^s_σ and in Theorem 1.1 may depend on $\varrho \in \Sigma$. Besides, to face the nonlinearity, we shall need H^s_σ to be an algebra, that is granted, as shown in § 2, if we shall assume further ψ is sub-additive. In turn, this will lead us to strengthen the assumptions (1.8), (1.9), (1.10). Precisely, we are able to treat (1.1) in two particular cases. The first case is when

(1.12) for all $\varrho \in \Sigma$ we may choose the neighborhood Γ_ϱ and the factors $a_j(x, \xi)$, $j = 1, \dots, k$, in (1.8) so that $a_j(x, \xi)$ is real-valued and $\partial_{\xi_n} a_j(x, \xi) > 0$ in Γ_ϱ.

Observe that the local coordinates x are now assumed to be fixed independent of $\varrho \in \Sigma$, and we cannot change the role of the dual variable ξ_n when changing ϱ and Γ_ϱ. Since $a_j(x, \xi)$ is real-valued in (1.12), the assumption (1.10) is trivially satisfied.

In the second case we shall allow $\operatorname{Im} a_j(x, \xi)$ to be not identically zero. We assume for $x = (x_1, x_2)$ in a neighborhood Ω of the origin in $\mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \backslash 0$ we have a global factorization of type (1.8):

$$p_m(x, \xi) = e_{m-k}(x, \xi)(\xi_2 + \lambda_1(x)\,\xi_1)\dots(\xi_2 + \lambda_k(x)\,\xi_1) \tag{1.13}$$

with $e_{m-k}(x, \xi)$ elliptic and $\lambda_1(x), \dots, \lambda_k(x)$ analytic in Ω satisfying

$$\operatorname{Im} \lambda_j(x) \geqslant 0 \qquad \text{for all } j = 1, \dots, k \text{ and } x \in \Omega\,. \tag{1.14}$$

Therefore (1.9), (1.10) are valid for $\xi_1 > 0$, and (1.9), (1.11) for $\xi_1 < 0$.

THEOREM 1.2. – *Let* $p_m(x, \xi)$ *satisfy* (1.12), *or else* (1.13), (1.14). *Then the semilinear equation*

$$P(x, D)\, v + F(x, \partial^\alpha v)_{|\alpha| \leq m-1} = \varepsilon f(x)\,,$$

where $\varepsilon > 0, f \in G^\sigma$, $1 < \sigma < k/(k-1)$, *compactly supported in a neighborhood of the origin* $\Omega = \{|x| < \delta\}$, *admits a solution* v *in* Ω, *if* ε *and* δ *are sufficiently small.*

We recall that here F is assumed to be entire function with respect to $\partial^\alpha v$, analytic with respect to x with $F(x, 0) = 0$, and the coefficients of $P(x, D)$ are analytic in Ω. The solution v is classical, i.e. $v \in C^m(\Omega)$.

It will be not actually necessary that both ε and δ are small; for a precise bound involving ε and δ see § 4. In our proof in § 4 we shall avoid the use of Nash-Moser method, but rely on the classical iterative procedure.

A model equation satisfying (1.13), (1,14) is a nonlinear perturbation of the m-th power of the Mizohata operator

$$(D_{x_2} + i x_2^{2h} D_{x_1})^m v + F(x, \partial^\alpha v)_{|\alpha| \leq m-1} = \varepsilon f(x)\,.$$

Observe that, if we replace $2h$ by an odd exponent, the corresponding linear equation is not solvable in C^∞, neither in G^σ, $1 < \sigma < \infty$; see F. Cardoso [3], M. Cicognani and L. Zanghirati [7], R. Goldman [19], T. Gramchev [23].

Concerning (1.12), we observe that in this case we are very near to the classical results of Leray and Ohya. Precisely, if we assume further (1.8) is valid globally for $\xi \in \mathbb{R}^n \backslash 0$ with $e_{m-k} = 1$, our equation is hyperbolic, having smooth characteristics of multiplicity $\leq k$, and the Cauchy problem with G^σ-data, $1 < \sigma < k/(k-1)$, is well posed (J. Leray and Y. Ohya [32]; see K. Kajitani [30] for non-smooth characteristics). Local solvability is then obvious.

Let us come now to the case of the Gevrey nonlinearity. Changing notations, we shall write t for the «time» variable, and denote by x the «space» variables in $\mathbb{R}^n$. We shall limit here attention to linear parts satisfying (1.12) or (1.13), (1.14) of a particular form, with coefficients depending only on t. Precisely we are assuming that

$$P(t, \partial_t, \partial_x) = \mathcal{L}_m \circ \ldots \circ \mathcal{L}_1 \tag{1.15}$$

where

$$\mathcal{L}_j = \partial_t + \langle \boldsymbol{a}^j(t), \partial_x \rangle, j = 1, \ldots m\,. \tag{1.16}$$

We may assume the vectors $\boldsymbol{a}^j(t)$ are C^∞ in a neighborhood I of $t = 0$.

The nonlinear term is supposed to be a G^θ function depending on $\partial_x^\alpha u$, $\alpha \in \mathbb{Z}_+^n$ for some $\theta \in [1, m/(m-1)]$, namely $F \in G^\theta(\mathbb{C}^N: \mathbb{C})$, $F(0) = 0$. According to (1.12), (1.13), (1.14), we assume:

$$\text{(1.17)} \qquad \text{all } \boldsymbol{a}^j \text{ are real-valued}, \qquad \text{or else } n=1 \text{ and } \operatorname{Im} a^j(t) \geqslant 0 \quad (\text{or} \leqslant 0) \quad \text{for all } j=1, \ldots, m\,.$$

We consider the equation

$$\text{(1.18)} \qquad P(t, \partial_t, \partial_x)\, u + F(u, \ldots, \partial_x^\alpha u, \ldots)\,|_{|\alpha| \leqslant m-1} = f(t, x)\,,$$

where x is in an open neighborhood Ω of the origin in $\mathbb{R}^n$, and if all $\boldsymbol{a}^j$ are real-valued, we consider initial data

$$\text{(1.19)} \qquad \partial_t^j u\,|_{t=0} = u_j^0(x)\,, \qquad j = 0, 1, \ldots, m-1\,.$$

Now we state the main result for a Gevrey nonlinearity.

THEOREM 1.3. – *Let* $F \in G^\theta(\mathbb{C}^N : \mathbb{C})$, $F(0)=0$, $1 \leqslant \theta \leqslant \sigma \leqslant m/(m-1)$. *Assume* (1.17) *is satisfied for* P *as in* (1.15), (1.16). *Let* $f(t, x) \in C^0([-T_0, T_0] : G_0^\sigma(\Omega))$, $T_0 > 0$ *and* $u_j^0(x) \in G_0^\sigma(\Omega)$, $j = 0, 1, \ldots, m-1$. *If* $\sigma = \theta$ *we require that*

$$\text{(1.20)} \qquad \sup_{x \in \Omega} |\partial_x^\alpha u_j^0(x)| \leqslant \kappa^{|\alpha|+1} (\alpha!)^\sigma\,, \qquad j = 0, 1, \ldots, m-1\,, \qquad \alpha \in \mathbb{Z}_+^n\,,$$

$$\text{(1.21)} \qquad \sup_{|t| \leqslant T_0,\, x \in \Omega} |\partial_x^\alpha f(t, x)| \leqslant \kappa^{|\alpha|+1} (\alpha!)^\sigma\,, \qquad \alpha \in \mathbb{Z}_+^n\,,$$

where $\kappa > 0$ *is a constant depending on the nonlinear term* F. *Then we can find* $T_0' \in]0, T_0[$ *such that there exists* $u(t, x) \in C^m(]-T_0', T_0'[: G^\sigma(\Omega))$ *solution to* (1.18) (*respectively to the Cauchy problem* (1.18), (1.19) *provided all* $\boldsymbol{a}^j$ *are real-valued*).

More precise statements will be given in § 9. A natural question is whether Theorem 1.2 is valid in the case of a Gevrey nonlinearity; at this moment we are not able to extend in this direction the proof of Theorem 1.3, which takes advantage of the particular form (1.15), (1.16), (1.18).

Finally, concerning the solvability of (1.1) in the case when $f \in C^\infty$ or $f \in G^\sigma$ with $\sigma > m/(m-1)$ we expect that, to obtain positive results, we shall have to impose conditions of Levi type on the nonlinear perturbation (for second order nonlinear hyperbolic equations see [43] and the discussion and the references in [13]). We purpose to discuss the problem in the future. Preliminary results in this direction are for example in G. Garello [18] concerning local solvability. About Sobolev-C^∞ well-posedness of the Cauchy problem for weakly hyperbolic nonlinear equations there exist other results, with applications to different models in Mathematical Physics, see for example W. Craig [4].

PART I: THE CASE OF ANALYTIC NONLINEARITY

2. – Gevrey-Sobolev spaces and non-linear operators.

We write

$$x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n), \qquad \text{and} \qquad \xi = (\xi', \xi_n) = (\xi_1, \dots, \xi_{n-1}, \xi_n)$$

for the dual variables. In this section $\delta > 0$ is fixed, and we argue for $x = (x', x_n) \in \mathbb{R}^{n-1} \times]-\delta, \delta[$.

DEFINITION 2.1. – *We say that $\psi(x_n, \xi') \in C^\infty(]-\delta, \delta[\times \mathbb{R}^{n-1})$ is a weight function of order ϱ, $0 < \varrho < 1$, if $\psi(x_n, \xi') \geq 0$ and for some positive constants C and r:*

$$(2.1) \qquad |D_{x_n}^j D_{\xi'}^\beta \psi(x_n, \xi')| \leq C^{j+|\beta|+1} j! \beta! (1 + |\xi'|)^{\varrho - |\beta|}$$

$$\text{for all } \beta \in \mathbb{Z}_+^{n-1}, j \in \mathbb{Z}_+, x_n \in]-\delta, \delta[, \xi' \in \mathbb{R}^{n-1}, |\xi'| > r.$$

DEFINITION 2.2. – *Let ψ be a weight function of order $\varrho = 1/\sigma$, $1 < \sigma < \infty$; fix $s > 0$, $\tau > 0$. We write $H_{\tau, \sigma}^{s, \psi}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ for the space of all functions f in $L^2(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ such that*

$$(2.2) \qquad \|f\|_{H_{\tau,\sigma}^{s,\psi}} = \|e^{\tau\psi(x_n, D')} f(x)\|_{H^s(\mathbb{R}^{n-1} \times]-\delta, \delta[)} < \infty$$

where

$$(2.3) \qquad e^{\tau\psi(x_n, D')} f(x) = (2\pi)^{-n+1} \int e^{ix'\xi'} e^{\tau\psi(x_n, \xi')} \tilde{f}(\xi', x_n)\, d\xi'.$$

We have denoted by $\tilde{f}$ the Fourier transform of f with respect to x', and by H^s the standard Sobolev spaces.

Gevrey-Sobolev spaces of similar, and even more general type, were studied by several authors; see in particular Kajitani [30] and Kajitani-Wakabayashi [31] for a systematic presentation.

As for infinite order pseudo-differential operators of the type (2.3), see also Rodino [44]. Since ψ is assumed here (x', ξ_n)-independent, an inverse of $e^{\tau\psi(x_n, D')}$ is given in the present case by the operator

$$(2.4) \qquad e^{-\tau\psi(x_n, D')} f(x) = (2\pi)^{-n+1} \int e^{ix'\xi'} e^{-\tau\psi(x_n, \xi')} \tilde{f}(\xi', x_n)\, d\xi'.$$

So we have the isometry between Hilbert spaces:

$$(2.5) \qquad e^{\tau\psi(x_n, D')} : H_{\tau, \sigma}^{s, \psi}(\mathbb{R}^{n-1} \times]-\delta, \delta[) \to H^s(\mathbb{R}^{n-1} \times]-\delta, \delta[)$$

with inverse

$$e^{-\tau\psi(x_n, D')}: H^s(\mathbb{R}^{n-1}\times]-\delta,\delta[)\to H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1}\times]-\delta,\delta[). \tag{2.6}$$

For the benefit of the non-specialists, we give a self-contained proof of the following basic proposition, showing in particular that local solvability for $H^{s,\psi}_{\sigma,\tau}$-data implies solvability for G^σ-data as considered in (1.3).

PROPOSITION 2.1. – *Let f be in $G^\sigma_0(\mathbb{R}^{n-1}\times]-\delta,\delta[)$, space of all the functions satisfying* (1.3) *for $x\in\mathbb{R}^{n-1}\times]-\delta,\delta[$, with compact support there. Then for every weight function ψ of order $\varrho=1/\sigma$ and for all $s>0$, we can find $\tau>0$ such that $f\in H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1}\times]-\delta,\delta[)$.*

The proof is a consequence of the following elementary lemmas.

LEMMA 2.2. – *For every multi-order $\beta=(\beta_1,\dots,\beta_{n-1})\in\mathbb{Z}^{n-1}_+$ and $j\in\mathbb{Z}_+$, we have*

$$D^\beta_{x'}e^{\tau\psi(x_n,D')}=e^{\tau\psi(x_n,D')}D^\beta_{x'}, \tag{2.7}$$

$$D^j_{x_n}e^{\tau\psi(x_n,D')}=\sum_{0\le h\le j}q_{(j-h)\varrho}(x_n,D')\,e^{\tau\psi(x_n,D')}D^h_{x_n} \tag{2.8}$$

where $q_m(x_n,D')$, $m=0,\varrho,\dots,j\varrho$ are pseudo-differential operators with symbol satisfying for suitable constants $c_{l\gamma}$ the estimates

$$|D^l_{x_n}D^\gamma_{\xi'}q_m(x_n,\xi')|\le c_{l\gamma}(1+|\xi'|)^{m-|\gamma|} \tag{2.9}$$

for all $\gamma\in\mathbb{Z}^{n-1}_+$, $l\in\mathbb{Z}_+$, $x_n\in]-\delta,\delta[$, $\xi'\in\mathbb{R}^{n-1}$. In particular we have

$$q_0(x_n,\xi')=1\,,\qquad q_\varrho(x_n,\xi')=j\tau(D_{x_n}\psi)(x_n,\xi'). \tag{2.10}$$

PROOF. – A direct proof of (2.7) is obvious, granted the standard properties of the oscillatory integrals (2.3). As for (2.8), (2.9), we may obtain it by induction on j, since from (2.3)

$$D_{x_n}e^{\tau\psi(x_n,D')}=\tau(D_{x_n}\psi)(x_n,D')\,e^{\tau\psi(x_n,D')}+e^{\tau\psi(x_n,D')}D_{x_n}$$

and

$$D^{j+1}_{x_n}e^{\tau\psi(x_n,D')}=\sum_{0\le h\le j}q_{(j-h)\varrho}(x_n,D')\,e^{\tau\psi(x_n,D')}D^{h+1}_{x_n}+$$

$$\sum_{0\le h\le j}[(D_{x_n}q_{(j-h)\varrho})(x_n,D')+\tau q_{(j-h)\varrho}(x_n,D')(D_{x_n}\psi)(x_n,D')]\,e^{\tau\psi(x_n,D')}D^h_{x_n}.$$

The same argument gives (2.10).

LEMMA 2.3. – *Let $q_m(x_n, \xi')$ satisfy the estimates* (2.9), $m \geqslant 0$, *and let s be an integer with $m \leqslant s$. Then*

$$\|q_m(x_n, D')f\| \leqslant C \sum_{|v| \leqslant s} \|D_{x'}^v f\| \tag{2.11}$$

where norms are taken in $L^2(\mathbb{R}^{n-1} \times]-\delta, \delta[)$.

The proof is obvious by Fourier transform.

LEMMA 2.4. – *When $s \geqslant 0$ is an integer, the following can be taken as equivalent norms in $H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$:*

(i) $\sum_{|\alpha| \leqslant s} \|D^\alpha e^{\tau\psi(x_n, D')} f\|$,

(ii) $\sum_{|\alpha| \leqslant s} \|e^{\tau\psi(x_n, D')} D^\alpha f\|$,

(iii) $\sum_{|\alpha+\beta| \leqslant s} \|D^\alpha e^{\tau\psi(x_n, D')} D^\beta f\|$.

PROOF. – The norm (i) corresponds to our very definition in (2.2). On the other hand, considering norm in $L^2(\mathbb{R}^{n-1} \times]-\delta, \delta[)$, we may estimate using Lemma 2.2 and splitting $\alpha = (\alpha', \alpha_n)$

$$\sum_{|\alpha| \leqslant s} \|D^\alpha e^{\tau\psi(x_n, D')} f\| \leqslant \sum_{|\alpha'| + \alpha_n \leqslant s} \sum_{0 \leqslant h \leqslant \alpha_n} \|q_{(\alpha_n - h)\varrho}(x_n, D') e^{\tau\psi(x_n, D')} D^{(\alpha', h)} f\|.$$

Then we apply Lemma 2.3 to estimate

$$\|q_{(\alpha_n - h)\varrho}(x_n, D') e^{\tau\psi(x_n, D')} D^{(\alpha', h)} f\| \leqslant C \sum_{|\beta| \leqslant \alpha_n - h} \|D_{x'}^\beta e^{\tau\psi(x_n, D')} D^{(\alpha', h)} f\|.$$

Using (2.7), observing that $|(\alpha' + \beta, h)| \leqslant |(\alpha', \alpha_n)| \leqslant s$ and combining with the preceding inequality, we have proved that the norm (i) can be estimated by (ii). In the same way we obtain the converse. Similar arguments apply to (iii).

PROOF OF PROPOSITION 2.1. – Referring to the norm (ii) in Lemma 2.4, we have

$$\|f\|_{H^{s,\psi}_{\tau,\sigma}} = \sum_{|\alpha| \leqslant s} \|e^{\tau\psi(x_n, \xi')} (D^\alpha f)\hat{}(\xi', x_n)\|_{L^2(\mathbb{R}^{n-1}_{\xi'} \times]-\delta, \delta[)}.$$

On the other hand, it follows from the assumption (1.3):

$$|D_{x'}^\beta D^\alpha f(x)| \leqslant C_s^{|\beta|+1} (\beta!)^\sigma, \qquad x \in \mathbb{R}^{n-1} \times]-\delta, \delta[,$$

where we limit consideration to $|\alpha| \leqslant s$, and $\beta \in \mathbb{Z}_+^{n-1}$. Fourier transforming with respect to x', we have

$$|(\xi')^\beta (D^\alpha f)\hat{}(\xi', x_n)| = \left| \int e^{-ix'\xi'} D_{x'}^\beta D^\alpha f(x)\, dx \right| \leqslant C_s^{|\beta|+1} (\beta!)^\sigma$$

for a new constant C_s depending on supp f. This implies as standard (cf. for example § 1.6 in Rodino [44]):

$$|(D^\alpha f)^\frown(\xi', x_n)| \leq C'_s\,(C'_s/(1+|\xi'|))^M (M!)^\sigma$$

for any positive integer M, and hence

$$|(D^\alpha f)^\frown(\xi', x_n)| \leq \inf_M C'_s\,(C'_s/(1+|\xi'|))^M (M!)^\sigma \leq C''_s\, e^{-\varepsilon(1+|\xi'|)^\varrho}$$

where $\varrho = 1/\sigma$, and ε is a suitable positive constant. For τ sufficiently small $\tau\psi(x_n, \xi') \leq \varepsilon(1+|\xi'|)^\varrho/2$ and we conclude

$$|e^{\tau\psi(x_n, \xi')}(D^\alpha f)^\frown(\xi', x_n)| \leq C''_s\, e^{-\varepsilon(1+|\xi'|)^\varrho/2}$$

for $x_n \in]-\delta, \delta[$, $\xi' \in \mathbb{R}^{n-1}$. Therefore $\|f\|_{H^{s,\psi}_{\tau,\sigma}}$ is bounded, and we obtain Proposition 2.1.

It is evident from the proof of Proposition 2.1 that the assumption of G^σ-regularity could be relaxed with respect to the x_n-variable. We now turn attention to the main theme of this paragraph; precisely, we want to study when $H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ is an algebra.

To this end we first produce the following definitions and examples.

DEFINITION 2.3. – *The non-negative function Φ on $\mathbb{R}^N$ is said to be sub-additive if*

$$\Phi(s+t) \leq \Phi(s) + \Phi(t) \qquad \textit{for all } s, t \in \mathbb{R}^N.$$

EXAMPLE 2.1. – $\Phi(t) = |t|^\varrho$ is sub-additive in $\mathbb{R}^N$ for $0 \leq \varrho \leq 1$. The following functions are sub-additive for $t \in \mathbb{R}$, $0 \leq \varrho \leq 1$:

$$\Phi(t) = t_+^\varrho\,, \qquad \text{i.e. } \Phi(t) = 0 \quad \text{for } t \leq 0, = t^\varrho \quad \text{for } t > 0\,;$$

$$\Phi(t) = t_-^\varrho\,, \qquad \text{i.e. } \Phi(t) = (-t)^\varrho \quad \text{for } t < 0, = 0 \quad \text{for } t > 0\,.$$

Let us observe that if $\Phi(t)$ is a sub-additive function in $\mathbb{R}^N$, and $c \geq 0$, then also $c\Phi(t)$ is sub-additive. Moreover, if $\Phi_1(t)$ and $\Phi_2(t)$ are sub-additive, also $\Phi_1(t) + \Phi_2(t)$ is sub-additive. It follows in particular that, for $c, d \in \mathbb{R}_+ \cup \{0\}$ with $c \neq d$ and $0 \leq \varrho \leq 1$, the function $\Phi(t) = ct_-^\varrho + dt_+^\varrho$ is sub-additive in $\mathbb{R}$. In terms of the function sign $t = t/|t|$, setting $c = A - B$, $d = A + B$, we can rewrite $\Phi(t) = (A + B\,\text{sign}\,t)\,|t|^\varrho$, with $A > |B|$.

Other examples can be obtained by observing that if $t = (t_1, t_2) \in \mathbb{R}^N = \mathbb{R}^{N_1}_{t_1} \times \mathbb{R}^{N_2}_{t_2}$, and $\Phi(t_1)$ is sub-additive in $\mathbb{R}^{N_1}_{t_1}$, then $\Phi(t_1)$ in $\mathbb{R}^N$, i.e. $\Phi(t_1) \times 1_{t_2}$, is also sub-additive.

DEFINITION 2.4. – *The non-negative function Φ on $\mathbb{R}^N$ is said to be essentially sub-additive if for some $C > 0$*

$$\Phi(s+t) \leqslant \Phi(s) + \Phi(t) + C \quad \textit{for all } s,\, t \in \mathbb{R}^N .$$

EXAMPLE 2.2. – Let Φ be continuous sub-additive in $\mathbb{R}^N$; let $\widetilde{\Phi}$ be continuous non-negative with $\widetilde{\Phi}(t) = \Phi(t)$ when $|t| \geqslant r$, for some $r > 0$; then $\widetilde{\Phi}$ is essentially sub-additive in $\mathbb{R}^N$, as it is easy to prove. Therefore, using Examples 2.1 and letting $\varphi(t) \in C^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, $\varphi(t) = 1$ for $t \geqslant 1$, $\varphi(t) = 0$ for $|t| \leqslant 1/2$, we obtain that the C^∞ functions

$$\Phi(t) = C\varphi(t)\,|t|^\varrho , \qquad C \in \mathbb{R}_+ , \qquad 0 \leqslant \varrho \leqslant 1 , \qquad t \in \mathbb{R}^N ,$$

$$\Phi(t) = (A + B\,\mathrm{sign}\, t)\,\varphi(t)\,|t|^\varrho , \qquad A > |B| , \qquad 0 \leqslant \varrho \leqslant 1 , \qquad t \in \mathbb{R} ,$$

are essentially sub-additive.

THEOREM 2.5. – *Let the weight function $\psi(x_n, \xi') \in C^\infty(]-\delta, \delta[\times \mathbb{R}^{n-1})$ be essentially sub-additive with respect to ξ', i.e.*

$$\psi(x_n, \xi' + \eta') \leqslant \psi(x_n, \xi') + \psi(x_n, \eta') + C \tag{2.12}$$

for some constant C independent of ξ', $\eta' \in \mathbb{R}^{n-1}$ and $x_n \in]-\delta, \delta[$.

Then for $s \geqslant n+3$ the space $H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ is an algebra, and for a suitable constant C_s we have

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leqslant C_s \|u\|_{H^{s,\psi}_{\tau,\sigma}} \|v\|_{H^{s,\psi}_{\tau,\sigma}} . \tag{2.13}$$

PROOF. – We limit ourselves to consider integers $s \geqslant n+3$, applying then interpolation for arbitrary $s > n+3$. Referring to the norm (ii) in Lemma 2.4 we have

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} = \sum_{|\alpha| \leqslant s} \|e^{\tau\psi(x_n, D')} D^\alpha(uv)\| \leqslant C_s \sum_{|\alpha| \leqslant s} \sum_{\beta+\gamma=\alpha} \|e^{\tau\psi(x_n, D')} D^\beta u D^\gamma v\| =$$

$$C'_s \sum_{|\beta+\gamma| \leqslant s} \|e^{\tau\psi(x_n, D')}(e^{-\tau\psi(x_n, D')} u'_\beta\, e^{-\tau\psi(x_n, D')} v'_\gamma)\|$$

were we take L^2-norms in $\mathbb{R}^{n-1}_{x'} \times]-\delta, \delta[$ and

$$u'_\beta(x) = e^{\tau\psi(x_n, D')} D^\beta u , \qquad v'_\gamma(x) = e^{\tau\psi(x_n, D')} D^\gamma v .$$

Applying Fourier transform with respect to x', we obtain

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leqslant C'_s \sum_{|\beta+\gamma| \leqslant s} \left\| \int H(x_n, \xi', \eta')\, \tilde{u}'_\beta(\xi' - \eta', x_n)\, \tilde{v}'_\gamma(\eta', x_n)\, d\eta' \right\|_{L^2(\mathbb{R}^{n-1}_{\xi'} \times]-\delta, \delta[)}$$

where $H(x_n, \xi', \eta') = \exp[\tau\psi(x_n, \xi') - \tau\psi(x_n, \xi' - \eta') - \tau\psi(x_n, \eta')]$.

Since H is bounded in $]-\delta, \delta[\times \mathbb{R}^{n-1}_{\xi'} \times \mathbb{R}^{n-1}_{\eta'}$, in view of the assumption

(2.12), we conclude

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leqslant C''_s \sum_{|\beta+\gamma| \leqslant s} \left\| \int |\tilde{u}'_\beta(\xi'-\eta', x_n)|\,|\tilde{v}'_\gamma(\eta', x_n)\,d\eta' \right\|_{L^2(\mathbb{R}^{n-1}_{\xi'} \times]-\delta,\delta[)}.$$

If $s \geqslant n+3$, $|\gamma|+|\beta| \leqslant s$, then one at least of the inequalities $|\beta| \leqslant s-(n-1)/2-2$, $|\gamma| \leqslant s-(n-1)/2-2$ must be satisfied, and we may write using Young estimates and setting $t = s-(n-1)/2-2$

$$\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leqslant C''_s \sum_{\substack{|\beta| \leqslant t \\ |\gamma| \leqslant s}} \left(\int_{-\delta}^{\delta} \|\tilde{u}'_\beta(\xi', x_n)\|^2_{L^1(\mathbb{R}^{n-1}_{\xi'})} \|\tilde{v}'_\gamma(\xi', x_n)\|^2_{L^2(\mathbb{R}^{n-1}_{\xi'})}\,dx_n \right)^{1/2} +$$

$$C''_s \sum_{\substack{|\beta| \leqslant s \\ |\gamma| \leqslant t}} \left(\int_{-\delta}^{\delta} \|\tilde{u}'_\beta(\xi', x_n)\|^2_{L^2(\mathbb{R}^{n-1}_{\xi'})} \|\tilde{v}'_\gamma(\xi', x_n)\|^2_{L^1(\mathbb{R}^{n-1}_{\xi'})}\,dx_n \right)^{1/2} \leqslant$$

$$C''_s \sum_{\substack{|\beta| \leqslant t \\ |\gamma| \leqslant s}} \sup_{|x_n| \leqslant \delta} \|\tilde{u}'_\beta(\xi', x_n)\|_{L^1(\mathbb{R}^{n-1}_{\xi'})} \|v'_\gamma\| + C''_s \sum_{\substack{|\beta| \leqslant s \\ |\gamma| \leqslant t}} \sup_{|x_n| \leqslant \delta} \|\tilde{v}'_\gamma(\xi', x_n)\|^2_{L^1(\mathbb{R}^{n-1}_{\xi'})} \|u'_\beta\|.$$

Since $u, v \in H^{s,\psi}_{\tau,\sigma}$ and $|\beta| \leqslant s$, $|\gamma| \leqslant s$, we have from Lemma 2.4, (ii) $u'_\beta, v'_\gamma \in L^2(\mathbb{R}^{n-1}_x \times]-\delta, \delta[)$ and $\|u'_\beta\| \leqslant \|u\|_{H^{s,\psi}_{\tau,\sigma}}$, $\|v'_\gamma\| \leqslant \|v\|_{H^{s,\psi}_{\tau,\sigma}}$. On the other hand, we may estimate as standard

$$\sup_{|x_n| \leqslant \delta} \|\tilde{u}'_\beta(\xi', x_n)\|_{L^1(\mathbb{R}^{n-1}_{\xi'})} \leqslant C_s \sup_{|x_n| \leqslant \delta} \sum_{|\varrho| \leqslant (n-1)/2+1} \|D^\varrho_{x'} u'_\beta(x', x_n)\|_{L^2(\mathbb{R}^{n-1}_{x'})} \leqslant$$

$$\leqslant C'_s \sum_{|v| \leqslant (n-1)/2+2} \|D^v u'_\beta\| \leqslant C'_s \|u\|_{H^{s,\psi}_{\tau,\sigma}}$$

if $|\beta| + (n-1)/2 + 2 \leqslant s$, in view of Lemma 2.4 (iii). Estimating in the same way $\sup_{|x_n| \leqslant \delta} \|\tilde{v}'_\gamma(\xi', x_n)\|_{L^1(\mathbb{R}^{n-1}_{\xi'})}$ we conclude $\|uv\|_{H^{s,\psi}_{\tau,\sigma}} \leqslant C_s \|u\|_{H^{s,\psi}_{\tau,\sigma}} \|v\|_{H^{s,\psi}_{\tau,\sigma}}$ for a new constant C_s.

EXAMPLE 2.3. – In the proof of Theorem 1.2 we shall apply Theorem 2.5 for the following two weight functions, with $\varphi \in C^\infty(\mathbb{R}^{n-1})$, $0 \leqslant \varphi(\xi') \leqslant 1$, $\varphi(\xi') = 1$ for $|\xi'| \geqslant 1$, $\varphi(\xi') = 0$ for $|\xi'| \leqslant 1/2$:

$$\psi(x_n, \xi') = \left(1 + \frac{x_n}{\delta_0}\right) \varphi(\xi')\,|\xi'|^\varrho, \tag{2.14}$$

$$\psi(x_2, \xi_1) = \left(1 + \frac{x_2}{\delta_0} \operatorname{sign} \xi_1\right) \varphi(\xi_1)\,|\xi_1|^\varrho, \tag{2.15}$$

which are essentially sub-additive in view of Examples 2.2 if $\delta_0 > \delta$. Theorem 2.5 for the weight (2.14) was already in Kajitani [30].

We now consider functions $E(x, z)$: $\mathbb{R}^{n-1}_{x'} \times]-\delta, \delta[\times \mathbb{C}^N_z \to \mathbb{C}$, compactly supported with respect to the variables x. More precisely we assume for $1 < \sigma' < \sigma$

$$E(x, z) = \sum_\beta f_\beta(x) z^\beta, \qquad f_\beta \in G^{\sigma'}(\mathbb{R}^{n-1}_{x'} \times]-\delta, \delta[), \qquad z \in \mathbb{C}^N, \tag{2.16}$$

where $\operatorname{supp} f_\beta \subset K \subset\subset \mathbb{R}^{n-1}_{x'} \times]-\delta, \delta[$ and

$$\sup |\partial^\alpha f_\beta(x)| \leq \lambda_\beta A^{|\alpha|} (\alpha!)^{\sigma'} \tag{2.17}$$

with positive constants A and λ_β, the function

$$\widetilde{E}(z) = \sum_\beta \lambda_\beta z^\beta \tag{2.18}$$

being entire.

THEOREM 2.6. – *Let $E(x, z)$ be of the form* (2.16), (2.17), (2.18). *Let*

$$v_j \in H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[), \qquad j = 1, \ldots, N,$$

with $s \geq n+3$ and ψ satisfying (2.12); *set $V = (v_1, \ldots, v_N)$. Then $E(x, V)$ is in $H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$.*

LEMMA 2.7. – *Let f be in $G^{\sigma'}_0(\mathbb{R}^{n-1} \times]-\delta, \delta[)$, with $1 < \sigma' < \sigma$. Then*

$$f \in H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[),$$

for every weight function ψ of order $\varrho = 1/\sigma$, for all $\tau > 0$ and $s \geq 0$. Moreover, if

$$\sup |\partial^\alpha f(x)| \leq \lambda A^{|\alpha|} (\alpha!)^{\sigma'}, \tag{2.19}$$

then $\|f\|_{H^{s, \psi}_{\tau, \sigma}} \leq \lambda A_s$, where A_s depends only on A from the right-hand side of (2.19) *and* $\operatorname{supp} f$.

PROOF OF LEMMA 2.7. – Arguing as in the proof of Proposition 2.1, we obtain for $|\alpha| \leq s$:

$$|(D^\alpha f)^\sim(\xi', x_n)| \leq \lambda C_s e^{-\varepsilon(1 + |\xi'|)^{\varrho'}},$$

where $\varrho' = 1/\sigma'$, and C_s, ε depend on A and $\operatorname{supp} f$. Referring to the norm (ii) in Lemma 2.4 and observing that $\varrho' > \varrho$, we get the the conclusion.

PROOF OF THEOREM 2.6. – We write with obvious vectorial notation

$$\|E(x, V)\|_{H^{s, \psi}_{\tau, \sigma}} \leq \sum_\beta \|f_\beta V^\beta\|_{H^{s, \psi}_{\tau, \sigma}} \leq \sum_\beta C_s^{|\beta|} \|f_\beta\|_{H^{s, \psi}_{\tau, \sigma}} \|V\|^\beta_{H^{s, \psi}_{\tau, \sigma}}$$

where we have applied Theorem 2.5 with C_s as in (2.13). Using then Lemma 2.7, we conclude

$$\|E(x, V)\|_{H^{s,\psi}_{\tau,\sigma}} \leq A_s \sum_\beta \lambda_\beta (C_s \|V\|_{H^{s,\psi}_{\tau,\sigma}})^\beta = A_s \widetilde{E}(C_s \|V\|_{H^{s,\psi}_{\tau,\sigma}}).$$

Let Ω' be a bounded open set, say a neighborhood of the origin, in $\mathbb{R}^{n-1}_{x'}$. We define $\Omega = \Omega' \times]-\delta, \delta[$ and

$$H^{s,\psi}_{\tau,\sigma,\mathrm{comp}}(\Omega) = \{f \in H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[), \ \operatorname{supp} f \text{ is a compact subset of } \Omega\};$$

$$H^{s,\psi}_{\tau,\sigma,\mathrm{loc}}(\Omega) = \{f \in \mathcal{D}'(\Omega), \ \varphi f \in H^{s,\psi}_{\tau,\sigma,\mathrm{comp}}(\Omega) \text{ for every } \varphi \in G_0^{\sigma'}(\Omega), \ 1 < \sigma' < \sigma\};$$

$$H^{s,\psi}_{\tau,\sigma}(\Omega) = \{f \text{ is the restriction to } \Omega \text{ from } H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)\}.$$

We understand from now on $s \geq n+3$, as assumed in Theorem 2.5. Then $H^{s,\psi}_{\tau,\sigma}(\Omega)$, endowed by the standard quotient norm, is an algebra and (2.13) is valid with the same constant C_s. To prepare the applications in the next sections, le us consider as in § 1

$$J(v) = F(x, \partial^\alpha v(x))_{|\alpha| \leq m-1} \tag{2.20}$$

where F is entire function with respect to $z = (\partial^\alpha v) \in \mathbb{C}^N$, and we allow now $G^{\sigma'}$-regularity with respect to x in a neighborhood $\Omega = \Omega' \times]-\delta, \delta[$ of the origin, according to (2.17), (2.18). Let us assume further for some integer $k \geq 0$:

$$\partial_z^\gamma F(x, 0) = 0 \quad \text{if } |\gamma| \leq k, \qquad x \in \Omega. \tag{2.21}$$

PROPOSITION 2.8. – *Under the preceding assumptions for F, there exist two entire functions $F_1(w)$, $F_2(w)$ in $\mathbb{C}$,*

$$F_1(w) = \sum_{h \geq k+1} \lambda_{1h} w^h, \quad F_2(w) = \sum_{h \geq k} \lambda_{2h} w^h \ \text{ with } \lambda_{1h} \geq 0, \quad \lambda_{2h} \geq 0, \tag{2.22}$$

such that for all $v, v_1, v_2 \in H^{s+m-1,\psi}_{\tau,\sigma}(\Omega)$

$$\|J(v)\|_{H^{s,\psi}_{\tau,\sigma}} \leq F_1(\|v\|_{H^{s+m-1,\psi}_{\tau,\sigma}}) \tag{2.23}$$

$$\|J(v_1) - J(v_2)\|_{H^{s,\psi}_{\tau,\sigma}} \leq \|v_1 - v_2\|_{H^{s+m-1,\psi}_{\tau,\sigma}} F_2(\max\{\|v_1\|_{H^{s+m-1,\psi}_{\tau,\sigma}}, \|v_2\|_{H^{s+m-1,\psi}_{\tau,\sigma}}\}) \tag{2.24}$$

where norms are in $H^{s,\psi}_{\tau,\sigma}(\Omega)$ and the hypotheses of Theorem 2.5 on ψ and s are assumed to be satisfied.

PROOF. – From (iii) in Lemma 2.4 we have

$$\|\partial^\alpha v\|_{H^{s,\psi}_{\tau,\sigma}} \leq \|v\|_{H^{s+m-1,\psi}_{\tau,\sigma}} \qquad \text{for } |\alpha| \leq m-1.$$

Applying the proof of Theorem 2.6 to $V = (\partial^\alpha v)_{|\alpha| \leq m-1}$, we then get (2.23) with

$$F_1(w) = A_s \sum_{|\beta| > k} \lambda_\beta (C_s w)^{|\beta|} .$$

As for $F_2(w)$, we use Taylor formula to write:

$$F(x, z^1) - F(x, z^2) = \langle z^1 - z^2, G(x, z^1, z^2) \rangle ,$$

$$G(x, z^1, z^2) = \int_0^1 F_z(x, z^1 + t(z^2 - z^1))\, dt , \qquad z^1, z^2 \in \mathbb{C}^N ,$$

and then we apply Theorem 2.6 to $G(x, \partial^\alpha v_1, \partial^\beta v_2)_{|\alpha|, |\beta| \leq m-1}$ in order to define $F_2(w)$.

3. – Proof of Theorem 1.1 and local solvability of the linear equation.

Since we are looking for local, or microlocal, solvability at the origin, it will be not restrictive to multiply the coefficients $c_\alpha(x)$ of $P(x, D)$ in (1.1), (1.2) by a function $\chi \in G_0^{\sigma'}(\Omega)$, with $1 \leq \sigma' < \sigma$, $\chi(x) = 1$ in a smaller neighborhood of the origin. From now we shall then assume $c_\alpha(x) \in G_0^{\sigma'}(\Omega)$ for $|\alpha| \leq m$, $\Omega = \Omega' \times]-\delta, \delta[$, with Ω' bounded neighborhood of the origin in $\mathbb{R}^{n-1}$ and $\delta > 0$.

Following the notations and the definitions of the preceding § 2, we consider here a weight function $\psi(x_n, \xi')$ of order $\varrho = 1/\sigma$ satisfying (2.12), and the corresponding Gevrey-Sobolev spaces $H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$, $H^{s, \psi}_{\tau, \sigma, \mathrm{comp}}(\Omega)$, $H^{s, \psi}_{\tau, \sigma, \mathrm{loc}}(\Omega)$, $H^{s, \psi}_{\tau, \sigma}(\Omega)$, where $\tau > 0$ and $s \geq n + 3$.

From Lemma 2.4, Theorem 2.5, Lemma 2.7 we have that for $s \geq m + n + 3$

$$P(x, D): H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) \to H^{s-m, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) ;$$

we may also regard $P(x, D)$ as continuous map:

$$P(x, D): H^{s, \psi}_{\tau, \sigma, \mathrm{loc}}(\Omega) \to H^{s-m, \psi}_{\tau, \sigma, \mathrm{comp}}(\Omega) ,$$

$$P(x, D): H^{s, \psi}_{\tau, \sigma}(\Omega) \to H^{s-m, \psi}_{\tau, \sigma}(\Omega) .$$

Consider

$$\tilde{P} = e^{\tau \psi(x_n, D')} P(x, D) e^{-\tau \psi(x_n, D')} ; \tag{3.1}$$

in view of (2.5), (2.6) we have

$$\widetilde{P}\colon H^s(\mathbb{R}^{n-1}\times]-\delta,\delta[)\to H^{s-m}(\mathbb{R}^{n-1}\times]-\delta,\delta[)$$

and

$$P=e^{-\tau\psi(x_n,D')}\widetilde{P}e^{\tau\psi(x_n,D')}\,. \tag{3.2}$$

We can now give the following more precise statement of Theorem 1.1. Let us fix attention on a weight function of type (2.14):

$$\psi(x_n,\xi')=\left(1+\frac{x_n}{2\delta}\right)\varphi(\xi')\,|\xi'|^{\varrho}\,, \tag{3.3}$$

where $\varrho=1/\sigma$ and $\varphi\in C_0^\infty(\mathbb{R}^{n-1})$, $0\leqslant\varphi\leqslant 1$, $\varphi(\xi')=1$ for $|\xi'|\geqslant 1$, $\varphi(\xi)=0$ for $|\xi'|\leqslant 1/2$.

THEOREM 3.1. – *Let the principal symbol $p_m(x,\xi)$ of $P(x,D)$ satisfy* (1.8), (1.9), (1.10) *in a conic neighborhood Γ of the point (x_0,ξ_0), with $x_0=0$ say, $\xi_0'\neq 0$. Let $\tau>0$ be fixed and let $\psi(x_n,\xi')$ be chosen as in* (3.3) *with $1/\varrho=\sigma<k/(k-1)$. Let $\Omega=\Omega'\times]-\delta,\delta[$ be sufficiently small. Define $\widetilde{P}$ according to* (3.1). *Then there exists a linear map*

$$\widetilde{E}\colon H^s(\mathbb{R}^{n-1}\times]-\delta,\delta[)\to H^{s+m-k(1-1/\sigma)}(\mathbb{R}^{n-1}\times]-\delta,\delta[)$$

such that

$$\widetilde{P}\widetilde{E}=\chi(x)\lambda(D)+\widetilde{R}\,, \tag{3.4}$$

where $\chi(x)$ is fixed arbitrary in $C_0^\infty(\Omega)$ with support in a neighborhood of x_0 and $\lambda(\xi)$ in $C^\infty(\mathbb{R}^n)$ homogeneous of degree 0 for large $|\xi|$, arbitrary with support in a conic neighborhood of ξ_0. Moreover $\widetilde{R}$ is a linear regularizing map, i.e.

$$\widetilde{R}\colon H^s(\mathbb{R}^{n-1}\times]-\delta,\delta[)\to H^t(\mathbb{R}^{n-1}\times]-\delta,\delta[)$$

for all $t\geqslant 0$.

Coming back to the equation

$$P(x,D)v=f\in H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1}\times]-\delta,\delta[)\,,$$

we may rewrite it in the form

$$\widetilde{P}e^{\tau\psi(x_n,D')}v=e^{\tau\psi(x_n,D')}f\in H^s(\mathbb{R}^{n-1}\times]-\delta,\delta[)\,.$$

Setting

$$E = e^{-\tau\psi(x_n, D')} \widetilde{E} e^{\tau\psi(x_n, D')}, \qquad R = e^{-\tau\psi(x_n, D')} \widetilde{R} e^{\tau\psi(x_n, D')},$$

we then obtain Theorem 1.1 with

$$v = Ef \in H^{s+m-k(1-1/\sigma),\, \psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[).$$

In fact, in view of (3.4), the function v is a microlocal solution in the sense that

$$P(x, D)\, v = e^{-\tau\psi(x_n, D')} \chi(x)\, \lambda(D)\, e^{\tau\psi(x_n, D')} f + Rf, \tag{3.5}$$

where we take $\chi(x) = 1$ in a neighborhood of x_0, $\lambda(\xi) = 1$ in a conic neighborhood of ξ_0, and

$$R\colon H^{s,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) \to H^{t,\psi}_{\tau,\sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$$

for all $t \geq 0$.

Aiming to applications to semilinear equations, we shall content here with (3.5); we observe however that a more explicit meaning to (3.5) could be given in terms of the Gevrey-Sobolev wave front sets of Chen Hua-Rodino [6].

The proof of Theorem 3.1 will be based on the following preliminary results.

DEFINITION 3.1. – *We say that $q(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ is in the class $\Theta^{m,M}$, $m \geq 0$ integer, $M \in \mathbb{R}$, if $q(x, \xi)$ is a polynomial with respect to ξ_n:*

$$q(x, \xi) = \sum_{j=0}^{m} q_{M-j}(x, \xi')\, \xi_n^j, \tag{3.6}$$

where the symbols $q_r(x, \xi')$, $r = M - j$, $j = 0, 1, \ldots, m$, are compactly supported with respect to x and satisfy the estimates

$$|D_x^\alpha D_{\xi'}^\beta q_r(x, \xi')| \leq C_{\alpha\beta}(1 + |\xi'|)^{r - |\beta|} \tag{3.7}$$

for $\alpha \in \mathbb{Z}_+^n$, $\beta \in \mathbb{Z}_+^{n-1}$, with suitable constants $C_{\alpha\beta}$ independent of $x \in \Omega$, $\xi' \in \mathbb{R}^{n-1}$. When $m = 0$ we understand $q(x, \xi)$ is ξ_n-independent satisfying estimates (3.7) *with $r = M$.*

The corresponding pseudo-differential operators $q(x, D)$ map $H^\lambda_{\mathrm{loc}}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ into $H^{\lambda-N}_{\mathrm{comp}}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$, $N = \max(m, M)$, since $\Theta^{m,M}$ is included in the class $S^N_{0,0}$, cf. Hörmander [29]. It is known that the rules of the symbolic calculus do not apply in full force to operators with symbol in $S^N_{0,0}$. However we may note that if $q^{(j)}(x, \xi) \in \Theta^{m_j, M_j}$ $j = 1, 2$, then we get $q(x, D) = q^{(1)}(x, D)\, q^{(2)}(x, D)$ with $q(x, \xi) \in \Theta^{m_1+m_2, M_1+M_2}$. Moreover in any conic subset Γ of $\Omega \times \mathbb{R}^n$ where $|\xi'| \geq \varepsilon |\xi_n|$ for some $\varepsilon > 0$, we have $\Theta^{m,M} \subset S^M_{1,0}$, i.e.

(3.6), (3.7) imply

$$|D_x^\alpha D_\xi^\gamma q(x, \xi)| \leq C_{\alpha\gamma}(1 + |\xi|)^{M - |\gamma|} \qquad \text{for } (x, \xi) \in \Gamma .$$

At a symbolic level, this will allow applications in Γ of the standard $S^m_{\varrho, \delta}$ theory for construction of parametrices, see again Hörmander [29].

PROPOSITION 3.2. – *Let $P(x, D)$ be a linear partial differential operator with coefficients in $G_0^{\sigma'}(\Omega)$, $\Omega = \Omega' \times]-\delta, \delta[$, $1 < \sigma' < \sigma$, and let $\psi(x_n, \xi')$ be a weight function of order $\varrho = 1/\sigma$ satisfying* (2.12). *Fix $\tau > 0$. Then the operator $\widetilde{P}$ defined by* (3.1) *can be written as a pseudo-differential operator with symbol*

$$\tilde{p}(x, \xi) \in \Theta^{m, m} .$$

Precisely we have

$$\tilde{p}(x, \xi) = p_m(x, \xi) + q_{m - (1 - \varrho)}(x, \xi) + q_{m - 2(1 - \varrho)}(x, \xi)$$

where:

i) $p_m(x, \xi)$ *is the principal symbol of* $P(x, D)$;

ii) $q_{m - (1 - \varrho)}(x, \xi) \in \Theta^{m, m - (1 - \varrho)}$ *is given by*

$$i\tau \partial_{\xi_n} p_m(x, \xi)\, \partial_{x_n} \psi(x_n, \xi') - i\tau \sum_{j=1}^{n-1} \partial_{x_j} p_m(x, \xi)\, \partial_{\xi_j} \psi(x_n, \xi') ; \tag{3.8}$$

iii) $q_{m - 2(1 - \varrho)}(x, \xi) \in \Theta^{m, m - 2(1 - \varrho)}$.

We point out that this proposition is, essentially, a particular case of Proposition 2.13 in Kajitani-Wakabayashi [31], see also Proposition 2.5 in Kajitani [30]. Aiming to a self contained proof, for benefit of non-specialists, we observe first that Lemma 2.2 gives for $\alpha = (\alpha', \alpha_n) \in \mathbb{Z}^n_+$, $\alpha_n \geq 1$:

$$\begin{aligned} &e^{\tau\psi(x_n, D')} D_x^\alpha e^{-\tau\psi(x_n, D')} = \\ &\qquad = D_x^\alpha - \alpha_n \tau (D_{x_n}\psi)(x_n, D')\, D_{x'}^{\alpha'} D_{x_n}^{\alpha_n - 1} + q_{|\alpha| - 2(1 - \varrho)}(x_n, D) \end{aligned} \tag{3.9}$$

where $q_{|\alpha| - 2(1 - \varrho)} \in \Theta^{|\alpha|, |\alpha| - 2(1 - \varrho)}$. This corresponds to Proposition 3.2 in the case $P = D^\alpha$. Moreover we have the following:

LEMMA 3.3. – *Let* $a \in G_0^{\sigma'}(\Omega)$, $1 < \sigma' < \sigma$. *Then, for* $\psi(x_n, \xi')$ *as in Proposition* 3.2:

$$e^{\tau\psi(x_n, D')} a(x) e^{-\tau\psi(x_n, D')} = a(x) + \tau \sum_{j=1}^{n-1} D_{x_j} a(x)\, \partial_{\xi_j} \psi(x_n, D') + q_{-2(1-\varrho)}(x, D'), \tag{3.10}$$

where $q_{-2(1-\varrho)}(x, \xi') \in \Theta^{0,\, -2(1-\varrho)}$.

PROOF. – We may write

$$e^{\tau\psi(x_n, D')} a(x)\, e^{-\tau\psi(x_n, D')} = q(x, D')$$

where the symbol of the pseudo differential operator $q(x, D')$ is given by the oscillatory integral

$$q(x, \xi') = (2\pi)^{-n+1} \int e^{ix'\eta'} e^{\tau\psi(x_n, \xi'+\eta') - \tau\psi(x_n, \xi')} \tilde{a}(\eta', x_n)\, d\eta' . \tag{3.11}$$

As before $\tilde{a}$ denotes the Fourier transform of $a(x', x_n)$ with respect to x'.

Using Taylor formula, we may write for every $N \geq 2$

$$e^{\tau\psi(x_n, \xi'+\eta') - \tau\psi(x_n, \xi')} = 1 + \sum_{j=1}^{n-1} \tau \partial_{\xi_j} \psi(x_n, \xi')\, \eta_j + \sum_{2 \leq |\beta| < N} (\beta!)^{-1} \lambda_\beta(x_n, \xi')\, \eta'^\beta + r_N(x_n, \xi', \eta')$$

where

$$r_N(x_n, \xi', \eta') = \sum_{|\beta|=N} \frac{\eta'^\beta}{\beta!} \int_0^1 \lambda_\beta(x_n, \xi'+t\eta')\, e^{\tau\psi(x_n, \xi'+t\eta') - \tau\psi(x_n, \xi')}\, dt \tag{3.12}$$

with

$$\lambda_\beta(x_n, \xi') = e^{-\tau\psi(x_n, \xi')} \partial_{\xi'}^\beta e^{\tau\psi(x_n, \xi')} . \tag{3.13}$$

Inserting in (3.11) we obtain (3.10) with

$$q_{-2(1-\varrho)}(x, \xi') = \sum_{2 \leq |\beta| < N} (\beta!)^{-1} D_x^\beta a(x)\, \lambda_\beta(x_n, \xi') + R_N(x, \xi'),$$

where

$$R_N(x, \xi') = (2\pi)^{-n+1} \int e^{ix'\eta'} r_N(x_n, \xi', \eta')\, \tilde{a}(\eta', x_n)\, d\eta' .$$

To prove $q_{-2(1-\varrho)} \in \Theta^{0,\, -2(1-\varrho)}$, we observe first that $\lambda_\beta(x_n, \xi') \in \Theta^{0,\, -|\beta|(1-\varrho)}$, as it is easy to deduce by induction. Therefore it will be sufficient to estimate R_N. From (2.12) and (2.1) we have

$$\psi(x_n, \xi'+t\eta') \leq \psi(x_n, \xi') + C(1+|\eta'|)^\varrho \tag{3.14}$$

for a suitable constant C; applying in (3.12) we obtain

$$|r_N(x_n, \xi', \eta')| \leq C'(1+|\xi'|)^{-N(1-\varrho)}(1+|\eta'|)^{2N} e^{C(1+|\eta'|)^\varrho} .$$

On the other hand, from the proof of Lemma 2.7 we have

$$|(D^\gamma a)\tilde{}(\eta', x_n)| \leq C''_\gamma e^{-\varepsilon(1+|\eta'|)^{\varrho'}}, \tag{3.15}$$

where $\varepsilon > 0$ and $\varrho' = 1/\sigma' > \varrho$. Inserting in the expression of $R_N(x, \xi')$ we conclude

$$|R_N(x, \xi')| \leq C'''(1+|\xi'|)^{-N(1-\varrho)} .$$

In general, for $\alpha = (\alpha', \alpha_n) \in \mathbb{Z}^n_+$, $\beta \in \mathbb{Z}^{n-1}_+$, we have

$$D^\alpha_x D^\beta_{\xi'} R_N(x, \xi') =$$

$$(2\pi)^{-n+1} \sum_{j+h=\alpha_n} \frac{\alpha_n!}{j!\,h!} \int e^{ix'\eta'} \eta^{\alpha'} D^j_{x_n} D^\beta_{\xi'} r_N(x_n, \xi', \eta')\, D^h_{x_n} \tilde{a}(\eta', x_n)\, d\eta' .$$

Now from (3.12) and (3.14) we obtain easily for a suitable $M > 0$

$$|D^j_{x_n} D^\beta_{\xi'} r_N(x_n, \xi', \eta')| \leq C'_{j\beta}(1+|\xi'|)^{-(N+|\beta|)(1-\varrho)+j\varrho}(1+|\eta'|)^M e^{C(1+|\eta'|)^\varrho} ;$$

hence using (3.15) we get the rough estimates

$$|D^\alpha_x D^\beta_{\xi'} R_N(x, \xi')| \leq C_{\alpha\beta}(1+|\xi'|)^{-N(1-\varrho)+|\alpha|} .$$

By taking N large enough, these are sufficient to conclude

$$|D^\alpha_x D^\beta_{\xi'} q_{-2(1-\varrho)}(x, \xi')| \leq C'_{\alpha\beta}(1+|\xi'|)^{-2(1-\varrho)-|\beta|} .$$

Lemma 3.3 is therefore proved.

PROOF OF PROPOSITION 3.2. – Using (3.9), Lemma 3.3 and the remarks after Definition 3.1, we have for $\alpha = (\alpha', \alpha_n)$, $|\alpha| \leq m$, $\alpha_n \geq 1$:

$$e^{\tau\psi(x_n, D')} c_\alpha(x)\, D^\alpha e^{-\tau\psi(x_n, D')} = c_\alpha(x)\, D^\alpha_x + i\tau\alpha_n c_\alpha(x)(\partial_{x_n}\psi)(x_n, D')\, D^{\alpha'}_{x'} D^{\alpha_n - 1}_{x_n} -$$

$$i\tau \sum_{j=1}^{n-1} \partial_{x_j} c_\alpha(x)\, \partial_{\xi_j}\psi(x_n, D')\, D^\alpha + q_{|\alpha|-2(1-\varrho)}(x, D),$$

with $q_{|\alpha|-2(1-\varrho)} \in \Theta^{|\alpha|, |\alpha|-2(1-\varrho)}$. This proves Proposition 3.2 for an operator with symbol $c_\alpha(x)\xi^\alpha$. Summing for $|\alpha| \leq m$, we obtain the conclusion in general.

We may now return to Theorem 3.1. Also for this proof we note similarity with the arguments of Kajitani-Wakabayashi [31], concerning micro-hyperbolic operators. We shall deduce Theorem 3.1 from the following:

LEMMA 3.4. – *Let the principal symbol* $p_m(x, \xi)$ *of* $P(x, D)$ *satisfy* (1.8), (1.9), (1.10) *in a conic neighborhood* Γ *of the point* (x_0, ξ_0), $x_0 = 0$, $\xi_0 \neq 0$. *Let* $\psi(x_n, \xi')$ *be chosen as in* (3.3) *with* $1/\varrho = \sigma < k/(k-1)$ *and* $\tau > 0$ *be fixed. Define* $\tilde{P}$ *according to* (3.1) *and let* $\tilde{p}(x, \xi) \in \Theta^{m, m}$ *be the corresponding symbol. Then, possibly after a shrinking of* Γ *and for large* $|\xi|$, *we have*

$$\tilde{p}(x, \xi) = g_{m-k}(x, \xi) \prod_{j=1}^{k} l^{(j)}(x, \xi) + \tilde{p}_{m-1}(x, \xi), \qquad (x, \xi) \in \Gamma \tag{3.16}$$

where:

(i) $g_{m-k}(x, \xi) = e_{m-k}(x, \xi) + g_{m-k-(1-1/\sigma)}(x, \xi)$, *with*

(I) $e_{m-k}(x, \xi) \in \Theta^{m-k, m-k}$ *homogeneous of order* $m-k$ *with respect to* ξ *and elliptic in* Γ;

(II) $g_{m-k-(1-1/\sigma)}(x, \xi) \in \Theta^{m-k, m-k-(1-1/\sigma)}$.

(ii) *For* $j = 1, \dots, k$ *we have*

$$l^{(j)}(x, \xi) = \xi_n + v_1^{(j)}(x, \xi') + v_{1/\sigma}^{(j)}(x, \xi') + v_{1-2(1-1/\sigma)}^{(j)}(x, \xi') \ \text{with:}$$

(I) $v_1^{(j)}(x, \xi')$ *homogeneous of order* 1 *with respect to* ξ', *satisfying*

$$\operatorname{Im} v_1^{(j)}(x, \xi') \geq 0 \ \text{for all } j = 1, \dots, k \text{ and } (x, \xi) \in \Gamma;$$

(II) $v_{1/\sigma}^{(j)}(x, \xi') = i(\tau/2\delta)|\xi'|^{1/\sigma} + \tau(1 + x_n/2\delta)\, \tilde{v}_{1/\sigma}^{(j)}(x, \xi')$, *where* $\tilde{v}_{1/\sigma}^{(j)}(x, \xi') \in \Theta^{0, 1/\sigma}$ *is independent of* τ, δ (*as well as* $e_{m-k}(x, \xi)$, $v_1^{(j)}(x, \xi')$ *are*);

(III) $v_{1-2(1-1/\sigma)}^{(j)}(x, \xi') \in \Theta^{0, 1-2(1-1/\sigma)}$.

(iii) $\tilde{p}_{m-1}(x, \xi) \in \Theta^{m, m-1}$.

PROOF. – In view of the assumption (1.9), we may rewrite (1.8):

$$p_m(x, \xi) = e_{m-k}(x, \xi) \prod_{j=1}^{k} (\xi_n - v_1^{(j)}(x, \xi')), \tag{3.17}$$

where $e_{m-k}(x, \xi)$ is a new elliptic factor in Γ and $v_1^{(j)}(x, \xi')$, $j = 1, \dots, k$, are homogeneous of order 1 with respect to ξ'. Since (1.10) is invariant under the multiplication by elliptic factors preserving (1.9), we have $\operatorname{Im} v_1^{(j)}(x, \xi) \geq 0$ for all $j = 1, \ldots, k$ and $(x, \xi) \in \Gamma$. Applying Proposition 3.2 we then obtain with the present choice of $\psi(x_n, \xi')$:

$$\tilde{p}(x, \xi) = e_{m-k}(x, \xi) \prod_{j=1}^{k} (\xi_n - v_1^{(j)}(x, \xi')) + i \frac{\tau}{2\delta} |\xi'|^{1/\sigma} \times$$

$$\left\{ \partial_{\xi_n} e_{m-k}(x, \xi) \prod_{j=1}^{k} (\xi_n - v_1^{(j)}(x, \xi')) + e_{m-k}(x, \xi) \sum_{j=1}^{k} \prod_{h \neq j} (\xi_n - v_1^{(h)}(x, \xi') \right\} +$$

$$\tau \left(1 + \frac{x_n}{2\delta} \right) q_{m-(1-1/\sigma)}(x, \xi) + q_{m-2(1-1/\sigma)}(x, \xi),$$

where $q_{m-(1-1/\sigma)} \in \Theta^{m, m-(1-1/\sigma)}$, $q_{m-2(1-1/\sigma)} \in \Theta^{m, m-2(1-1/\sigma)}$, and $q_{m-(1-1/\sigma)}$ does not depend on τ and δ. We may then impose (3.16), and determine $g_{m-k-(1-1/\sigma)}$, $v^{(j)}_{1-2(1-1/\sigma)}$, $\tilde{p}_{m-1}$ by a straightforward algebraic computation.

Before proving Theorem 3.1, we recall some basic facts concerning $S^m_{\varrho,\delta}$-classes. We argue on a symbol $q(x, \xi) \in S^m_{1,0}(\Gamma)$, that is we assume estimates as those in the remark after Definition 3.1 are satisfied. We say that $q(x, \xi)$ is of type (m, m', ϱ, δ) in Γ, $m' \leq m$, $0 \leq \delta < \varrho \leq 1$, if for suitable positive constants c, C, $c_{\alpha\beta}$ we have in Γ

(3.18) $$|q(x, \xi)| \geq c|\xi|^{m'} \quad \text{for } |\xi| \geq C,$$

(3.19) $$|D_x^\alpha D_\xi^\beta q(x, \xi)| \leq c_{\alpha\beta} |q(x, \xi)| (1 + |\xi|)^{-\varrho|\beta| + \delta|\alpha|}.$$

If $q(x, \xi)$ is of type (m, m', ϱ, δ) then it admits an inverse $q'(x, \xi) \in S^{-m'}_{\varrho,\delta}(\Gamma)$, i.e.

$$|D_x^\alpha D_\xi^\beta q'(x, \xi)| \leq c'_{\alpha\beta} (1 + |\xi|)^{-m' - \varrho|\beta| + \delta|\alpha|}$$

and $q \# q' = \sum (\alpha!)^{-1} \partial_\xi^\alpha q(x, \xi) D_x^\alpha q'(x, \xi) \sim 1$, cf. Hörmander [29]. We observe the following:

(3.20) If $q(x, \xi)$ is of type (m, m', ϱ, δ) and $\lambda(x, \xi) \in S^{m'-\varepsilon}_{1,0}(\Gamma)$ for some $\varepsilon > 0$, then $q(x, \xi) + \lambda(x, \xi)$ is also of type (m, m', ϱ, δ).

(3.21) Assume $q_j(x, \xi)$ is of type $(m_j, m_j', \varrho, \delta)$, $j = 1, 2$. Then $q(x, \xi) = q_1(x, \xi)\, q_2(x, \xi)$ is of type $(m_1 + m_2, m_1' + m_2', \varrho, \delta)$.

Assume $q(x, \xi)$ is of type $(m, m', \varrho, 1-\varrho)$ with $1/2 < \varrho \leq 1$, in Γ. Let $x = x(y, \eta)$, $\xi = \xi(y, \eta)$ be a C^∞ diffeomorphism, with components homogeneous in η of degree $0, 1$ respectively, defined from a cone Λ to the cone Γ. Then

(3.22) $r(y, \eta) = q(x(y, \eta), \xi(y, \eta))$ is of type $(m, m', \varrho, 1-\varrho)$ in Λ.

PROOF OF THEOREM 3.1. – As in Lemma 3.4, consider $\tilde{p}(x, \xi) \in \Theta^{m, m}$, symbol of $\tilde{P}$, which we regard as element of $S^m_{1,0}(\Gamma)$, according to the remark after Definition 3.1. Let us prove that $\tilde{p}(x, \xi)$ is of type $(m, m-k(1-1/\sigma), 1/\sigma, 1-1/\sigma)$ in Γ. Expressing $\tilde{p}(x, \xi)$ as in (3.16), we first observe that we may ignore $\tilde{p}_{m-1}(x, \xi)$ in view of (3.20). Moreover, in view of (3.21), it will be sufficient to check that every $l^{(j)}(x, \xi)$ is of type $(1, 1/\sigma, 1/\sigma, 1-1/\sigma)$, since $g_{m-k}(x, \xi)$ is of type $(m-k, m-k, 1, 0)$ because of the ellipticity of its principal symbol $e_{m-k}(x, \xi)$. As for $l^{(j)}(x, \xi)$, we may apply again (3.20) and limit ourselves to study the factors

$$l'(x, \xi) = \xi_n + v_1(x, \xi') + i\frac{\tau}{2\delta}|\xi'|^{1/\sigma} + \tau\left(1 + \frac{x_n}{2\delta}\right)\tilde{v}_{1/\sigma}(x, \xi'),$$

where we omit the index j for simplicity of notations.

Let us write

$$\mu_{1/\sigma}(x, \xi') = \frac{\tau}{2\delta}|\xi'|^{1/\sigma} + \tau\left(1 + \frac{x_n}{2\delta}\right)\operatorname{Im}\tilde{v}_{1/\sigma}(x, \xi')$$

and observe that

$$\mu_{1/\sigma}(x, \xi') \geqslant \frac{\tau}{4\delta}|\xi'|^{1/\sigma} \tag{3.23}$$

if δ is chosen sufficiently small. Let us also write

$$\nu_{1/\sigma}(x, \xi') = \tau\left(1 + \frac{x_n}{2\delta}\right)\operatorname{Re}\tilde{v}_{1/\sigma}(x, \xi'),$$

$$v_1(x, \xi') = \nu_1(x, \xi) + i\mu_1(x, \xi').$$

It will be now useful to perform a change of variables $y(x, \xi)$, $\eta(x, \xi)$ with C^∞-inverse $x = x(y, \eta)$, $\xi = \xi(y, \eta)$ as in (3.22), by imposing

$$\eta_n = \xi_n - \nu_1(x, \xi')$$

(cf. Hörmander [29], Egorov [15], where the map $y = y(x, \xi)$, $\eta = \eta(x, \xi)$ was constructed to be symplectic, that is not necessary here). In view of (3.22), we are therefore reduced to consider in the corresponding cone Λ

$$\tilde{l}(y, \eta) = \eta_n + i\tilde{\mu}_1(y, \eta) + i\tilde{\mu}_{1/\sigma}(y, \eta) + \tilde{\nu}_{1/\sigma}(y, \eta), \tag{3.24}$$

where from Lemma 3.4 we have

$$\tilde{\mu}_1(y, \eta) = \mu_1(x(y, \eta), \xi'(y, \eta)) \geqslant 0, \tag{3.25}$$

and moreover for some $C > 0$

$$\tilde{\mu}_{1/\sigma}(y, \eta) = \mu_{1/\sigma}(x(y, \eta), \xi'(y, \eta)) \geq C|\eta|^{1/\sigma} \tag{3.26}$$

in view of (3.23). As for $\tilde{\nu}_{1/\sigma}(y, \eta) = \nu_{1/\sigma}(x(y, \eta), \xi'(y, \eta))$, we simply observe that it is real valued, homogeneous with respect to η of order $1/\sigma$.

We have to prove that $\tilde{l}(y, \eta)$ in (3.24) is of type $(1, 1/\sigma, 1/\sigma, 1 - 1/\sigma)$.

To this end, we first observe that for positive constants c, C and large $|\eta|$ we have in Λ

$$c(|\eta_n| + |\eta'|^{1/\sigma} + \tilde{\mu}_1(y, \eta)) \leq |\tilde{l}(y, \eta)| \leq C(|\eta_n| + |\eta'|^{1/\sigma} + \tilde{\mu}_1(y, \eta)) \tag{3.27}$$

in view of (3.25), (3.26). Therefore obviously

$$|\tilde{l}(y, \eta)| \geq c|\eta|^{1/\sigma}.$$

On the other hand for $\beta \neq 0$

$$|D_y^\alpha D_\eta^\beta \tilde{l}(y, \eta)| \leq c_{\alpha\beta}(1 + |\eta|)^{1 - |\beta|} \leq c'_{\alpha\beta} |\tilde{l}(y, \eta)| (1 + |\eta|)^{1 - 1/\sigma - |\beta|} \leq$$

$$c''_{\alpha\beta} |\tilde{l}(y, \eta)| (1 + |\eta|)^{-|\beta|/\sigma},$$

whereas in the case $\beta = 0$, $\alpha \neq 0$

$$|D_y^\alpha \tilde{l}(y, \eta)| \leq c_\alpha(1 + |\eta|) \leq c'_\alpha |\tilde{l}(y, \eta)| (1 + |\eta|)^{1 - 1/\sigma} \leq c''_\alpha |\tilde{l}(y, \eta)| (1 + |\eta|)^{(1 - 1/\sigma)|\alpha|}.$$

We have then proved that $\tilde{l}(y, \eta)$ is of type $(1, 1/\sigma, 1/\sigma, 1 - 1/\sigma)$ in Λ and therefore $\tilde{p}(x, \xi)$ is of type $(m, m - k(1 - 1/\sigma), 1/\sigma, 1 - 1/\sigma)$ in Γ. We may then construct in Γ

$$q(x, \xi) \in S^{m, m - k(1 - 1/\sigma)}_{1/\sigma, 1 - 1/\sigma}$$

such that $\tilde{p} \# q \sim 1$. Consider in the same class the symbol

$$\tilde{e}(x, \xi) = q \# (\chi(x) \lambda(\xi))$$

which, because of the assumptions on $\chi(x)$ and $\lambda(\xi)$, is well defined in $\Omega \times \mathbb{R}^n$. Let finally $\widetilde{E}$ be a properly supported pseudo-differential operator with symbol $\tilde{e}(x, \xi)$. If $\lambda(\xi) = 0$ in a conic neighborhood of the manifold $\xi' = 0$, the symbol of $\widetilde{P}\widetilde{E}$ can be computed as standard and (3.4) is satisfied for a suitable regularizing map $\widetilde{R}$. Since $\widetilde{E}$ has the required continuity property, Theorem 3.1 is proved.

REMARK 3.1. – If in the hypotheses of Theorem 3.1 we replace (1.10) with (1.11), then the conclusions keep valid provided in the statement we refer to the weight function

$$\psi(x_n, \xi') = \left(1 - \frac{x_n}{2\delta}\right) \varphi(\xi') |\xi'|^{\varrho} .$$

In fact, it is sufficient to observe that this choice of $\psi(x_n, \xi')$ gives in the preceding proof

$$\tilde{\mu}_{1/\sigma}(y, \eta) \leqslant -C|\eta|^{1/\sigma} \tag{3.28}$$

and from (1.11) we have

$$\tilde{\mu}_1(y, \eta) \leqslant 0 . \tag{3.29}$$

The basic estimate (3.27) reads then

$$c(|\eta_n| + |\eta'|^{1/\sigma} - \tilde{\mu}_1(y, \eta)) \leqslant |\tilde{l}(y, \eta)| \leqslant C(|\eta_n| + |\eta'|^{1/\sigma} - \tilde{\mu}_1(y, \eta)) \tag{3.30}$$

and we may conclude as before.

We begin now to prove Theorem 1.2 in the linear case.

THEOREM 3.5. – *Let the principal symbol $p_m(x, \xi)$ of $P(x, D)$ satisfy* (1.12), *or else* (1.13), (1.14). *Let $\tau > 0$ be fixed and let $\psi(x_n, \xi')$ be chosen as in* (3.3) *with $1/\varrho = \sigma < k/(k-1)$, in the case when* (1.12) *is satisfied*; *when* (1.13), (1.14) *are satisfied, define instead*

$$\psi(x_2, \xi_1) = \left(1 + \frac{x_2}{2\delta} \operatorname{sign} \xi_1\right) \varphi(\xi_1) |\xi_1|^{\varrho} ,$$

cf. Example 2.3. *Let $\Omega = \Omega' \times]-\delta, \delta[$ be sufficiently small. Then there exists a linear map*

$$E \colon H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) \to H^{s+m-k(1-1/\sigma)}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[)$$

such that

$$P(x, D)\, Eu = \chi(x) u + Ru ,$$

where we may fix $\chi(x)$ arbitrary in $G_0^{\sigma'}(\Omega)$, $1 < \sigma' < \sigma$, with $\chi(x) = 1$ in a smaller neighborhood of the origin. Moreover R is a linear regularizing map, in the sense that for all $t \geqslant 0$

$$R \colon H^{s, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) \to H^{t, \psi}_{\tau, \sigma}(\mathbb{R}^{n-1} \times]-\delta, \delta[) .$$

Shrinking further Ω, we obtain $E \colon H^{s, \psi}_{\tau, \sigma, \mathrm{comp}}(\Omega) \to H^{s+m-k(1-1/\sigma), \psi}_{\tau, \sigma}(\Omega)$ such that $P(x, D)\, E = Id + R$, with R regularizing as before.

PROOF. – Using the notations in the first part of this section, we begin by constructing $\widetilde{E}$ such that

$$\widetilde{P}\widetilde{E} = \chi(x) + \widetilde{R}\,, \tag{3.31}$$

where $\widetilde{E}$ and $\widetilde{R}$ have the continuity properties in Theorem 3.1. The proof of Theorem 3.1 and the Remark 3.1 give actually $\widetilde{E}_1$ such that

$$\widetilde{P}\widetilde{E}_1 = \chi(x)\,\lambda(D) + \widetilde{R}_1\,,$$

where, in view of (1.12), or (1.13), (1.14), we may take any $\lambda(\xi) \in C^\infty(\mathbb{R}^n)$ homogeneous of degree zero with $\lambda(\xi) = 0$ in a conic neighborhood of the manifold $\xi' = 0$, and $\widetilde{R}_1$ is regularizing (on the standard Sobolev spaces). On the other hand we have

$$\tilde{p}(x,\xi) = \sum_{j=0}^{m} q_{m-j}(x,\xi')\,\xi_n^j$$

with $q_{m-j}(x,\xi') \in \Theta^{0,\,m-j}$. It is not restrictive to assume $q_0(x,\xi') = 1$; therefore from Proposition 3.2

$$\tilde{p}(x,\xi) = p_m(x,\xi) + q_{m-(1-\varrho)}(x,\xi)$$

where $q_{m-(1-\varrho)} \in \Theta^{m-1,\,m-(1-\varrho)}$, hence $\in S^{m-(1-\varrho)}_{0,0}$ in view of the remarks after Definition 3.1. Assume supp $(1-\lambda(\xi))$ is included in a sufficiently small neighborhood Γ_0 of the manifold $\xi' = 0$, so that $p_m(x,\xi)$ is elliptic in Γ_0. We can then construct $\widetilde{E}_2$ such that

$$\widetilde{P}\widetilde{E}_2 = \chi(x)(1-\lambda(D)) + \widetilde{R}_2\,,$$

where $\widetilde{R}_2$ is regularizing. The symbol $\tilde{e}_2(x,\xi)$ of $\widetilde{E}_2$ will be computed in $S^{-m}_{0,0}$ by taking $p_m^{-1}(x,\xi)$ as principal part in Γ_0. Observing that

$$\tilde{p}(x,\xi)\,\#\,p_m^{-1}(x,\xi) = 1 + r_{-(1-\varrho)}(x,\xi)$$

with $r_{-(1-\varrho)} \in S^{-(1-\varrho)}_{0,0}$ in Γ_0, we can find by standard iteration $s_{-(1-\varrho)} \in S^{-(1-\varrho)}_{0,0}$ such that

$$(1 + r_{-(1-\varrho)}(x,\xi))\,\#\,(1 + s_{-(1-\varrho)}(x,\xi)) \sim 1$$

and define

$$\tilde{e}_2(x,\xi) = p_m^{-1}(x,\xi)\,\#\,(1 + s_{-(1-\varrho)}(x,\xi))\,\#\,\chi(x)(1-\lambda(\xi))\,.$$

Taking $\widetilde{E} = \widetilde{E}_1 + \widetilde{E}_2$ we get (3.31). Returning to the operator $P(x,D)$ we find then E and R as required, with

$$P(x,D)\,Eu = e^{-\tau\psi(x_n,D')}\chi(x)\,e^{\tau\psi(x_n,D')}u + Ru\,.$$

As final step, we rewrite the preceding formula with $\chi(x)$ replaced by $\chi_0(x) \in G_0^{\sigma'}(\Omega)$, $\chi_0(x) = 1$ for $x \in \operatorname{supp}\chi$. Replacing further here E by $E\chi$, it remains to compute

$$e^{-\tau\psi(x_n, D')}\chi_0(x)\, e^{\tau\psi(x_n, D')}\chi(x) = \chi(x) - e^{-\tau\psi(x_n, D')}\widetilde{R}_3\, e^{\tau\psi(x_n, D')}$$

where

$$\widetilde{R}_3 = (1 - \chi_0(x))\, e^{\tau\psi(x_n, D')}\chi(x)\, e^{-\tau\psi(x_n, D')}$$

maps $H^s(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ into $H^t(\mathbb{R}^{n-1} \times]-\delta, \delta[)$ for all t, as it follows easily from the proof of Lemma 3.3. This concludes the proof of Theorem 3.5.

REMARK 3.2. – The two weight functions $\psi(x_n, \xi')$ considered in Theorem 3.5 are essentially sub-additive; that is necessary in our paper for the application to the semilinear case. Looking only for local solvability of linear equations, Theorem 3.5 could be extended to more general $P(x, D)$ by using non-sub-additive weight functions.

4. – Local solvability for semilinear equations.

The purpose of this section is to prove Theorem 1.2. In fact, a more precise assertion will be given.

First we state an abstract theorem on solvability in the framework of the Gevrey-Sobolev spaces $H^{s,\psi}_{\tau,\sigma}$. Let $\tau > 0$, $\sigma > 1$ be fixed, and let s and ψ satisfy the hypotheses of Theorem 2.5. We impose on the nonlinearity F the same assumption as in Theorem 2.6, Proposition 2.8, namely

$$F(x, \cdot) \in G^{\sigma'}(\Omega : \mathcal{H}(\mathbb{C}^N))\,, \qquad N = \sum_{\alpha \in \mathbb{Z}^n_+,\ |\alpha| \leqslant m-1} 1\,, \qquad 1 \leqslant \sigma' < \sigma\,, \tag{4.1}$$

where $\mathcal{H}(\mathbb{C}^N)$ is the space of the entire functions in $\mathbb{C}^N$. We require further that

$$F(x, 0) = 0\,, \qquad x \in \Omega\,. \tag{4.2}$$

We may then apply Proposition 2.8 with $k = 0$ and obtain (2.23), (2.24) for suitable functions F_1, F_2.

Our main assumption is the existence of an operator E right parametrix of the linear operator P in the following sense $P \circ E = Id + R$, where E and the remainder R satisfy the following properties: for every $\tau > 0$, $s \geqslant n + 3$ there exists a positive nondecreasing continuous function $\mathcal{C} : [0, \delta_0] \to [0, +\infty[$,

$\mathcal{C}(0) = 0$ such that

$$\|Rw\|_{H^{s,\psi}_{\tau,\sigma}} \leq \mathcal{C}(\delta)\|w\|_{H^{s,\psi}_{\tau,\sigma}}, \qquad w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta), \qquad 0 < \delta \leq \delta_0, \tag{4.3}$$

$$\|Ew\|_{H^{s+m-1,\psi}_{\tau,\sigma}} \leq \mathcal{C}(\delta)\|w\|_{H^{s,\psi}_{\tau,\sigma}}, \qquad w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta), \qquad 0 < \delta \leq \delta_0, \tag{4.4}$$

where $\Omega_\delta = \{|x| < \delta\}$.

In view of (4.3) and (4.4) we can define

$$a_s(\delta) := \sup_{0 \neq w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta)} \frac{\|Rw\|_{H^{s,\psi}_{\tau,\sigma}}}{\|w\|_{H^{s,\psi}_{\tau,\sigma}}} \leq \mathcal{C}(\delta),$$

$$b_s(\delta) := \sup_{0 \neq w \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta)} \frac{\|Ew\|_{H^{s+m-1,\psi}_{\tau,\sigma}}}{\|w\|_{H^{s,\psi}_{\tau,\sigma}}} \leq \mathcal{C}(\delta),$$

for every $\delta \in]0, \delta_0]$. Now we can state our main abstract theorem.

THEOREM 4.1. – *Under the hypothesis* (4.1)-(4.4) *we claim that for every*

$$f \in H^{s,\psi}_{\tau,\sigma}(\Omega_\delta) \cap \mathcal{E}'(\Omega_\delta), \qquad s \geq n+3,\ \tau > 0,\ 0 < \delta \leq \delta_0$$

there exists a solution $v \in H^{s+m-1,\psi}_{\tau,\sigma}(\Omega_\delta)$ *of the semilinear PDE*

$$P(x, D)\, v + F(x, v, \ldots, \partial_x^\alpha v, \ldots)\,|_{|\alpha| \leq m-1} = f(x), \qquad x \in \Omega_\delta \tag{4.5}$$

of the form $v = Ew$ *with* $w \in G_0^\sigma(\Omega_\delta)$ *satysfying*

$$\|w - f\|_{H^{s,\psi}_{\tau,\sigma}} \leq 1 \tag{4.6}$$

provided δ *and* $\|f\|_{H^{s,\psi}_{\tau,\sigma}}$ *satisfy the relations*

$$\begin{cases} a_s(\delta)(\|f\|_{H^{s,\psi}_{\tau,\sigma}} + 1) + F_1(b_s(\delta)(\|f\|_{H^{s,\psi}_{\tau,\sigma}} + 1)) \leq 1, \\ a_s(\delta) + b_s(\delta)\, F_2(b_s(\delta)(\|f\|_{H^{s,\psi}_{\tau,\sigma}} + 1)) < 1. \end{cases} \tag{4.7}$$

PROOF. – We shall reduce the problem to the application of the fixed point theorem in a suitable Banach space. We look for a solution $v(x)$ to (4.5) in the following form $v(x) = Ew(x)$. Then the equation (4.5) is reduced to

$$\begin{cases} w(x) = \mathcal{K}w(x) + f(x), \\ \mathcal{K}w(x) = -Rw - F(x, Ew, \ldots, \partial_x^\alpha Ew, \ldots)\,|_{|\alpha| \leq m-1}. \end{cases} \tag{4.8}$$

We write for simplicity $B^s = B^s(\Omega_\delta)$ instead of $H^{s,\psi}_{\tau,\sigma}(\Omega_\delta)$. Set

$$X_\delta := \{w \in G_0^\sigma(\Omega_\delta) \cap B^s(\Omega_\delta) : \|w - f\|_{B^s} \leq 1\}.$$

We will show that under the hypotheses of Theorem 4.1 the operator $\Theta w := \mathcal{K}w + f$ is contraction in X_δ provided the restrictions (4.7) hold.

a) First we prove that Θ preserves X_δ. Indeed, taking into account (2.23), the definition of a_s, b_s and the first inequality in (4.7) we have for $w \in X_\delta$

$$\text{(4.9)} \qquad \|\Theta w - f\|_{B^s} = \|\mathcal{K}w\|_{B^s} \leqslant \|Rw\|_{B^s} + \|F(\cdot, Ew, \ldots, \partial_x^\alpha Ew, \ldots)|_{|\alpha| \leqslant m-1}\|_{B^s} \leqslant$$

$$a_s(\delta)\|w\|_{B^s} + F_1(b_s(\delta)\|w\|_{B^s}) \leqslant a_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}}) + F_1(b_s(\delta)(1 + \|f\|_{H^{s,\psi}_{\tau,\sigma}})) \leqslant 1\,.$$

Hence $\Theta(X_\delta) \subset X_\delta$.

b) Now we deduce the contraction property. Using again the definition of a_s, b_s, (2.24) and the second inequality in (4.7) we obtain the following estimate

$$\|\mathcal{K}(w_1) - \mathcal{K}(w_2)\|_{B^s} \leqslant \|w_1 - w_2\|_{B^s}(a_s(\delta) + b_s(\delta)\, F_2(b_s(\delta)(\|f\|_{H^{s,\psi}_{\tau,\sigma}} + 1))) = q\|w_1 - w_2\|_{B^s}$$

for every all $w_1, w_2 \in X_\delta$ with $q = a_s(\delta) + b_s(\delta)\, F_2(b_s(\delta)(\|f\|_{H^{s,\psi}_{\tau,\sigma}} + 1)) < 1$. Theorem 4.1 is proved.

PROOF OF THEOREM 1.2. – We want to apply Theorem 4.1. Observe first that the assumptions on the nonlinearity F in Theorem 1.2 can be relaxed as in (4.1). Secondly, using Proposition 2.1, we have that $f \in G_0^s(\Omega)$ implies $f \in H^{s,\psi}_{\tau,\sigma}$, for ψ defined as in Theorem 3.5, $1 < \sigma < k/(k-1)$, a suitable $\tau > 0$ and any $s \geqslant n+3$. Then, applying Theorem 3.5 to the linear part, we check that (4.3), (4.4) are satisfied, if δ_0 is fixed sufficiently small, cf. the proof of Theorem 3.1 and the arguments used in [24]. At this moment, we fix δ and ε in Theorem 1.2 in such a way that (4.7) is satisfied. We then obtain from Theorem 4.1 a solution $v \in H^{s+m-1,\psi}_{\tau,\sigma}(\Omega_\delta)$. Since $H^{s+m-1,\psi}_{\tau,\sigma}(\Omega_\delta) \subset H^{s+m-1}(\Omega_\delta) \subset C^m(\Omega_\delta)$ under our assumption on s, Theorem 1.2 is proved.

PART II: THE CASE OF GEVREY NONLINEARITY

5. – Banach spaces of Gevrey functions.

Our aim in the present second part of the paper is to prove Theorem 1.3. We begin by a new analysis of Gevrey-Sobolev spaces; more precisely, in the following definitions the exponential weight of Part I is replaced by infinite sums of L^p-norms, cf. J. Leray and Y. Ohya [32]. We also mention that infinite sums of L^2-norms have been used by P. D'Ancona and S. Spagnolo [12] in order to study the lifespan for second order nonlinear hyperbolic equations with analytic data while C. Wagschal [47] and D. Gourdin et M. Mechab [21] relied upon formal norms of Gevrey type for the study of the Goursat problems in Gevrey classes. This frame will allow to treat Gevrey nonlinearity, cf. Sections 6, 7, 8, limiting however applications to linear principal parts depending on t only, see Section 9 where the proof of Theorem 1.3 is concluded by direct estimates.

First we introduce certain Banach spaces of Gevrey functions. Let us fix $\sigma \geqslant 1$ and let $\Omega \subset \mathbb{R}^n$ be an open set. For each $1 \leqslant p \leqslant \infty$, $T > 0$ we set

$$E_\sigma(L^p; T) = E_\sigma(L^p(\Omega); T) = \{f(x) \in G^\sigma(\Omega): |\!| f |\!|^\sigma_{L^p, T} < \infty\} \tag{5.1}$$

where

$$|\!| f |\!|^\sigma_{L^p, T} = \sum_{k=0}^{\infty} \frac{T^k}{(k!)^\sigma} \|\partial^k f\|_{L^p}, \tag{5.2}$$

with

$$\|\partial^k f\|_{L^p} = \begin{cases} \max\limits_{\alpha \in \mathbb{Z}^n_+,\ |\alpha| = k} |\partial_x^\alpha f(\cdot)|_{L^p(\Omega)} & \text{if } 1 \leqslant p < \infty, \\ \max\limits_{\alpha \in \mathbb{Z}^n_+,\ |\alpha| = k} |\partial_x^\alpha f(\cdot)|_{L^\infty(\Omega)} & \text{if } p = \infty \end{cases} \tag{5.3}$$

and if $p = \infty$ we will write $|\!| f |\!|_T := |\!| f |\!|_{L^\infty, T}$ and $E_\sigma(T) := E_\sigma(L^\infty; T)$.

Furthermore, for any given nonnegative integer s we define in $K \subset\subset \mathbb{R}^n$:

$$E_\sigma(H^s_p; T) = \{f(x) \in G^\sigma(K): |\!| f |\!|^\sigma_{H^s_p, T} < \infty\} \tag{5.4}$$

where

$$|\!| f |\!|^\sigma_{H^s_p, T} = \omega_{s,p,n} \sum_{k=0}^{\infty} \frac{T^k}{(k!)^\sigma} \|\partial^k f\|_{H^s_p}, \tag{5.5}$$

with

$$\|g\|_{H^s_p} := \sum_{j=0}^{s} \|\partial^j g\|_{L^p} \tag{5.6}$$

and $\omega_{s,p,n}$ is a positive constant, which usually will be assumed to be equal one, unless specified for $s > n/p$ as

$$\omega_s = \omega_{s,p,n} := \sup_{u, v \in H^s_p \setminus 0} \left(\frac{\|uv\|_{H^s_p}}{\|u\|_{H^s_p} \|v\|_{H^s_p}} \right). \tag{5.7}$$

In our application for equations of the type (1.18) we have a special variable t. For that reason we introduce another type of Banach spaces which could be viewed as analogues of the weighted spaces in part I.

For any given $\mu \in \mathbb{Z}_+$ and $T \in]0, T_0[$ we denote

$$(5.8) \quad E_\sigma(C^\mu(H_p^s); T) = \{ f \in C^\mu([-T/2, T/2]: G^\theta(K)) : \| f \|^\sigma_{C^\mu(H_p^s); T} < \infty \}$$

where

$$(5.9) \qquad \| f \|^\sigma_{C^\mu(H_p^s), T} = \omega_{s,p,n} \sum_{\nu=0}^{\mu} \sum_{k=0}^{\infty} \sup_{|t| \leq T/2} \left(\frac{(T-t)^{k+\nu}}{(k+\nu)!^\sigma} \| \partial_t^\nu \partial^k f(t) \|_{H_p^s} \right).$$

Finally, if in addition $m \in \mathbb{Z}_+$, we define

$$(5.10) \qquad E_\sigma(m, C^\mu(H_p^s); T) = \{ f \in C^\mu([-T/2, T/2]: G^\theta(K)) : \| f \|^\sigma_{m, C^\mu(H_p^s); T} < \infty \}$$

where

$$(5.11) \qquad \| f \|^\sigma_{m, C^\mu(H_p^s), T} = \omega_{s,p,n} \sum_{\nu=0}^{\mu} \sum_{k=0}^{\infty} \sup_{|t| \leq T/2} \left(\frac{(T-t)^{k+\nu+m}}{(k+\nu+m)!^\sigma} \| \partial_t^\nu \partial^k f(t) \|_{H_p^s} \right).$$

Clearly

$$(5.12) \qquad \begin{cases} E_\sigma(C^\mu(H_p^s); T) \subset E_\sigma(m, C^\mu(H_p^s); T) \quad \text{and} \\ \| f \|^\sigma_{m, C^\mu(H_p^s); T} \leq \dfrac{3^m T^m}{2^m (m!)^\sigma} \| f \|^\sigma_{C^\mu(H_p^s); T} \end{cases}$$

for all $T > 0$, $m \in \mathbb{Z}_+$, $\mu \in \mathbb{Z}_+$.

We have

THEOREM 5.1. – *Let $\sigma \geq 1$, $T > 0$, $\mu \in \mathbb{Z}_+$ and either $1 < p < \infty$, $s > n/p$ or $p = 1$, $s \geq n$ or $p = \infty$, $s \geq 0$. Then $E_\sigma(H_p^s; T)$ and $E_\sigma(C^\mu(H_p^s); T)$ are Banach algebras provided $\omega_{s,p,n}$ is given by* (5.7).

PROOF. – Let $f, g \in E_\sigma(H_p^s; T)$. We write

$$(5.13) \qquad \| fg \|_{H_p^s, T} = \omega_s \sum_{k=0}^{\infty} \frac{T^k \| \partial^k (fg) \|_{H_p^s}}{(k!)^\sigma} \leq \omega_s \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \frac{T^k \| \partial^j f \partial^{k-j} g \|_{H_p^s}}{(k!)^\sigma} \leq$$

$$\omega_s^2 \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(j!)^{\sigma-1} ((k-j)!)^{\sigma-1}}{(k!)^{\sigma-1}} \frac{T^j \| \partial^j f \|_{H_p^s}}{(j!)^\sigma} \frac{T^{k-j} \| \partial^{k-j} g \|_{H_p^s}}{((k-j)!)^\sigma} \leq \| f \|_{H_p^s, T} \| g \|_{H_p^s, T}.$$

In a similar way we show that $E_\sigma(C^\mu(H_p^s); T)$ is Banach algebra. The proof is complete.

To prepare the results of the next sections, we start by improving well-known results for the action of smooth functions in the classical Sobolev spaces as well some Moser type estimates.

LEMMA 5.2. – *Given two integers $s \geqslant 1$ and $r > n/p + [s/2]$ we cand find a positive constant $C > 0$ depending on s, p, r and n only such that*

$$\|g \circ \boldsymbol{u}\|_{H_p^s} \leqslant \|\boldsymbol{u}\|_{H_p^s} (N \|g\|_{C^1} + C \|g\|_{C^s} (\|\boldsymbol{u}\|_{H_p^r})^{s-1}) \tag{5.14}$$

for all $g \in C^s(\mathbb{R}^N : \mathbb{R})$, $g(0) = 0$ and $\boldsymbol{u} \in (H_p^t(\Omega))^N$, $t = \max\{s, r\}$. Here $\|g\|_{C^s}$ stands for $\|g\|_{C^s(K)}$ with $K = \boldsymbol{u}(\Omega) \subset \mathbb{R}^N$.

PROOF. – First we note that the Taylor formula yields

$$\|g(\boldsymbol{u})\|_{L^p} \leqslant N \|g\|_{C^1} \|\boldsymbol{u}\|_{L^p}. \tag{5.15}$$

Let now $\alpha \in \mathbb{Z}_+^n$, $1 \leqslant k = |\alpha| \leqslant s$. We have from Faà di Bruno formula

$$\|\partial^k (g \circ \boldsymbol{u})\|_{L^p} \leqslant \max_{|\alpha| = k} \sum_{\nu=1}^{N} \|\partial_{z_\nu} g\|_{L^\infty} \|\partial_x^\alpha u_\nu\|_{L^p} + \sum_{j=2}^{k} \frac{4^{(N-1)j} \|\partial^j g\|_{L^\infty}}{j!} \times$$

$$\sum_{\substack{k_1 + \ldots + k_j = k \\ k_1 \geqslant 1, \ldots, k_j \geqslant 1}} \frac{k!}{k_1! \ldots k_j!} \|\prod \partial^{k_j} \boldsymbol{v}_j\|_{L^p}$$

where the second term is further estimated by

$$C_k \|g\|_{C^k} \sum_{j=2}^{k} \sum_{\substack{k_1 + \ldots + k_j = k \\ k_1 \geqslant 1, \ldots, k_j \geqslant 1}} \|\prod \partial^{k_j} \boldsymbol{v}_j\|_{L^p} \tag{5.16}$$

with the convention that $\sum\limits_{j=2}^{1}$ equals 0.

Given $2 \leqslant j \leqslant k$ let us consider a fixed j-uple $(k_1, \ldots, k_j)$ with $k_1 + \ldots + k_j = k$, $k_\nu \geqslant 1$ for $\nu = 1, \ldots, j$. Choose and fix k_μ to satisfy

$$k_\mu = \max\{k_1, \ldots, k_j\}. \tag{5.17}$$

Since $k_1 + \ldots + k_j = k \leqslant s$ the definition of k_μ implies

$$k_\nu \leqslant [s/2] \quad \text{for } \nu \neq \mu, \quad \nu = 1, \ldots, j. \tag{5.18}$$

Now we get, using the embedding theorems for the Sobolev spaces, the following chain of inequalities with $\delta = r - [s/2] - n/p > 0$

$$\|\prod \partial^{k_j} \boldsymbol{v}_\nu\|_{L^p} \leqslant \|\partial^{k_\mu} \boldsymbol{v}_\mu\|_{L^p} \prod_{1 \leqslant \nu \leqslant j, \, \nu \neq \mu} \|\partial^{k_\nu} \boldsymbol{v}_\nu\|_{L^\infty} \leqslant \tag{5.19}$$

$$C^{j-1} \|\boldsymbol{u}\|_{H_p^s} \prod_{1 \leqslant \nu \leqslant j, \, \nu \neq \mu} \|\boldsymbol{v}_\nu\|_{H_p^{[s/2] + n/p + \delta}} \leqslant C^{j-1} \|\boldsymbol{u}\|_{H_p^s} (\|\boldsymbol{u}\|_{H_p^r})^{j-1}.$$

Observing that

$$\max_{|\alpha|=k} \sum_{\nu=1}^{N} \|\partial_{z_\nu} g\|_{L^\infty} \|\partial_x^\alpha u_\nu\|_{L^p} \leq N \|g\|_{C^1} \|\partial^k \boldsymbol{u}\|_{L^p} \tag{5.20}$$

we conclude the proof by substituting (5.19) in (5.16) and summing from 0 to s.

LEMMA 5.3. – *Let $K \subset\subset \mathbb{R}^d$, $f \in G^\theta(K\colon \mathbb{R})$ and $f(0)=0$. Then for every $R>0$, $r, s \in \mathbb{N}$ such that $B_R(0) \subset\subset K$, $s > n/p$ and $r > n/p + [s/2]$ one can find $C>0$ satisfying*

$$\|(\partial_z^\alpha f)(\boldsymbol{u}(\cdot))\|_{H_p^s} \leq C^{|\alpha|+1}(|\alpha|!)^\theta \|\boldsymbol{u}\|_{H_p^s}(1+\|\boldsymbol{u}\|_{H_p^r})^{s-1} \tag{5.21}$$

provided

$$\boldsymbol{u} \in (H_p^s(\mathbb{R}^n))^d, \qquad \|\boldsymbol{u}\|_{L^\infty} \leq R\,. \tag{5.22}$$

PROOF. – We apply Lemma 5.2 to $g(z) = \partial_z^\alpha f(z)$ and use that fact one can find $C_s > 0$ such that

$$\|\partial_z^\alpha f(\cdot)\|_{C^s} \leq C_s^{|\alpha|+1}(|\alpha|!)^\theta \tag{5.23}$$

for all $\alpha \in \mathbb{Z}_+^d$. The proof of is complete.

REMARK 5.1. – The interesting case in (5.21) is of course $r < s$. It seems that one can extend the polynomial estimates for s positive real numbers (see the results of J. Rauch and M. Reed [42] in the framework of the L^2 Sobolev spaces H^s, $s > n/2$). In fact, one could prove more precise results than (5.14), namely by using the results for the multiplication in the Sobolev spaces H_p^s.

Next we present a generalization of the Moser type estimates for the product of two functions (e.g. see (A.5.3) Lemma, p. A.12, J. Goodman and D. Yang [20])

LEMMA 5.9. – *Given j smooth functions $h_1, \ldots, h_j$ on $K \subset\subset \mathbb{R}^n$ and two real numbers $s \geq 0$ and $r > n/p$ one can find a positive constant C depening on s, r, p and n only such that*

$$\|h_1 h_2 \ldots h_j\|_{H_p^s} \leq C^j \sum_{\mu=1}^{j} \|h_\mu\|_{H_p^s} \left(\prod_{\substack{1 \leq \nu \leq j \\ \nu \neq \mu}} \|h_\gamma\|_{H_p^r} \right) \tag{5.24}$$

for all $j \in \mathbb{N}$ and all smooth functions $h_1, \ldots, h_j$ on K.

PROOF. – Let $A = A_{s,\,p,\,n} > 0$ be the positive constant verifying the Moser estimate

$$\|uv\|_{H_p^s} \leq A(\|u\|_{H_p^s}\|v\|_{H_p^r} + \|u\|_{H_p^r}\|v\|_{H_p^s}), \qquad u,\ v \in C^\infty(K). \tag{5.25}$$

We proceed by induction, observing that

$$\|h_1 h_2 \ldots h_{j+1}\|_{H_p^s} \leq A\|h_{j+1}\|_{H_p^s}\|h_1 \ldots h_j\|_{H_p^r} + A\|h_1 \ldots h_j\|_{H_p^s}\|h_{j+1}\|_{H_p^r}. \tag{5.26}$$

Next we use the supposed validity of (5.24) for j and the Schauder lemma

$$\sup_{j \geq 1}\left(\frac{\|u_1 \ldots u_j\|_{H_p^r}}{\|u_1\|_{H_p^r} \ldots \|u_j\|_{H_p^r}}\right)^{1/j} =: \omega < \infty \tag{5.27}$$

for all smooth $u_\mu \not\equiv 0$, $\mu = 1, 2, \ldots$ and conclude that

$$\|h_1 h_2 \ldots h_{j+1}\|_{H_p^s} \leq C^{j+1}\prod_{i=1}^{j+1}\|h_i\|_{H_p^r}\sum_{\mu=1}^{j+1}\frac{\|h_\mu\|_{H_p^s}}{\|h_\mu\|_{H_p^r}}\left(\frac{A\omega^j}{C^{j+1}} + \frac{A}{C}\right) \tag{5.28}$$

and the induction holds provided $C > 0$ is chosen to satisfy $C \geq 2\max\{A, \omega\}$.

6. – Nonlinear maps in Gevrey Banach spaces.

The main aim of this section is to study nonlinear superpositions in Gevrey Banach spaces.

Let us begin with a more precise analysis of the Faà di Bruno formula. We recall the if $\alpha \in \mathbb{Z}_+^n \setminus 0$ and $k := |\alpha|$ the Taylor formula implies

$$\partial_x^\alpha(f(g(x))) = \sum_{j=1}^{k}\sum_{|\beta|=j}\frac{f^{(\beta)}(g(x))}{\beta!}\,\partial_y^\alpha((g(y) - g(x))^\beta)|_{y=x}. \tag{6.1}$$

For fixed $\beta \in \mathbb{Z}_+^d$, $j = |\beta|$ we use the lexicographical order and write

$$g^\beta = g_1^{\beta_1} \ldots g_d^{\beta_d} = h_1 \ldots h_j. \tag{6.2}$$

Using standard combinatorial arguments we get

$$g_{\alpha,\,\beta}(x) = \partial_y^\alpha((g(y) - g(x))^\beta)|_{y=x} = \sum_{(k,\,j)} M_j^k(k_1, \ldots, k_j; g), \tag{6.3}$$

$$\sum_{(k,\,j)} = \sum_{\substack{k_1 + \ldots + k_j = k \\ k_1 \geq 1, \ldots, k_j \geq 1}}, \tag{6.4}$$

where $M_j^k(k_1, \ldots, k_j; g)$ is a sum of $k!/(k_1! \ldots k_j!)$ terms of the type

$$\partial^{\gamma^1} h_1(x) \ldots \partial^{\gamma^j} h_j(x)$$

with $\gamma^\mu = (\gamma_1^\mu, \ldots, \gamma_n^\mu) \in \mathbb{Z}_+^n \setminus 0$, $|\gamma^\mu| = k_\mu$, $\gamma^\mu \leqslant \alpha$ for $1 \leqslant \mu \leqslant j$. We will write for brevity in the following symbolic way

$$M_j^k(k_1, \ldots, k_j; g) = \frac{k!}{k_1! \ldots k_j!} \partial^{k_1} h_1(x) \ldots \partial^{k_j} h_j(x). \tag{6.5}$$

If $g(t) = g(t, x) \in (C^q(I: C^\infty(\Omega)))^d$, $I = [-T, T]$ for some $T > 0$ we will write analogously for $k = |\alpha|$, $k + q \geqslant 1$, $|\beta| = j \geqslant 1$

$$g_{\alpha, \beta}^q(x) = \partial_{\tau, y}^{(q, \alpha)}((g(\tau, y) - g(t, x))^\beta)|_{\tau = t,\, y = x} = \sum_{(k, q, j)} M_j^{k, q}(k_1, q_1, \ldots, k_j, q_j; g) \tag{6.6}$$

where

$$\sum_{(k, q, j)} = \sum_{\substack{k_1 + q_1 + \ldots + k_j + q_j = k + q \\ k_1 + q_1 \geqslant 1, \ldots, k_j + q_j \geqslant 1}}, \tag{6.7}$$

$$M_j^{k, q}(k_1, q_1, \ldots, k_j, q_j; g) = \frac{(k + q)!}{(k_1 + q_1)! \ldots (k_j + q_j)!} \prod_{\nu = 1}^{j} \partial_t^{q_\nu} \partial_x^{k_\nu} h_\nu(t, x). \tag{6.8}$$

We have

Proposition 6.1. – *Let $1 \leqslant p \leqslant \infty$, $r, s \in \mathbb{N}$ satisfying $s > n/p$ and $r > n/p + [s/2]$ if $1 \leqslant p < \infty$ and $s \geqslant 0$, $r \geqslant [s/2]$ if $p = \infty$. Then we can find a positive constant C, depending on s, p, r and n only, having the following properties*:

i) *for given $g \in (C^\infty(\Omega))^d$, $\alpha \in \mathbb{Z}_+^n$, $k = |\alpha| \geqslant 1$, $\beta \in \mathbb{Z}_+^d$, $1 \leqslant j = |\beta| \leqslant k$ the following inequality holds*

$$\|g_{\alpha, \beta}\|_{H_p^s} \leqslant C^j \sum_{\substack{k_1 + \ldots + k_j = k \\ k_1 \geqslant 1, \ldots, k_j \geqslant 1}} \frac{k!}{k_1! \ldots k_j!} \sum_{\nu = 1}^{j} \left(\|\partial^{k_\nu} g\|_{H_p^s} \prod_{\substack{1 \leqslant \mu \leqslant j \\ \mu \neq \nu}} \|\partial^{k_\mu} g\|_{H_p^r} \right); \tag{6.9}$$

ii) *for given $g(t) \in C^q(I: C^\infty(\Omega))^d$, $t \in I = [-T, T]$ for some $T > 0$, $q \in \mathbb{Z}_+$ $\alpha \in \mathbb{Z}_+^n$, $k = |\alpha|$, $k + q \geqslant 1$, $\beta \in \mathbb{Z}_+^d$, $1 \leqslant j = |\beta| \leqslant k$ the following estimate* *is true*

$$\|g^q_{\alpha,\beta}(t)\|_{H^s_p} \leqslant C^j \sum_{\substack{k_1+q_1+\ldots+k_j+q_j=k+q \\ k_1+q_1\geqslant 1,\,\ldots,\,k_j+q_j\geqslant 1}} \frac{(k+q)!}{(k_1+q_1)!\ldots(k_j+q_j)!} \times \tag{6.10}$$

$$\sum_{\nu=1}^{j}\left(\|\partial_t^{q_\nu}\partial^{k_\nu}g(t)\|_{H^s_p}\prod_{\substack{1\leqslant\mu\leqslant j \\ \mu\neq\nu}}\|\partial_t^{q_\mu}\partial^{k_\mu}g(t)\|_{H^r_p}\right);$$

PROOF. – We note that (6.2), (6.4) and (6.5) yield

$$\|g_{\alpha,\beta}\|_{H^s_p} \leqslant \sum_{\substack{k_1+\ldots+k_j=k \\ k_1\geqslant 1,\,\ldots,\,k_j\geqslant 1}} \frac{k!}{k_1!\ldots k_j!}\left\|\prod_{\nu=1}^{j}\partial^{k_\nu}h_\nu\right\|_{H^s_p}. \tag{6.11}$$

Next we apply Lemma 5.4 from the previous section and take into account that the definition of h_j implies that for all $\gamma\geqslant 0$

$$\|\partial^{k_\nu}h_\nu\|_{H^\gamma_p}\leqslant\|\partial^{k_\nu}g\|_{H^\gamma_p}, \qquad \nu=1,\,\ldots,\,j\,.$$

In similar way we deal with part ii). The proof is complete.

We need another auxiliary assertion, where again the interesting case is $r<s$.

POPOSITION 6.2. – *Let* $1\leqslant p\leqslant\infty$, $r,\,s\in\mathbb{N}$ *satisfying* $s>n/p$ *and* $r>n/p+[s/2]$ *if* $1\leqslant p<\infty$ *and* $s\geqslant 0$, $r\geqslant[s/2]$ *if* $p=\infty$. *Let* $C>0$ *be the corresponding constant from Proposition* 6.1. *Then we claim*:

i) *for given* $g\in(C^\infty(\Omega))^d$, $f\in C^\infty(K\colon\mathbb{R})$, $K:=g(\Omega)\subset\mathbb{R}^d$, $k\geqslant 1$, *the following inequality holds*

$$\|\partial^k(f\circ g)\|_{H^s_p}\leqslant \tag{6.12}$$

$$A(1+\|g\|_{H^r_p})^{s-1}\sum_{j=1}^{k}\frac{C^j}{j!}\|f\|_{C^{j+s}}\left(\|g\|_{H^s_p}S^0_{k,j}(g)+\|g\|_{H^r_p}\sum_{\nu=1}^{j}S^\nu_{k,j}(g)\right)$$

where

$$S^0_{k,j} = \sum_{(k,j)} \frac{k!}{k_1! \dots k_j!} \prod_{1 \leq \mu \leq j} \| \partial^{k_\mu} g \|_{H^r_p} ; \tag{6.13}$$

$$S^\nu_{k,j} = \| \partial^{k_\nu} g \|_{H^s_p} \sum_{(k,j)} \frac{k!}{k_1! \dots k_j!} \prod_{\substack{1 \leq \mu \leq j \\ \mu \neq \nu}} \| \partial^{k_\mu} g \|_{H^r_p} \tag{6.14}$$

for $\nu = 1, \dots, j$;

ii) *for given* $g(t) \in C^q(I : C^\infty(\Omega))^d$, $t \in I = [-T, T]$ *for some* $T > 0$, $q \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, $k + q \geq 1$, *the following estimate is true*

$$\| \partial^q_t \partial^k (f \circ g)(t) \|_{H^s_p} \leq \tag{6.15}$$

$$A(\|g\|_{C^q(H^r_p)})^{s-1} \sum_{j+l=1}^{k+q} \frac{C^j}{(j+l)!} \|f\|_{C^{j+s}} \left(\|g\|_{C^q(H^s_p)} T^0_{k,j,l}(g) + \|g\|_{C^q(H^r_p)} \sum_{\nu=1}^{j+l} T^\nu_{k,j,l}(g) \right)$$

where

$$T^0_{k,j,l} = \sum_{(k,q,j)} \frac{(k+q)!}{(k_1+q_1)! \dots (k_j+q_j)!} \prod_{1 \leq \mu \leq j+q} \| \partial^{k_\mu} g \|_{C^{q_\mu}(H^r_p)} ; \tag{6.16}$$

$$T^\nu_{k,j,l} = \| \partial^{k_\nu} g \|_{C^{q_\nu}(H^s_p)} \sum_{(k,q,j)} \frac{(k+q)!}{(k_1+q_1)! \dots (k_j+q_j)!} \prod_{\substack{1 \leq \mu \leq j \\ \mu \neq \nu}} \| \partial^{k_\mu} g \|_{C^{q_\mu}(H^r_p)} \tag{6.17}$$

for $\nu = 1, \dots, j+q$.

PROOF. – Let us limit ourselves to i). In view of (6.1), (6.3) we can write

$$\| \partial^k (f \circ g) \|_{H^s_p} \leq \sum_{|\alpha| = k} \sum_{j=1}^{k} \sum_{|\beta| = j} \frac{1}{\beta!} \| f^{(\beta)}(g) \, g_{\alpha\beta} \|_{H^s_p} . \tag{6.18}$$

Applying the Moser estimate (5.25), we further estimate by

$$C \sum_{|\alpha| = k} \sum_{j=1}^{k} \sum_{|\beta| = j} \frac{1}{\beta!} \left(\| f^{(\beta)}(g) \|_{H^s_p} \| g_{\alpha\beta} \|_{H^r_p} + \| f^{(\beta)}(g) \|_{H^r_p} \| g_{\alpha\beta} \|_{H^s_p} \right) . \tag{6.19}$$

By the Schauder lemma, (6.2) and (6.5)

$$\| g_{\alpha\beta} \|_{H^r_p} \leq \omega^j_r \sum_{(k,j)} \frac{k!}{k_1! \dots k_j!} \prod_{\mu=1}^{j} \| \partial^{k_\mu} g \|_{H^r_p} . \tag{6.20}$$

Moreover $\| f^{(\beta)}(g) \|_{H^s_p}$, $\| f^{(\beta)}(g) \|_{H^r_p}$ are estimated by Lemma 5.2, while for $\| g_{\alpha\beta} \|$

we apply Proposition 6.1, i). Plugging in, we conclude the proof by means of the Stirling formula.

We fix now a closed set $K \subset \mathbb{R}^d$ with smooth boundary and we take $f(z) \in G^\theta(K: \mathbb{R})$. Let $1 \leqslant p \leqslant \infty$ and $s \in \mathbb{N}$, with $s > n/p$ if $1 \leqslant p < \infty$ while in the case $p = \infty$ we require $s \geqslant 0$. We assume that

$$\limsup_{k \to \infty} \left(\frac{\| \partial^{k+s} f \|_{L^p}}{(k!)^\theta} \right)^{1/k} = \kappa_f = \kappa(f, K, s, p, n, \theta) < \infty . \tag{6.21}$$

We note that if K is compact, the restriction (6.13) is superfluous.

Let $g \in C^\infty(\Omega: \mathbb{R}^d) = (C^\infty(\Omega))^d$ and g belongs to some Banach space X. If $K \subset \mathbb{R}^d$ is a neighborhood of the origin we will denote

$$X^K := \{ g \in X: g(x) \in K, \text{ for } x \in \Omega \} . \tag{6.22}$$

REMARK 6.4. – In particular, if the hypercube $B_R(0; d)$ in $\mathbb{R}^d$ centered at the origin with a side $R > 0$ is contained in K, then

$$\{ g \in X: \|g\|_{L^\infty} \leqslant R \} \subset X^K . \tag{6.23}$$

Furthermore, if $g \in E^\sigma(H^s_p; T; \mathbb{R}^d)$, (respectively $g \in E^\sigma(m, C^\mu(H^s_p); T; \mathbb{R}^d)$, $m, \mu \in \mathbb{Z}_+$) and $s > n/p$ if $1 \leqslant p < \infty$ and $s \geqslant 0$ when $p = \infty$ then $g \in X^K$ provided $| g |_{H^s_p, T}$ (respectively $T^{-m} | g |_{m, C^\mu(H^s_p), T}$) is small enough. This is a consequence from the Sobolev embedding theorems and the definition of the Gevrey Banach spaces.

Now we state the first result on nonlinear maps in the scale of the Gevrey Banach spaces.

THEOREM 6.3. – *Let $\Omega \subset \mathbb{R}^n$ and let $f(z) \in G^\theta(K: \mathbb{R})$ for some $\theta \geqslant 1$, $K \subset \mathbb{R}^d$. Let $1 \leqslant p \leqslant \infty$ and $s, r \in \mathbb{N}$, with $s > r > n/p + [s/2]$ if $1 \leqslant p < \infty$ while in the case $p = \infty$ we require that $s \geqslant r \geqslant s/2 \geqslant 0$. Let us fix $\kappa > \kappa_f$. Then we have*:

i) *if $\sigma = \theta$ we can find a constant $C' > 0$, depending on κ, s, r, n and p only, such that*

$$| f \circ g |^\sigma_{H^s_p, T} \leqslant C' | g |^\sigma_{H^s_p, T} \frac{(1 + \|g\|_{H^r_p})^{s-1}}{1 - C\kappa | g |^\sigma_{H^r_p, T}} \tag{6.24}$$

provided $T > 0$, $g \in E_\sigma(H^s_p; T; \mathbb{R}^d)^K$ and $| g |^\sigma_{H^r_p, T} < C^{-1} 4^{1-d} \kappa^{-1}$;

ii) *if* $\sigma > \theta$ *there exists a constant* $C' > 0$, *depending on* κ, s, r, n *and* p *only, such that for all* $T > 0$ *and* $g \in E_\sigma(H_p^s; T; \mathbb{R}^d)^K$

$$(6.25) \qquad |\!|\!| f \circ g |\!|\!|^{\sigma}_{H_p^s, T} \leqslant C' |\!|\!| g |\!|\!|^{\sigma}_{H_p^s, T} (1 + \|g\|_{H_p^r})^{s-1} \exp\left(\gamma (|\!|\!| g |\!|\!|^{\sigma}_{H_p^r, T})^{1/(\sigma - \theta)}\right)$$

where $\gamma > 0$ *depends only on* σ, θ, C *and* κ. *Here* $C > 0$ *is the constant from Proposition* 6.1.

PROOF. – First we show a combinatorial inequality. Let $j \geqslant 2$ and $k_1, \ldots, k_j \in \mathbb{N}$ be fixed. We claim that

$$(6.26) \qquad \frac{j!\, k_1! \ldots k_j!}{(k_1 + \ldots + k_j)!} = \frac{k_1! \ldots k_j!}{(j+1) \ldots (j + k_1 + \ldots + k_j - j)} \leqslant 1 .$$

Indeed, without loss of generality we may assume $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_j$. If $k_1 = 1$ clearly $k_1 + \ldots + k_j = j$ and (6.16) holds. Let now $k_1 \geqslant 2$ and let r be the largest index s such that $k_s \geqslant 2$. Hence we can write

$$\frac{k_1! \ldots k_j!}{(j+1) \ldots (j + k_1 + \ldots + k_j - j)} = \frac{k_1! \ldots k_r!}{(j+1) \ldots (j + k_1 + \ldots + k_r - r)}$$

and since

$$k_1! \ldots k_j! = \prod_{\nu=1}^{r} (2 \times \ldots \times k_\nu) \leqslant 2 \times 3 \times \ldots (1 + k_1 + \ldots + k_r - r)$$

we obtain (6.16) in view of the inequality $j \geqslant 1$.

Let us choose and fix $\bar{\kappa} \in]\kappa_f, \kappa[$. We observe that the choice of $\bar{\kappa}$ implies that there exists $C_1 > 0$ satisfying

$$(6.27) \qquad \|f\|_{C^{j+s}} \leqslant C_1 \bar{\kappa}^j (j!)^\theta , \qquad j \in \mathbb{Z}_+ .$$

Combining (6.17) with (6.12), cf. Lemma 5.3, we obtain with $C > 0$ being the constant in Proposition 6.1 and $A > 0$ being the constant in Proposition 6.2

$$(6.28) \qquad \sum_{k=1}^{\infty} \frac{T^k}{(k!)^\sigma} \|\partial^k (f \circ g)\|_{H_p^s} \leqslant$$

$$A (1 + \|g\|_{H_p^r})^{s-1} \sum_{k=1} \sum_{j=1}^{k} \frac{T^k C^j}{k!\, j!} \|f\|_{C^{j+s}} \left(\|g\|_{H_p^s} S^0_{k,j}(g) + \|g\|_{H_p^r} \sum_{\nu=1}^{j} S^\nu_{k,j}(g) \right) \leqslant$$

$$A C_1 (1 + \|g\|_{H_p^r})^{s-1} \sum_{j=1}^{\infty} \frac{\bar{\kappa}^j}{(j!)^{(\sigma - \theta)}} \left(\sum_{\mu=0}^{j} G^j_\mu \right)$$

where

$$G_0^j = \|g\|_{H_p^s} \sum_{k_\varrho \geqslant 1,\, \varrho = 1,\, \dots,\, j} \left(\frac{j!\, k_1! \dots k_j!}{(k_1 + \dots + k_j)!} \right)^{\sigma - 1} \prod_{\nu=1}^{j} \frac{T^{k_\nu} \|\partial^{k_\nu} g\|_{H_p^r}}{(k_\nu!)^\sigma} \leqslant \|g\|_{H_p^s} (\| g \|^\sigma_{H_p^r,\, T})^j\,, \tag{6.29}$$

$$G_\mu^j = \|g\|_{H_p^r} \sum_{q=1}^{\infty} \frac{T^{k_\mu} \|\partial^{k_\mu} g\|_{H_p^s}}{(q!)^\sigma} \sum_{k_\varrho \geqslant 1,\, \varrho = 1,\, \dots j,\, \varrho \neq \mu} \left(\frac{j!\, k_1! \dots k_j!}{(k_1 + \dots + k_j)!} \right)^{\sigma - 1} \times \prod_{1 \leqslant \nu \leqslant j,\, \nu \neq \mu} \frac{T^{k_\nu} \|\partial^{k_\nu} g\|_{H_p^r}}{(k_\nu!)^\sigma} \leqslant \| g \|^\sigma_{H_p^s,\, T} \|g\|_{H_p^r} (\| g \|^\sigma_{H_p^r,\, T})^{j-1} \leqslant \| g \|^\sigma_{H_p^s,\, T} (\| g \|^\sigma_{H_p^r,\, T})^j \tag{6.30}$$

for $\mu = 1, \dots, j$. Hence,

$$\sum_{\mu=1}^{j} G_\mu^j \leqslant \| g \|^\sigma_{H_p^s,\, T} (j (\| g \|^\sigma_{H_p^r,\, T})^j) \tag{6.31}$$

and plugging (6.27) and (6.21) into (6.18) we get

$$\sum_{k=1}^{\infty} \frac{T^k}{(k!)^\sigma} \|\partial^k (f \circ g)\|_{H_p^s} \leqslant A C_1 (1 + \|g\|_{H_p^r})^{s-1} \|g\|_{H_p^s} \sum_{j=1}^{\infty} \frac{(C\overline{\kappa})^j}{(j!)^{\sigma - \theta}} (\| g \|^\sigma_{H_p^r,\, T})^j + A C_1 (1 + \|g\|_{H_p^r})^{s-1} \| g \|^\sigma_{H_p^s,\, T} \sum_{j=1}^{\infty} \frac{j (C\overline{\kappa})^j}{(j!)^{\sigma - \theta}} (\| g \|^\sigma_{H_p^r,\, T})^j \leqslant 2 C'' A C_1 (1 + \|g\|_{H_p^r})^{s-1} \| g \|^\sigma_{H_p^s,\, T} \sum_{j=1}^{\infty} \frac{\kappa^j (\| g \|^\sigma_{H_p^r,\, T})^j}{(j!)^{\sigma - \theta}} \tag{6.32}$$

where $C'' = \sup_{j \geqslant 1} ((1 + j)(\overline{k}/k)^j) < +\infty$. Taking into account the estimate of $\|f \circ g\|_{H_p^s}$ in Lemma 5.2 (or Lemma 5.3 for $\alpha = 0$) we observe that evidently if $\sigma = \theta$ (6.22) leads to the proof of the assertion in part i) while in the case $\sigma > \theta$ ii) follows from the fact that for any $\varrho > 0$ there exist $a > 0$ and $b > 0$ such that

$$\sum_{j=0}^{\infty} \frac{z^j}{(j!)^\varrho} \leqslant a \exp(b z^{1/\varrho})\,, \qquad z \geqslant 0\,. \tag{6.33}$$

The validity of (6.33) for $z \geqslant 1$ follows readily from the Stirling formula.

The proof is complete.

7. – Estimates with loss of derivatives.

Let now $m \in \mathbb{N}$, $m \geqslant 2$, $I \times \Omega \subset \mathbb{R}_t \times \mathbb{R}_x^n$. Set $d := \sum_{\alpha \in \mathbb{Z}_+^n,\ |\alpha| \leqslant m-1} 1$ and let $K \subset \mathbb{R}^d$ be a neighborhood of the origin. We claim

PROPOSITION 7.1. – *Let n, s and m be positive integers, $1 \leqslant p \leqslant \infty$, $m \geqslant 2$, $s > 2n/p + 2m - 2$. Let $f \in C^s(\mathbb{R}^d : \mathbb{R})$. For given $u(t, x) \in C^{m-1}(I : C^\infty(\Omega))$, we set*

$$U(t, x) = (U_1(t, x), \ldots, U_d(t, x)) = \{\partial_x^\alpha u(t, x)\}_{|\alpha| \leqslant m-1} \tag{7.1}$$

and

$$f_{m-1}(U) = f(u, \ldots, \partial_x^\alpha u, \ldots)\big|_{|\alpha| \leqslant m-1}\,. \tag{7.2}$$

We will suppose that

$$U(t, x) \in K\,, \qquad \text{for } t \in I,\ x \in \Omega\,. \tag{7.3}$$

Then we can find a positive constant C, depending on s, p, n, m and K such that for every $j \in \mathbb{N}$ and all

$$h_\mu(t, x) = U_{\omega(\mu)}(t, x)\,, \qquad \omega(\mu) \in \{1, \ldots, d\}\,, \qquad \mu = 1, \ldots, j$$

the following a priori estimate holds

$$\begin{aligned} &\|f_{m-1}(U)\, \partial^{k_1} h_1 \ldots \partial^{k_j} h_j\|_{C^0(H_p^s)} \leqslant \\ &C^{j+1} \|f\|_{C^s} \Bigg(\|u\|_{C^0(H_p^{s+m-1})} (1 + (\|u\|_{C^0(H_p^s)})^{s-1}) \prod_{\mu=1}^{j} \|\partial^{k_\mu} u\|_{C^0(H_p^s)} + \\ &\|u\|_{C^0(H_p^s)} (1 + (\|u\|_{C^0(H_p^s)})^{s-1}) \sum_{\mu=1}^{j} \|\partial^{k_\mu} u\|_{C^0(H_p^{s+m-1})} \prod_{1 \leqslant \nu \leqslant j,\ \nu \neq \mu} \|\partial^{k_\nu} u\|_{C^{m-1}(H_p^s)} \Bigg) \end{aligned} \tag{7.4}$$

for all positive integers $k_1, \ldots, k_j$.

PROOF. – In view of the choice of s, m and p we can find $r \in \mathbb{Z}_+$ satisfying

$$s - m + 1 \geqslant r > n/p + [s/2] \ \text{ i.e. } \ s \geqslant r + m - 1 > n/p + [s/2] + m - 1\,. \tag{7.5}$$

Applying Lemma 5.4 we get

$$(7.6)\qquad \|f_{m-1}(\boldsymbol{U})\,\partial^{k_1}h_1\ldots\partial^{k_j}h_j\|_{C^{m-1}(H_p^s)}\leqslant$$

$$C^{j+1}\|f_{m-1}(\boldsymbol{U})\|_{C^{m-1}(H_p^s)}\prod_{\mu=1}^{j}\|\partial^{k_\mu}h_\mu\|_{C^{m-1}(H_p^r)}+$$

$$\|f_{m-1}(\boldsymbol{U})\|_{C^{m-1}(H_p^r)}\sum_{\mu=1}^{j}\|\partial^{k_\mu}h_\mu\|_{C^{m-1}(H_p^s)}\prod_{1\leqslant\nu\leqslant j,\,\nu\neq\mu}\|\partial^{k_\nu}h_\mu\|_{C^{m-1}(H_p^r)}\,.$$

Now, the definition of h_μ and (7.5) imply that

$$(7.7)\ \|\partial^{k_\mu}h_\mu\|_{C^{m-1}(H_p^r)}=\|\partial^{k_\mu}U_{\omega(\mu)}\|_{C^{m-1}(H_p^r)}\leqslant\|\partial^{k_\mu}u\|_{C^{m-1}(H_p^{r+m-1})}\leqslant\|\partial^{k_\mu}u\|_{C^{m-1}(H_p^s)}$$

for all $\mu=1,\ldots,j$. On the other hand, Lemma 5.2 and (7.5) yield

$$(7.8)\qquad \|f_{m-1}(\boldsymbol{U})\|_{C^{m-1}(H_p^s)}\leqslant C\|\boldsymbol{U}\|_{C^{m-1}(H_p^s)}(d\|f\|_{C^1}+\Theta\|f\|_{C^s}(\|\boldsymbol{U}\|_{C^{m-1}(H_p^r)})^{s-1})\leqslant$$

$$C\|u\|_{C^{m-1}(H_p^{s+m-1})}(d\|f\|_{C^1}+\Theta\|f\|_{C^s}(\|u\|_{C^{m-1}(H_p^s)})^{s-1})\,.$$

Applying Lemma 5.2 with $r=s>n/p$ we get also

$$(7.9)\qquad \|f_{m-1}(\boldsymbol{U})\|_{C^{m-1}(H_p^r)}\leqslant C\|u\|_{C^{m-1}(H_p^s)}(d\|f\|_{C^1}+\Theta\|f\|_{C^s}(\|u\|_{C^{m-1}(H_p^s)})^{s-1})$$

Combining (7.5), (7.6), (7.7), (7.8) and (7.9), we conclude the proof.

If s and p are fixed we will write for brevity $|\!|\!| u |\!|\!|^{\sigma}_{m-1,T}:=|\!|\!| u |\!|\!|^{\sigma}_{m-1,C^{m-1}(H_p^s),T}$ for $m\in\mathbb{N}$, $T>0$, $I=[-T/2,\,T/2]$, where $|\!|\!| u |\!|\!|^{\sigma}_{m-1,C^{m-1}(H_p^s),T}$ is defined in (5.11).

The next theorem will play a crucial role for the solvability of (1.1) when the nonlinear term is Gevrey.

THEOREM 7.2. – *Let $s>2n/p+2m-2$ be an integer. Assume further that $f\in G^\theta(\mathbb{R}^d\colon\mathbb{R})$ and $f(0)=0$. Let $R>0$ and $K=B_R(0)$ be the ball with center* 0 *and radius R. Set $\kappa_f>0$ to be the constant defined in* (6.18). *Then for each $\kappa>\kappa_f$ there exists $C_1>0$ such that for every*

$$(7.10)\qquad u\in B_\sigma^{R/2}(C^{m-1}(H_p^s),\,T),\quad \text{i.e. } u\in E_\sigma(C^{m-1}(H_p^s),\,T)\,,\qquad |\!|\!| u |\!|\!|_{H_p^s,T}<R/2$$

the following properties hold (*we use the notation* (7.2), *writing for short u instead of $\boldsymbol{U}$*):

i) *if* $\theta = \sigma$ *then*

$$\Vert f_{m-1}(u) \Vert^{\sigma}_{m-1,\,T} \leq C_1 \Vert u \Vert^{\sigma}_{m-1,\,T} \frac{1}{1 - C\kappa \Vert u \Vert_{C^{m-1}(H^s_p),\,T}} \tag{7.11}$$

provided $C\kappa \Vert u \Vert_{C^{m-1}(H^s_p),\,T} < 1$;

ii) *if* $\sigma > \theta$ *then*

$$\Vert f_{m-1}(u) \Vert^{\sigma}_{m-1,\,T} \leq C_1 \Vert u \Vert^{\sigma}_{m-1,\,T} \exp\left(\gamma(\Vert u \Vert_{C^{m-1}(H^s_p),\,T})^{1/(\sigma-\theta)}\right). \tag{7.12}$$

Here $C > 0$ (*respectively* $\gamma > 0$) *is the constant in Proposition* 6.1 (*respectively Theorem* 6.3).

PROOF. – First, we introduce some notations. For given $\beta \in \mathbb{Z}^N_+$ we use the lexicographical order and write

$$\{\partial^{\alpha}_{t,\,x} u\}_{|\alpha| \leq m-1} = (v_1, \ldots, v_N)$$

and then for given $\beta = (\beta_1, \ldots \beta_N) \in \mathbb{Z}^N_+$ we put $j = |\beta|$ and write in the lexicographical order

$$v_1^{\beta_1} \ldots v_N^{\beta_N} = h_1 h_2 \ldots h_j$$

namely

$$\begin{cases} h_{\mu} = v_1\,, & \text{for } \mu = 1, \ldots, \beta_1\,, \\ h_{\beta_1 + \mu} = v_2\,, & \text{for } \mu = 1, \ldots, \beta_2\,, \\ \ldots & \\ h_{\beta_1 + \ldots + \beta_{N-1} + \mu} = v_N\,, & \text{for } \mu = 1, \ldots, \beta_N \end{cases} \tag{7.13}$$

with the evident convention to skip h_{μ} if $\beta_1 = 0$ etc.

Now the proof follows from Proposition 7.1 and evident modifications of the arguments used in the proof of Theorem 6.3 with r satisfying $s - m + 1 \geq r > n/p + [s/2]$ and taking into account the inequality

$$\Vert \partial^q_t \partial^k v(t) \Vert_{H^r_p} \leq \Vert \partial^q_t \partial^k u(t) \Vert_{H^{r+m-1}_p}. \tag{7.14}$$

8. – The fundamental estimate.

Here is the main result of the paper on estimates on superpositions in the Banach Gevrey spaces, which generalizes Theorem 7.2 and allows to use the contraction principle.

THEOREM 8.1. – *Let $g \in G^{\theta}(\mathbb{R}^N : \mathbb{C}^d)$, $N, d \in \mathbb{N}$ and let $n, s \in \mathbb{N}$, $m, \mu \in \mathbb{Z}_+$, $1 \leq p \leq \infty$, $s > 2n/p$. Then we can find two polynomials*

$$P_j(z) = p_0^j + p_{s-j}^j z^{s-j}, \quad p_{s-j}^j > 0, \quad p_0^j \geq 0, \quad j = 0, 1 \tag{8.1}$$

with p_0^j and p_{s-j}^j depending on s, n, p and g only, having the following properties:

i) *if $\theta = \sigma$ then for every $R > 0$ one can find $\kappa > 0$ such that*

$$\| Ug(V) \|^{\sigma}_{m, C^{\mu}(H_p^s), T} \leq \| U \|^{\sigma}_{m, C^{\mu}(H_p^s), T} \frac{P_0(\| V \|^{\sigma}_{C^{m-1}(H_p^s), T})}{1 - \kappa \| V \|^{\sigma}_{C^{\mu}(H_p^s), T}} + \tag{8.2}$$

$$\| V \|^{\sigma}_{m, C^{\mu}(H_p^s), T} \frac{\| U \|^{\sigma}_{C^{\mu}(H_p^s), T} P_1(\| V \|^{\sigma}_{C^{\mu}(H_p^s), T})}{1 - \kappa \| V \|^{\sigma}_{C^{\mu}(H_p^s), T}}$$

for all $U \in (E_{\sigma}(C^{\mu}(H_p^s), T))^d$, $V \in (B_{\sigma}^R(C^{\mu}(H_p^s), T))^N$, cf. the notation (7.11);

ii) *if $\sigma > \theta$ then*

$$\| Ug(V) \|^{\sigma}_{m, C^{\mu}(H_p^s), T} \leq \tag{8.3}$$

$$\| U \|^{\sigma}_{m, C^{\mu}(H_p^s), T} P_0(\| V \|^{\sigma}_{C^{m-1}(H_p^s), T}) \exp\left(C(\| V \|^{\sigma}_{C^{\mu}(H_p^s), T})^{1/(\sigma - \theta)}\right) +$$

$$\| V \|^{\sigma}_{m, C^{\mu}(H_p^s), T} \| U \|^{\sigma}_{C^{\mu}(H_p^s), T} P_0(\| V \|^{\sigma}_{C^{\mu}(H_p^s), T}) \exp\left(C(\| V \|^{\sigma}_{C^{\mu}(H_p^s), T})^{1/(\sigma - \theta)}\right)$$

for all $U \in (E_{\sigma}(C^{\mu}(H_p^s), T))^d$, $V \in (E_{\sigma}(C^{\mu}(H_p^s), T))^N$.

PROOF. – Fix $r = [n/p] + [s/2] + 1$. The choice of s implies $s > r$. We have

$$\| Ug(V) \|_{m, C^{\mu}(H_p^s), T} = \sum_{l=0}^{\mu} \sum_{k=0}^{\infty} \sup_{t \in I} \frac{(T-t)^{k+m+l} \| \partial_t^l \partial^k (U(t)\, g(V(t))) \|_{H_p^s}}{((k+l+m)!)^{\sigma}} . \tag{8.4}$$

On the other hand, there exists $C = C(r, s, n, p) > 0$ such that

$$\| \partial_t^l \partial^k (U(t)\, g(V(t))) \|_{H_p^s} \leq \sum_{q=0}^{l} \sum_{j=0}^{k} \binom{k}{j} \binom{l}{q} C \times \tag{8.5}$$

$$\left(\| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^s} \| \partial_t^q \partial^j (g(V(t))) \|_{H_p^r} + \| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^r} \| \partial_t^q \partial^j (g(V(t))) \|_{H_p^s} \right) .$$

Therefore, plugging (8.3) into (8.2), we get

$$\text{(8.6)} \qquad \| Ug(V) \|_{m,\, C^{\mu}(H_p^s),\, T} \leqslant C \sum_{l=0}^{\mu} \sum_{k=0}^{\infty} \sum_{q=0}^{p} \sum_{j=0}^{k} \binom{k}{j} \binom{l}{q} \mathcal{M}_{k,\mu,m}^{j,l} \times$$

$$\Bigg(\sup_{t \in I} \frac{(T-t)^{k-j+l-q+m}}{((k-j+l-q+m)!)^{\sigma}} \| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^s} \sup_{t \in I} \frac{(T-t)^{j+q}}{((j+q)!)^{\sigma}} \| \partial_t^q \partial^j g(V(t)) \|_{H_p^r} +$$

$$\sup_{t \in I} \frac{(T-t)^{k-j+l-q}}{((k-j+l-q)!)^{\sigma}} \| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^r} \sup_{t \in I} \frac{(T-t)^{j+q+m}}{((j+q+m)!)^{\sigma}} \| \partial_t^q \partial^j g(V(t)) \|_{H_p^s} \Bigg)$$

where $\mathcal{M}_{k,\mu,m}^{j,l}$ is the maximum between

$$\text{(8.7)} \qquad \frac{((j+q)!\,(k-j+l-q+m)!)^{\sigma}}{((k+l+m)!)^{\sigma}} \quad \text{and} \quad \frac{((j+q+m)!\,(k-j+l-q)!)^{\sigma}}{((k+l+m)!)^{\sigma}} .$$

We claim that for all $k \geqslant j \geqslant 1$, $\mu \geqslant q \geqslant 0$, $m \in \mathbb{Z}_+$

$$\text{(8.8)} \qquad \mathcal{M} := \mathcal{M}_{k,\mu,m}^{j,p} \binom{k}{j} \binom{p}{q} \leqslant 1 .$$

Indeed, the subsequent inequalities and (8.7) yield (8.8)

$$\text{(8.9)} \qquad \binom{k}{j} \binom{l}{q} \leqslant \binom{k+l}{j+q},$$

$$\text{(8.10)} \qquad \binom{k+l}{j+q} \frac{((j+q)!\,(k-j+l-q+m)!)^{\sigma}}{((k+l+m)!)^{\sigma}} \leqslant$$

$$\frac{((j+q)!\,(k-j+l-q)!)^{\sigma-1}}{((k+l)!)^{\sigma-1}} \frac{\prod_{\nu=1}^{m} (k-j+l-q+\nu)^{\sigma}}{\prod_{\nu=1}^{m} (k+l+\nu)^{\sigma}} \leqslant 1 ,$$

$$\text{(8.11)} \qquad \binom{k+p}{j+q} \frac{((j+q+m)!\,(k-j+p-q)!)^{\sigma}}{((k+p+m)!)^{\sigma}} \leqslant$$

$$\frac{((j+q)!\,(k-j+p-q)!)^{\sigma-1}}{((k+p)!)^{\sigma-1}} \frac{\prod_{\nu=1}^{m} (j+q+\nu)^{\sigma}}{\prod_{\nu=1}^{m} (k+p+\nu)^{\sigma}} \leqslant 1$$

for all integers $0 \leqslant j \leqslant k$, $0 \leqslant q \leqslant l \leqslant \mu$, $m \geqslant 0$.

Combining (8.6) and (8.7) we get

$$\| Ug(V) \|_{m, C^\mu(H_p^s), T} \leqslant C \sum_{l=0}^{\mu} \sum_{k=0}^{\infty} \sum_{q=0}^{l} \sum_{j=0}^{k} \times \tag{8.12}$$

$$\left(\sup_{t \in I} \frac{(T-t)^{k-j+l-q+m}}{((k-j+l-q+m)!)^\sigma} \| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^s} \sup_{t \in I} \frac{(T-t)^{j+q}}{((j+q)!)^\sigma} \| \partial_t^q \partial^j g(V(t)) \|_{H_p^r} + \right.$$

$$\left. \sup_{t \in I} \frac{(T-t)^{k-j+p-q}}{((k-j+l-q)!)^\sigma} \| \partial_t^{l-q} \partial^{k-j} U(t) \|_{H_p^r} \sup_{t \in I} \frac{(T-t)^{j+q+m}}{((j+q+m)!)^\sigma} \| \partial_t^q \partial^j g(V(t)) \|_{H_p^s} \right) =$$

$$C (\| U \|^\sigma_{m, C^\mu(H_p^s), T} \| g(V) \|_{C^\mu(H_p^r), T} + \| U \|_{C^\mu(H_p^r), T} \| g(V) \|^\sigma_{m, C^\mu(H_p^s), T}) .$$

Now we apply the arguments in the proofs of Theorem 6.3 and Theorem 7.2 in order to estimate

$$\| g(V) \|^\sigma_{m, C^\mu(H_p^s), T} \quad \text{and} \quad \| g(V) \|_{C^\mu(H_p^r), T}$$

and conclude the proof of Theorem 8.1.

9. – Proof of Theorem 1.3.

Let us now consider the equation (1.18) with P defined by (1.15) and (1.16). We will require less regularity, namely

$$\boldsymbol{a}^j(t) = (a_1^j(t), \ldots, a_n^j(t)) \in C^{m-1}(\mathbb{R} : \mathbb{C}^n), \quad j = 1, \ldots, m \tag{9.1}$$

We set

$$A^j(t) = \int_t^0 \boldsymbol{a}^j(\tau)\, d\tau , \quad j = 1, \ldots, m ; \tag{9.2}$$

$$A(z_1, \ldots, z_m) = A^1(z_1) + \ldots + A^m(z_m) . \tag{9.3}$$

In view of (1.17) if $\mathcal{L}$ is not hyperbolic we have $n = 1$ and

$$\operatorname{Im} (A^j(\tau) - A^j(t))\, \xi \geqslant 0 \quad \text{provided } (t - \tau)\, \xi > 0 . \tag{9.4}$$

Therefore, we define right inverse operators of $\mathcal{L}_j$ in the following way:

(9.5) $$\mathcal{L}_j^{-1} f(t, x) = \int_0^t f(\tau, x + A^j(\tau) - A^j(t))\, d\tau \quad \text{if all } \boldsymbol{a}^j(t) \in C^{m-1}(I: \mathbb{R}^n),$$

(9.6) $$\mathcal{F}_{x\to\xi}(\mathcal{L}_{j,c}^{-1} f(t, \cdot)) = \int_{\pm c}^t e^{i(A^j(\tau) - A^j(t))\xi} \hat{f}(\tau, \xi)\, d\tau, \qquad \pm\xi > 0 \text{ otherwise},$$

where c is a positive constant, $[-c, c] \subset I$, $f \in C_0^\infty(]-c, c[\times\mathbb{R}^n)$ and $|t| < c$. More precisely, one checks easily that $\mathcal{L}_j \circ \mathcal{L}_j^{-1} f = f$ (respectively $\mathcal{L}_j \circ \mathcal{L}_{j,c}^{-1} f = f$) for all $f \in C_0^\infty(\mathbb{R}^{n+1})$ (respectively $f \in C_0^\infty(\mathbb{R}^{n+1})$, $f = 0$ for $|t| \geqslant c$) if $\boldsymbol{a}^j \in C^{m-1}(\mathbb{R}: \mathbb{R}^n)$ (respectively $n = 1$ and $\operatorname{Im}(\boldsymbol{a}^j(t)) \not\equiv 0$). We set

(9.7) $$\mathcal{L}^{-1} := \mathcal{L}_1^{-1} \circ \dots \mathcal{L}_m^{-1} \quad \text{if all } \boldsymbol{a}_j(t), \quad 1 \leqslant j \leqslant m \text{ are real-valued},$$

(9.8) $$\mathcal{L}_c^{-1} := \mathcal{L}_{1,c}^{-1} \circ \dots \mathcal{L}_{m,c}^{-1} \quad \text{otherwise},$$

Now we can reduce the solution of (1.20) (respectively (1.20) and (1.21) if the equation is hyperbolic) to the integral equation

(9.9) $$w(t, x) = \mathcal{K} u(t, x) + U^0(t, x)$$

where

(9.10) $$\mathcal{K} u := \mathcal{L}^{-1} \circ F_{m-1}(u) \quad \text{if } \mathcal{L} \text{ is hyperbolic},$$

(9.11) $$\mathcal{K} u := \mathcal{L}_c^{-1} \circ F_{m-1}(u) \quad \text{otherwise},$$

while

(9.12) $$U^0(t, x) = \begin{cases} \mathcal{L}^{-1} f(t, x) + \widetilde{U}^0(t, x) & \text{if } \mathcal{L} \text{ is hyperbolic},\\ \mathcal{L}_c^{-1} f(t, x) & \text{if } \mathcal{L} \text{ is not hyperbolic},\end{cases}$$

where $\widetilde{U}^0$ stands for the unique solution of the Cauchy problem

(9.13) $$\begin{cases} \mathcal{L}_m \circ \mathcal{L}_{m-1} \circ \dots \circ \mathcal{L}_1 U = 0 & \text{for } t \in I,\ x \in \Omega,\\ \partial_t^j U(0, x) = u_j^0(x) & \text{for } x \in \Omega, \quad j = 0, 1, \dots, m-1.\end{cases}$$

in the hyperbolic case.

In particular, in the hyperbolic case we have with the notation (7.2)

(9.14) $$\mathcal{K} u(t, x) = \int_0^t \int_0^{z_1} \dots \int_0^{z_{m-1}} F_P(u; t, z, x)\, dz,$$

(9.15) $$F_P(u; z, \tau, x) = F_{m-1}(U(\tau, x + A(t, z') - A(z))$$

with $z = (z_1, \ldots, z_m)$, $z' = (z_1, \ldots, z_{m-1})$; while in the nonhyperbolic case we have

$$\widehat{\mathcal{K}u}(t, \xi) = \int_{\operatorname{sign}(\xi)c}^{t} \int_{\operatorname{sign}(\xi)c}^{z_1} \ldots \int_{\operatorname{sign}(\xi)c}^{z_{m-1}} e^{i\phi(t,z)\xi} \mathcal{F}_P(u; z_m, \xi)\, dz\,, \qquad \xi \neq 0\,, \tag{9.16}$$

$$\mathcal{F}_P(u; \tau, \xi) = \int_{\mathbb{R}} e^{-ix\xi} F_{m-1}(u(\tau, x))\, dx\,, \tag{9.17}$$

$$\phi(t, z) = A_1(t) - A_1(z_1) + \ldots + A_m(z_{m-1}) - A_m(z_m)\,. \tag{9.18}$$

We claim that

THEOREM 9.1. – *Let $F \in G^\theta(\mathbb{R}^N: \mathbb{C})$, $F(0) = 0$, $1 \leq \theta \leq \sigma \leq m/(m-1)$. Let $1 \leq p \leq \infty$, $s \in \mathbb{N}$, $s \geq 2n/p + 2m - 2$. Choose $R > 0$ and consider $B_0(R) \subset \mathbb{R}^N$. Then there exist two positive constants C and C_1, depending on s, n, p, m and κ_f only such that, with the notations of Theorem 7.2:*

i) *if $\theta = \sigma$ then*

$$\| F_{m-1}(u_1) - F_{m-1}(u_2) \|^{\sigma}_{m-1, T} \leq C_1 \| u_1 - u_2 \|^{\sigma}_{C^{m-1}(H^s_p), T} \frac{1 + \left(\max_{j=1,2} \{\| u_j \|^{\sigma}_{C^{m-1}(H^s_p), T}\}\right)^{s-1}}{1 - C \max_{j=1,2} \{\| u_j \|^{\sigma}_{C^{m-1}(H^s_p), T}\}} \tag{9.19}$$

for all $u_j \in B^R_\sigma(C^{m-1}(H^s_p), T)$ with $R \leq \kappa^{-1}$, $j = 1, 2$;

ii) *if $\sigma > \theta$ then*

$$\| F_{m-1}(u_1) - F_{m-1}(u_2) \|^{\sigma}_{m-1, T} \leq C_1 \| u_1 - u_2 \|^{\sigma}_{C^{m-1}(H^s_p), T} \times \left(1 + \left(\max_{j=1,2} \{\| u_j \|^{\sigma}_{C^{m-1}(H^s_p), T}\}\right)^{s-1}\right) \exp\left(C \max_{j=1,2} \{\| u_j \|^{\sigma}_{C^{m-1}(H^s_p), T}\}\right)^{1/(\sigma-\theta)} \tag{9.20}$$

for all $u_j \in B^R_\sigma(C^{m-1}(H^s_p): T)$, $j = 1, 2$.

PROOF. – Using the Taylor formula we obtain

$$F_{m-1}(u_1) - F_{m-1}(u_2) = \langle U_1 - U_2, G\rangle, \tag{9.21}$$

$$G = \int_0^1 \nabla_U F(U^\lambda)\, d\lambda\,, \tag{9.22}$$

where

$$U^\lambda = U_2 + \lambda(U_1 - U_2), \qquad \lambda \in [0, 1]\,. \tag{9.23}$$

Then we apply Theorem 8.1 and conclude the proof.

THEOREM 9.2. – *Let* $F \in G^{\theta}(\mathbb{R}^N : \mathbb{C})$, $F(0) = 0$, $1 \leq \theta \leq \sigma \leq m/(m-1)$. *Let* $1 \leq p \leq \infty$, $s \in \mathbb{N}$, $s \geq 2n/p + 2m - 2$. *Choose* $R > 0$ *and consider* $B_0(R) \subset \mathbb{R}^N$. *Furthermore, if* P *is not weakly hyperbolic, we assume that* $p = 2$. *Then there exist two positive constants* C *and* C_1, *depending on* s, n, p, m *and* κ_F *only such that, with the preceding notation*

i) *if* $\theta = \sigma$ *then*

$$\|\mathcal{K}u_1 - \mathcal{K}u_2\|^{\sigma}_{C^{m-1}(H^s_p),\,T} \leq \tag{9.24}$$

$$C_1 T \|u_1 - u_2\|^{\sigma}_{C^{m-1}(H^s_p),\,T} \, \frac{1 + \left(\max\limits_{j=1,2} \{\|u_j\|^{\sigma}_{C^{m-1}(H^s_p),\,T}\}\right)^{s-1}}{1 - C \max\limits_{j=1,2} \{\|u_j\|^{\sigma}_{C^{m-1}(H^s_p),\,T}\}}$$

for all $T > 0$, $u_j \in B^R_{\sigma}(C^{m-1}(H^s_p),\,T)$ *with* $R \leq \kappa^{-1}$, $j = 1, 2$;

ii) *if* $\sigma > \theta$ *then*

$$\|\mathcal{K}u_1 - \mathcal{K}u_2\|^{\sigma}_{C^{m-1}(H^s_p),\,T} \leq C_1 T \|u_1 - u_2\|^{\sigma}_{C^{m-1}(H^s_p),\,T} \times \tag{9.25}$$

$$\left(1 + \left(\max_{j=1,2} \{\|u_j\|^{\sigma}_{C^{m-1}(H^s_p),\,T}\}\right)^{s-1}\right) \exp\left(C \max_{j=1,2} \{\|u_j\|^{\sigma}_{C^{m-1}(H^s_p),\,T}\}\right)^{1/(\sigma-\theta)}$$

for all $T > 0$, $u_j \in B^R_{\sigma}(C^{m-1}(H^s_p),\,T)$, $j = 1, 2$.

PROOF. – We consider first the hyperbolic case. Straightforward calculations yield that for each integer $0 \leq l \leq m-1$ one can find $l+1$ continuous functions $C_{l,q}(t)$, $q = 0, 1, \ldots, l$ such that

$$\partial^l_t \partial^k_x \mathcal{K}u(t,x) = \sum_{q=0}^{l} C_{l,q}(t)\, \mathcal{H}^{q,l}_k(t,x), \tag{9.26}$$

$$\mathcal{H}^{q,l}_k = \int_0^t \ldots \int_0^{z_{m-1-q}} \partial^{l-q}_t \partial^k_x F^q_P(u; t, z_1 \ldots, z_{m-q}, x)\, dz_1 \ldots dz_{m-q}, \tag{9.27}$$

where

$$F^q_P(u; t, z_1 \ldots, z_{m-q}, x) = \tag{9.28}$$

$$F\big(U(t, x + \mathcal{A}^q(t, z_1, \ldots, z_{m-q-1}) - \mathcal{A}^q(z_1, \ldots, z_{m-q-1}, z_{m-q}))\big),$$

$$\mathcal{A}^q(z_1, \ldots, z_{m-q-1}, z_{m-q}) = \sum_{\nu=1}^{m-q} A^{\nu}(z_\nu). \tag{9.29}$$

On the other hand

$$\frac{(T-t)^{k+l}}{((k+l)!)^{\sigma}} \|\mathcal{K}_k^{q,\,l}(t)\|_{H_p^s} \leqslant \tag{9.30}$$

$$\frac{((k+l-q+m-1)!)^{\sigma}}{((k+l)!)^{\sigma}} \int_0^t \dots \int_0^{z_{m-1-q}} \frac{(T-t)^{k+l}}{(T-z_{m-q})^{k+q+m-1}}\, dz_1 \dots dz_{m-q} \times$$

$$\sup_{\tau \in I} \frac{(T-\tau)^{k+l-q+m-1}}{((k+l-q+m-1)!)^{\sigma}} \|\partial_\tau^{l-q} \partial^k (F \circ U)(\tau)\|_{H_p^s} .$$

We need two technical lemmas.

LEMMA 9.3. – *One can find a positive constant C such that*

$$\int_0^t \dots \int_0^{z_{m-1-q}} \frac{(T-t)^{k+l}}{(T-z_{m-q})^{k+l-q+m-1}}\, dz_1 \dots dz_{m-q} \leqslant \frac{CT}{\prod\limits_{\nu=1}^{m} (k+l-q-1+\nu)} \tag{9.31}$$

for all nonnegative integers $0 \leqslant q \leqslant l \leqslant m-1,\ k \geqslant 0$.

PROOF. – One checks by induction that for all integers $2 \leqslant \nu \leqslant \mu + 1$ and all real numbers $0 \leqslant t < T$ the following estimate holds

$$\int_0^t \dots \int_0^{z_{\nu-1}} \frac{1}{(T-z_\nu)^{\mu}}\, dz_1 \dots dz_\nu \leqslant \begin{cases} \dfrac{(T-t)^{-\mu+\nu}}{(\mu-\nu)(\mu-\nu+1)\dots(\mu-1)} & \text{if } \mu \geqslant \nu+1\,, \\[2ex] \dfrac{-\ln(1-t/T)}{(\nu-1)!} & \text{if } \mu = \nu\,, \\[2ex] \dfrac{\int_0^t -\ln(1-\tau/T)\, d\tau}{(\nu-2)!} & \text{if } \mu = \nu - 1\,. \end{cases} \tag{9.32}$$

Hence, we obtain

$$\int_0^t \dots \int_0^{z_{m-1-q}} \frac{(T-t)^{k+l}}{(T-z_{m-q})^{k+l-q+m-1}}\, dz_1 \dots dz_{m-q} \leqslant C_q^{k,\,l,\,m}(t,\,T)\,, \tag{9.33}$$

where

$$
\text{(9.34)}\quad C_q^{k,l,m}(t,T) := \begin{cases} \dfrac{(T-t)}{(k+l-1)\dots(k+l-q+m-2)} & \text{if } k+l \geqslant 2\,, \\ \dfrac{-(T-t)\ln(1-t/T)}{(m-q-1)!} & \text{if } k+l=1\,, \\ \dfrac{\int_0^t -\ln(1-\tau/T)\,d\tau}{(m-2)!} & \text{if } k=l=q=0\,, \end{cases}
$$

for $0 \leqslant t \leqslant T/2$. The proof is complete since (9.34) implies that there is a positive constant verifying

$$
\sup_{T>0} \frac{\max_{|t| \leqslant T/2} C_q^{k,l,m}(t,T)}{T} \leqslant \frac{C}{\prod_{\nu=1}^{m}(k+l-q-1+\nu)}
$$

for all nonnegative integers $0 \leqslant q \leqslant l \leqslant m-1$, $k \geqslant 0$.

Next we need a combinatorial estimate.

LEMMA 9.4. – *Set*

$$
\text{(9.35)}\qquad R_{q,\sigma}^{k,l,m} :=
$$

$$
\begin{cases} \dfrac{((k+l-q+m-1)!)^\sigma}{((k+l)!)^\sigma (k+l-1)\dots(k+l-q+m-2)} & \textit{if } k+l \geqslant 2\,, \\ \dfrac{((m-q)!)^\sigma}{(m-q-1)!} & \textit{if } k+l=1\,,\quad \textit{then } q=0 \textit{ or } q=1\,, \\ \dfrac{(m!)^\sigma}{(m-2)!} & \textit{if } k=l=0\,. \end{cases}
$$

Then

$$
\text{(9.36)}\qquad \sup_{k \in \mathbb{Z}_+} \max_{0 \leqslant q \leqslant l \leqslant m} R_{q,\sigma}^{k,l,m} := R_m^\sigma < \infty \qquad \textit{iff } \sigma \leqslant m/(m-1)\,.
$$

PROOF. – It is enough to consider the case $k+l \geqslant 3$, when we can write

$$
\text{(9.37)}\qquad R_{q,\sigma}^{k,l,m} = \frac{((k+l+1)\dots(k+l-q+m-1))^\sigma}{(k+l-2+1)\dots(k+l-q+m-2)} \leqslant
$$

$$
\frac{(k+l+1)^{(m-q-1)\sigma}((m-q-1)!)^\sigma}{(k+l-2)^{m-q}} = O(k^{(m-q-1)\sigma-(m-q)})
$$

for $k \to \infty$.

Easy calculations show that for an integer $m \geqslant 2$

$$\max_{q=0,1,\dots,m-1} \sup_{k \geqslant 1} k^{(m-1-q)\sigma-(m-q)} < \infty \qquad \text{iff } \sigma \leqslant m/(m-1) \tag{9.38}$$

and hence we get the desired conclusion of the lemma.

Now, coming back to the proof of the theorem, we note that Lemma 9.3, Lemma 9.4 and the summation of (9.30) imply (9.24) and (9.25).

If not all of the operators $\mathcal{L}_j$, $j=1,\dots,m$ are hyperbolic, we deduce the corresponding $H^s = H_2^s$ estimates using the Parceval identity and the representation (9.16) for the operator $\mathcal{K}$; details are left to the reader. The proof is complete.

Evidently Theorem 9.2 leads to local solvability for (1.20) and in the weakly hyperbolic case to local well-posedness of the Cauchy problem (1.20), (1.21). Theorem 1.3 is therefore proved.

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REFERENCES

[1] S. ALINHAC - P. GÉRARD, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Inter Edition, Editions du CNRS, Meudon, Paris (1991).

[2] J.-M. BONY, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sc. École Norm. Sup., **14** (1981), 161-205.

[3] F. CARDOSO, *A necessary condition of Gevrey solvability for differential operators with double characteristics*, Comm. Partial Diff. Eqs., **14** (1989), 981-1009.

[4] W. CRAIG, *Nonstrictly hyperbolic nonlinear systems*, Math. Ann., **277** (1987), 213-232.

[5] L. CATTABRIGA - L. RODINO - L. ZANGHIRATI, *Analytic-Gevrey hypoellipticity for a class of pseudodifferential operators with multiple characteristics*, Comm. Partial Diff. Eqs., **15** (1989), 81-96.

[6] CHEN-HUA - L. RODINO, *Micro-elliptic Gevrey regularity for nonlinear partial differential equations*, Boll. Un. Mat. Ital., **10-B** (1996), 199-232.

[7] M. CICOGNANI - L. ZANGHIRATI, *On a class of unsolvable operators*, Ann. Scuola Norm. Sup. Pisa, **20** (1993), 357-369.

[8] M. CICOGNANI - L. ZANGHIRATI, *Analytic regularity for solutions to semi-linear weakly hyperbolic equations*, Rend. Sem. Mat. Univ. Politec. Torino, **51** (1993), 387-396.

[9] A. CORLI, *On local solvability in Gevrey classes of linear partial differential operators with multiple characteristics*, Comm. Partial Diff. Eqs., **14** (1988), 1-25.

[10] A. CORLI, *On local solvability of linear partial differential operators with multiple characteristics*, J. Diff. Eqs., **81** (1989), 275-293.

[11] A. CORLI - L. RODINO, *Gevrey solvability for hyperbolic operators with costant multiplicity*, in *Recent Developments in Hyperbolic Equations*, Pitman Res. Notes in Math., **183**, 290-304, Longman, Harlow (1988).

[12] P. D'ANCONA - S. SPAGNOLO, *On the life span of the analytic solutions to quasilinear weakly hyperbolic equations*, Indiana Univ. Math. J., **40**, 1 (1991), 71-99.

[13] P. D'ANCONA - M. REISSIG, *New trends in the theory of nonlinear weakly hyperbolic equations of second order*, Proc. 2-nd WCNA, Athens, Greece, 10-17 July 1996, Nonl. Anal. TMA, **30**, 4 (1997), 2507-2515.

[14] B. DEHMAN, *Resolubilité local pour des équations semi-linéaires complexes*, Can. J. Math., **42** (1990), 126-140.

[15] Y. EGOROV, *Linear Differential Equations of Principal Type*, Nauka, Moscow (1994); Plenum Press, New York (1985).

[16] A. FERRARI - E. TITI, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Diff. Eqs., to appear.

[17] C. FOIAS - R. TEMAM, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal., **87** (1989), 359-369.

[18] G. GARELLO, *Local solvability for semilinear equations with multiple characteristics*, Ann. Univ. Ferrara, Sez VII, Sc. Mat. Suppl. Vol. **41**, 1995 (1997), 199-209.

[19] R. GOLDMAN, *A necessary condition for local solvability of a pseudo-differential equation having multiple characteristics*, J. Diff. Eqs., **19** (1975), 176-200.

[20] J. GOODMAN - D. YANG, *Local solvability of nonlinear partial differential equations of real principal type*, preprint.

[21] D. GOURDIN - M. MECHAB, *Problème de Goursat nonlinéaire dans les espaces de Gevrey pour les équations de Kirchhoff généralisées*, J. Math. Pures Appl., **75** (1996), 569-593.

[22] T. GRAMCHEV, *Powers of Mizohata type operators in Gevrey classes*, Boll. Un. Mat. Ital., (7) **5-B** (1991), 135-156.

[23] T. GRAMCHEV, *Nonsolvability for differential operators with multiple complex characteristics*, J. Math. Kyoto Univ., **33**, 4 (1993), 989-1002.

[24] T. GRAMCHEV - P. POPIVANOV, *Local solvability of semi-linear partial differential equations*, Ann. Univ. Ferrara Sez. VII (N.S.), **25** (1989), 147-154.

[25] T. GRAMCHEV - M. YOSHINO, *Rapidly convergent iteration method for simultaneous normal forms of commuting maps*, preprint, 1997.

[26] V. GRUSHIN, *On a class of elliptic pseudodifferential operators degenerate on a submanifold*, Mat. Sb., **84** (1971), 163-195 (in russian); Math. USSR Sb., **13** (1971), 155-185.

[27] J. HOUNIE, *Local solvability of partial differential equations*, Rev. Un. Mat. Argentina, **37** (1991), 77-86.

[28] J. HOUNIE - P. SANTIAGO, *On the local solvability of semilinear equations*, Comm. Partial Diff. Eqs., **20** (1995), 1777-1789.

[29] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators, I-IV*, Springer-Verlag, Berlin, 1983-85.

[30] K. KAJITANI, *Local solution of Cauchy problem for nonlinear hyperbolic systems in Gevrey classes*, Hokkaido Math. J., **12** (1983), 434-460.

[31] K. KAJITANI - S. WAKABAYASHI, *Microhyperbolic operators in Gevrey classes*, Publ. RIMS Kyoto Univ., **25** (1989), 169-221.

[32] J. LERAY - Y. OHYA, *Systèmes nonlinéaires hyperboliques nonstricts*, Math. Ann., **170** (1967), 167-205.

[33] C. D. LEVERMORE - M. OLIVER, *Analyticity of solutions for a generalized Euler equation*, J. Diff. Eqs., **133** (1997), 321-339.

[34] M. MASCARELLO - L. RODINO, *Partial Differential Equations with Multiple Characteristics*, Akademie Verlag-Wiley, Berlin (1997).

[35] A. MENIKOFF, *Some examples of hypoelliptic partial differential equations*, Math. Ann., **221** (1976), 167-181.

[36] Y. MEYER, *Remarques sur un théorème de J. M. Bony*, Rend. Circ. Mat. Palermo, bf 2, **1** (1981), 1-20.

[37] K. PAYNE, *Smooth tame Fréchet algebras and Lie groups of pseudodifferential operators*, Comm. Pure Appl. Math., **44** (1991), 309-337.

[38] T. OKAJI, *Gevrey hypoelliptic operators which are not C^∞-hypoelliptic*, J. Math. Kyoto Univ., **28** (1988), 311-322.

[39] P. POPIVANOV, *A class of differential operators with multiple characteristics which have not solutions*, Pliska Stud. Math. Bulg., **3** (1981), 47-60 (Russian).

[40] P. POPIVANOV, *On the local solvability of a class of PDE with double characteristics*, Trudy Sem. Petrovsk., **1** (1975), 237-278 (Russian); Amer. Math. Soc. Transl., **118**, 2 (1982), 51-89.

[41] K. PROMISLOW, *Time analyticity and Gevrey regularity for solutions of a class of dissipative partial differential equations*, Nonl. Anal., TMA, **16** (1991), 959-980.

[42] J. RAUCH - M. REED, *Nonlinear microlocal analysis of semi-linear hyperbolic systems in one space dimension*, Duke Math. J., **49** (1982), 397-475.

[43] M. REISSIG - K. YAGDJIAN, *Levi conditions and global Gevrey regularity for the solutions of quasilinear weakly hyperbolic equations*, Math. Nachr., **178** (1996), 285-307.

[44] L. RODINO, *Linear Partial Differential Operators in Gevrey Spaces*, World Scientific, Singapore, New Jersey, London, Hong Kong (1993).

[45] M. TAYLOR, *Pseudodifferential Operators and Nonlinear PDE*, Birkäuser, Boston (1991).

[46] F. TRÈVES, *Introduction to Pseudodifferential Operators and Fourier Integral Operators*, vol. I-II, Plenum Press, New York (1980).

[47] C. WAGSCHAL, *Le problème de Goursat non linéaire*, J. Math. Pures Appl., **58** (1979), 309-337.

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