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Ingham Type Theorems and Applications to Control Theory.

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Sunto. – *Ingham* [6] *ha migliorato un risultato precedente di Wiener* [23] *sulle serie di Fourier non armoniche. Modificando la sua funzione di peso noi otteniamo risultati ottimali, migliorando precedenti teoremi di Kahane* [9], *Castro e Zuazua* [3], *Jaffard, Tucsnak e Zuazua* [7] *e di Ullrich* [21]. *Applichiamo poi questi risultati a problemi di osservabilità simultanea.*

1. – Introduction.

Let Λ be a countable subset of $\mathbb{R}$ such that, with respect to a suitable $\gamma > 0$,

$$|\lambda - \mu| \geq \gamma \quad \text{for all } \lambda,\ \mu \in \Lambda \text{ with } \lambda \neq \mu . \tag{1.1}$$

We will be concerned with series like

$$f(t) = \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda \cdot t}$$

with $a_\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. They generalize the *Fourier series*, so it makes sense to ask if, for a sufficiently large interval $I \subset \mathbb{R}$, the $L^2(I)$ norm of f is equivalent to the l^2 norm of the sequence $\vec{\boldsymbol{a}} \equiv \{a_\lambda\}$ (Bessel type inequality). In 1936 Ingham [6] proved that, under the assumption (1.1)

$$\begin{cases} \text{for every } T > 0 \text{ there exists } c_T \text{ such that} \\ \|f\|_{L^2(I)} \leq c_T \|\vec{\boldsymbol{a}}\|_{l^2} \text{ if } |I| \leq 2T \end{cases} \tag{1.2}$$

and

$$\begin{cases} \text{for every } T > \pi/\gamma \text{ there exists } c_T \text{ such that} \\ \|\vec{\boldsymbol{a}}\|_{l^2} \leq c_T \|f\|_{L^2(I)} \text{ if } |I| \geq 2T ; \end{cases} \tag{1.3}$$

the restriction $T > \pi/\gamma$ in (1.3) being optimal.

Let us briefly recall the key idea in Ingham's proofs: if $k \colon t \mapsto k(t)$ is an

(*) Part of this work was accomplished during the visit of the second author at the Istituto per le Applicazioni del Calcolo «Mauro Picone» and at the Università di Roma «Tor Vergata» in 1997 and 1998. He thanks these institutions for their hospitality.

$L^1 \cap L^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$, then we have the identity

$$\int_{\mathbb{R}} k(t)\,|f(t)|^2\,dt = 2\pi \sum_{\lambda,\,\mu\in\Lambda} a_\lambda \overline{a_\mu} K(\mu-\lambda)\,, \tag{1.4}$$

where K denotes the Fourier transform of k. If we denote by $\boldsymbol{K}$ the matrix whose entries are $K(\lambda-\mu)$, under suitable assumptions on k,

$$\text{the map } \vec{\boldsymbol{a}} \to \boldsymbol{K}\cdot\vec{\boldsymbol{a}} \text{ is } \begin{cases} \text{(i) } \textit{bounded} \text{ from } l^2 \text{ into itself}\,, \\ \text{(ii) } \textit{coercive} \text{ on } l^2\,, \end{cases} \tag{1.5}$$

so that the right hand side of (1.4) is equivalent to $\|\vec{\boldsymbol{a}}\|^2_{l^2}$. Concerning (1.2) it will be sufficient to have

$$\begin{cases} k(t) \geqslant 0 & \text{for all } t\in\mathbb{R}\,, \\ \inf\limits_{t\in I} k(t) > 0 & \text{for some interval } I\,, \end{cases} \tag{1.6}$$

and the size of I is irrelevant (larger intervals will be divided into smaller ones, where the estimate holds true); while concerning (1.3) we need

$$k(t) \leqslant 0 \qquad \text{for } t \notin [-T, T]\,. \tag{1.7}$$

Of course one can realize both (1.6) and (1.7) by choosing $k \geqslant 0$ and compactly supported; it is one of the choices suggested by Ingham and followed by many authors. However, with such a choice, property (1.5) is very difficult to realize: the function K *cannot* have a compact support, and the matrix $\boldsymbol{K}$ will be «full»; in order to establish (1.5) we can only hope that, because of (1.1), the non-diagonal terms in $\boldsymbol{K}$ are small compared to the diagonal one $K(0)\ldots$.

In order to impose (1.5) it is more convenient to impose K (instead of k) to be compactly supported, e.g., by remarking that

$$\text{if } K(0) > 0 \text{ and } K(u) = 0 \text{ for } |u| \geqslant \gamma\,, \qquad \text{then (1.5) holds true}\,, \tag{1.8}$$

because of $\boldsymbol{K} = K(0)\,\boldsymbol{I}$. In fact, Ingham himself suggested (and used for a second proof of (1.3)) a choice of k satisfying (1.8): with the notation $\gamma = 1$ and $T = \pi + \varepsilon$, he defined

$$k_\varepsilon(t) := \frac{1-\cos(t)}{(\pi^2-t^2)^2}\,((\pi+\varepsilon)^2 - t^2) = 2\left(\frac{\cos(t/2)}{\pi^2-t^2}\right)^2 ((\pi+\varepsilon)^2-t^2)\,.$$

In the following three sections of this paper we adapt this proof in three different directions. First we improve some earlier extensions of Castro, Jaffard, Tucsnak and Zuazua [3], [7] who weakened the *gap condition* (1.1). Then we extend Ingham's theorem for exponential functions of several variables, improving thereby a former theorem of Kahane [9]. Finally, we give an optimal variant of our generalization of Kahane's theorem, using l_∞ norm in $\mathbb{R}^N$ instead of the Euclidean one, and also allowing more general sums where the coefficients a_n can be algebraic polynomials of the variable $t \in \mathbb{R}^N$. In the one-dimensional case this reduces to an earlier theorem of Ullrich [21].

In the last two sections we apply our results to solve some simultaneous observability problems.

Let us note that Ingham's theorem has already been generalized in many different directions before; see, e.g., [1], [4], [11], [12], [15], [16], [17], [19], [20], [24].

Throughout this paper every interval I is supposed to have a finite positive length $0 < |I| < \infty$ and all constants are assumed to be positive.

2. – A weakening of the gap condition.

Let

$$\dots < \lambda_{-1} < \lambda_0 < \lambda_1 < \dots$$

be a strictly increasing sequence of real numbers, and consider all sums of the form

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n t} \qquad (a_n \in \mathbb{C}). \tag{2.1}$$

Instead of (1.1) here we only assume the existence of a number $\gamma > 0$ such that

$$\lambda_{n+2} - \lambda_n \geq 2\gamma \text{ for all } n. \tag{2.2}$$

Introducing the sets

$$A = \{n \in \mathbb{Z} : \lambda_{n+1} - \lambda_n < \gamma\}$$

and

$$B = \{n \in \mathbb{Z} : n \notin A \text{ and } n-1 \notin A\}, \tag{2.19}$$

we have the

THEOREM 2.1. – (*a*) *For every interval I there exists a constant c_1 such that all finite sums* (2.1) *satisfy the* direct inequality

$$\int_I |f(t)|^2\, dt \leq c_1 \sum_{n\in B} |a_n|^2 + c_1 \sum_{n\in A} |a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2). \tag{2.3}$$

(*b*) *For every interval I of length $|I| > 2\pi/\gamma$ there exists a constant c_2*

such that all finite sums (2.1) *satisfy the* inverse inequality

$$(2.4)\qquad \sum_{n\in A} |a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2) +$$

$$\sum_{n\in B} |a_n|^2 \leqslant c_2 \int_I |f(t)|^2\,dt\,.$$

(*c*) *The estimate* (2.4) *can fail if* $|I| = 2\pi/\gamma$.

REMARK 2.2. – Under the stronger hypothesis

$$\lambda_{n+1} - \lambda_n \geqslant \gamma \quad \text{for all } n$$

instead of (2.2) this result reduces to Ingham's theorem. Hence part (*c*) follows at once.

REMARK 2.3. – Theorem 2.1 improves a former result of Castro and Zuazua [3] by weakening their assumption on the sequence (λ_n), and a subsequent theorem of Jaffard, Tucsnak and Zuazua [7] by improving their assumption $|I| > 3\sqrt{6}/\gamma$ for the inverse inequality. They also applied Ingham's second method with a different weight function. Our weight function below is closer to the Ingham's original one.

REMARK 2.4. – By a *finite sequence* we mean a sequence having only a finite number of nonzero elements. The estimates (2.3) and (2.4) extend easily to infinite sums for which the series on the right-hand side of (2.3) converges. Indeed, given such a complex sequence (a_n), set

$$f_m(t) = \sum_{n=-m}^{m} a_n e^{i\lambda_n t}, \qquad m = 1, 2, \ldots .$$

Applying (2.3) to the finite sums $f_p - f_m$ with $p > m$, we obtain that (f_m) is a Cauchy sequence and hence converges in $L^2(I)$ to some function f.

Next, applying (2.3) and (2.4) for every f_m and letting $m \to \infty$ we conclude that (2.3) and (2.4) hold true for f too, with the same constants c_1 and c_2.

An analogous remark holds for theorems 3.1 and 4.1 later.

The following four remarks will allow us to simplify the proof.

REMARK 2.5. – If we replace γ by some $0 < \delta < \gamma$ in the definition of the sets A and B, then the resulting inequalities (2.3) and (2.4) are equivalent to the original ones. Indeed, if

$$\delta < \lambda_{n+1} - \lambda_n \leqslant \gamma$$

for some n, then

$$|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2) \leqslant (2+\gamma^2)(|a_n|^2 + |a_{n+1}|^2)$$

and

$$|a_n|^2 + |a_{n+1}|^2 \leqslant \delta^{-2} (|a_n + a_{n+1}|^2 + |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2)) .$$

Hence (2.3) remains valid with $(2+\gamma^2)\, c_1$ instead of c_1 and (2.4) remains valid with $\max\{c_2, c_2 \delta^{-2}\}$ instead of c_2.

REMARK 2.6. – If the estimate (2.3) is satisfied for some interval I, then it is also satisfied for every translate $t' + I$ of I, with another constant c_1'. To show this, we shall need the inequality

$$|z_1 e^{i\mu_1} + z_2 e^{i\mu_2}|^2 \leqslant 2|z_1 + z_2|^2 + 2|\mu_1 - \mu_2|^2 |z_2|^2 \tag{2.5}$$

for all complex numbers z_1, z_2 and real numbers μ_1, μ_2. Indeed, using the triangle inequality and then the Lagrange mean value theorem we have

$$|z_1 e^{i\mu_1} + z_2 e^{i\mu_2}| \leqslant |(z_1 + z_2)\, e^{i\mu_1}| + (|z_2(e^{i\mu_2} - e^{i\mu_1})| \leqslant |z_1 + z_2| + |\mu_1 - \mu_2|\, |z_2| ,$$

and we conclude by applying Young's inequality. Now, given

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n t} \qquad (a_n \in \mathbb{C})$$

arbitrarily, setting

$$g(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n (t'+t)} = \sum_{n=-\infty}^{\infty} (a_n e^{i\lambda_n t'})\, e^{i\lambda_n t} =: \sum_{n=-\infty}^{\infty} a_n' e^{i\lambda_n t}$$

we have

$$\int_{t'+I} |f(t)|^2\, dt = \int_I |g(t)|^2\, dt \leqslant$$

$$c_1 \sum_{n\in B} |a_n'|^2 + c_1 \sum_{n\in A} |a_n' + a_{n+1}'|^2 + c_1 \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n'|^2 + |a_{n+1}'|^2) =$$

$$c_1 \sum_{n\in B} |a_n|^2 + c_1 \sum_{n\in A} |a_n e^{i\lambda_n t'} + a_{n+1} e^{i\lambda_{n+1} t'}|^2 +$$

$$c_1 \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2) .$$

Applying (2.5) for each $n \in A$ we obtain that

$$|a_n e^{i\lambda_n t'} + a_{n+1} e^{i\lambda_{n+1} t'}|^2 \leqslant 2|a_n + a_{n+1}|^2 + 2|t'|^2 |\lambda_{n+1} - \lambda_n|^2 |a_n|^2 .$$

Substituting into the preceding inequality we obtain

$$\int_{t'+I} |f(t)|^2\,dt \leq c_1 \sum_{n\in B} |a_n|^2 + 2c_1 \sum_{n\in A} |a_n + a_{n+1}|^2 +$$

$$c_1(1+2|t'|^2) \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2)\,.$$

Hence (2.3) is satisfied for $t' + I$ with $c_1' = \max\{2c_1,\, c_1 + 2|t'|^2\}$.

REMARK 2.7. – If the estimate (2.4) is satisfied for some interval I, then it is also satisfied for every translate $t' + I$ of I, with another constant c_2'. Indeed, introducing $g(t)$ as in the preceding remark, we have

$$\sum_{n\in B} |a_n|^2 + \sum_{n\in A} |a_n + a_{n+1}|^2 + \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2) =$$

$$\sum_{n\in B} |a_n'|^2 + \sum_{n\in A} |a_n' e^{-i\lambda_n t'} + a_{n+1}' e^{-i\lambda_{n+1} t'}|^2 +$$

$$\sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n'|^2 + |a_{n+1}'|^2) \leq \sum_{n\in B} |a_n'|^2 + 2 \sum_{n\in A} |a_n' + a_{n+1}'|^2 +$$

$$2|t'|^2 \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 |a_n'|^2 + \sum_{n\in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n'|^2 + |a_{n+1}'|^2) \leq$$

$$\max\{2,\, 1 + 2|t'|^2\}\, c_2 \int_I |g(t)|^2\,dt \leq \max\{2,\, 1 + 2|t'|^2\}\, c_2 \int_{t'+I} |f(t)|^2\,dt\,,$$

so that (2.4) is satisfied for $t' + I$ with $c_2' = c_2 \max\{2,\, 1 + 2|t'|^2\}$.

REMARK 2.8. – If the theorem holds true for some $\gamma > 0$, then it also holds for *all* $\gamma > 0$. To prove this, fix an arbitrary positive number p and set

$$\lambda_n' = p\lambda_n \qquad \text{for all } n\,.$$

The sequence (λ_n') satisfies a condition analogous to (2.2) with $\gamma' = p\gamma$ instead of γ. If the estimates (2.3) or (2.4) hold for some interval I, then on the interval $I' := p^{-1} I$ we have

$$\int_{I'} \left| \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n' t'} \right|^2 dt' = p^{-1} \int_I \left| \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n t} \right|^2 dt\,,$$

so that analogous estimates hold for the new sequence with c_1, c_2 replaced by c_1/p and $c_2 p$, respectively.

In view of the last three remarks it suffices to prove the theorem for intervals of the type $(-R, R)$, and for one particular value of $\gamma > 0$.

Now we turn to the proofs. The formula

$$H(x) := \begin{cases} \cos x & \text{if } -\pi/2 < x < \pi/2\,, \\ 0 & \text{otherwise} \end{cases} \qquad (x \in \mathbb{R}) \tag{2.6}$$

defines an even real function H in the Sobolev space $H_0^1(-\pi/2, \pi/2)$, whose inverse Fourier transform

$$h(t) := \int_{-\infty}^{\infty} e^{itx} H(x)\, dx = \frac{2\cos \pi t/2}{1-t^2} \qquad (t \in \mathbb{R})$$

is an even real function in $C^\infty(\mathbb{R})$. (Moreover, h extends to an entire analytic function by the Paley-Wiener theorem because H has a compact support.)

PROOF OF PART (a) OF THEOREM 2.1. – Set $K := H * H$ and denote its inverse Fourier transform by k. They are are even real functions having the following properties:

$$K \in H_0^1(-\pi, \pi)\,, \tag{2.7}$$

$$k \in C^\infty(\mathbb{R})\,, \tag{2.8}$$

$$k \geqslant 0 \quad \text{on } \mathbb{R}\,, \tag{2.9}$$

$$k \geqslant 1 \quad \text{on some interval } I\,. \tag{2.10}$$

Indeed, the first relation follows from the properties of the support of a convolution. The next two follow from the equality $k = h^2$. The last one holds for a sufficiently small interval around 0 because $k(0) = h(0)^2 = 4$ by the above explicit formula.

Assume that $\gamma = \pi$. A direct computation yields for $0 < x < \pi$ the explicit formulae

$$2K(x) = \sin x + (\pi - x)\cos x\,,$$

$$2K'(x) = (x - \pi)\sin x\,,$$

$$2K''(x) = \sin x + (x - \pi)\cos x\,.$$

Hence

$$K(0) > 0\,, \qquad K'(0) = 0\,, \qquad K''(0) < 0\,.$$

Applying Taylor's formula we conclude that

$$|K(x) - K(0)| \leqslant |K''(0)|\, x^2 \qquad \text{for all } x \in [-\delta, \delta] \tag{2.11}$$

for some suitable $0 < \delta \leqslant \gamma$. Let us change γ to δ in the definition of the sets A and B.

Observe that $n \in A$ implies $n+1 \notin A$. Indeed, if $n \in A$, then

$$\lambda_{n+1} - \lambda_n < \delta \leqslant \gamma ,$$

so that

$$\lambda_{n+2} - \lambda_{n+1} \geqslant 2\gamma - \gamma \geqslant \delta$$

by (2.2). Furthermore, (2.2) and (2.7) imply $K(\lambda_m - \lambda_n) = 0$ whenever $|m-n| \geqslant 2$. Furthermore, $K(\lambda_{n+1} - \lambda_n) = 0$ unless $n \in A$. Therefore we have the equality

$$\sum_{m,\, n=-\infty}^{\infty} K(\lambda_m - \lambda_n)\, a_m \overline{a_n} =$$

$$\sum_{n=-\infty}^{\infty} K(0)\, |a_n|^2 + \sum_{n \in A} K(\lambda_{n+1} - \lambda_n)(a_n \overline{a_{n+1}} + \overline{a_n} a_{n+1}) =$$

$$\sum_{n \in B} K(0)\, |a_n|^2 + \sum_{n \in A} K(0)\, |a_n + a_{n+1}|^2 +$$

$$\sum_{n \in A} (K(\lambda_{n+1} - \lambda_n) - K(0))(a_n \overline{a_{n+1}} + \overline{a_n} a_{n+1}) .$$

Using (2.11) hence we deduce the inequality

$$\sum_{m,\, n=-\infty}^{\infty} K(\lambda_m - \lambda_n)\, a_m \overline{a_n} \leqslant \sum_{n \in B} K(0)\, |a_n|^2 + K(0) \sum_{n \in A} |a_n + a_{n+1}|^2 +$$

$$|K''(0)| \sum_{n \in A} |\lambda_{n+1} - \lambda_n|^2 (|a_n|^2 + |a_{n+1}|^2) .$$

We conclude by noting that thanks to (2.9) and (2.10) we have

$$(2\pi) \sum_{m,\, n=-\infty}^{\infty} K(\lambda_m - \lambda_n)\, a_m \overline{a_n} = \int_{-\infty}^{\infty} k(t)\, |f(t)|^2\, dt \geqslant \int_I |f(t)|^2\, dt .$$

In the sequel we shall frequently use the powers of the function H. Let $H^M \colon \mathbb{R} \to \mathbb{R}$ be the Mth power of the function H introduced in (2.5), and let $h_M \colon \mathbb{R} \to \mathbb{R}$ denote its inverse Fourier transform given by

$$h_M(t) = \int_{-\infty}^{\infty} e^{itx} H^M(x)\, dx .$$

LEMMA 2.9. – *(a) H^M is not identically zero, even and real-valued.*

(b) H^M belongs to the Sobolev space $W_0^{M,\infty}(-\pi/2, \pi/2)$.

(c) h_M extends to an entire function $\mathbb{C} \to \mathbb{C}$.

(d) h_M is not identically zero, even and real-valued.

(*e*) *We have* $(H^M)'' + M^2 H^M = M(M-1) H^{M-2}$ *almost everywhere if* $M \geqslant 2$.

PROOF. – (*a*) and (*d*) are obvious.

(*b*) First we note that H^M belongs to $C^\infty(-\pi/2, \pi/2)$ and vanishes identically outside this interval. Therefore it is sufficient to verify that

$$(H^M)^{(j)}(\pm\pi/2) = 0\,, \qquad j = 0, 1, \ldots, M-1$$

and that the one-sided derivatives

$$(H^M)^{(M)}(\pi/2 - 0) \text{ and } (H^M)^{(M)}(-\pi/2 + 0)$$

exist and are finite. All these properties follows by applying the Leibniz rule. Indeed, differentiating $j < M$ times the product $\cos^M$, all terms of the resulting sum contains at least one factor cos which vanishes at $\pm\pi/2$. Furthermore, applying the same rule we obtain easily that

$$(H^M)^{(M)}(\pi/2 - 0) = (-1)^M \quad \text{and} \quad (H^M)^{(M)}(-\pi/2 + 0) = 1\,.$$

(*c*) This follows from (*b*) by the Paley-Wiener theorem.

(*e*) Outside $[-\pi/2, \pi/2]$ both sides vanish, so it is sufficient to verify the identity in $(-\pi/2, \pi/2)$. We have

$$\begin{aligned}
(H^M)''(x) &= (\cos^M x)'' = (-M\cos^{M-1} x \sin x)' = \\
&M(M-1)\cos^{M-2} x \sin^2 x - M\cos^M x = \\
&M(M-1)\cos^{M-2} x(1-\cos^2 x) - M\cos^M x = \\
&M(M-1)\cos^{M-2} x - M^2\cos^M x = M(M-1)(H^{M-2})(x) - M^2(H^M)(x)\,.
\end{aligned}$$

PROOF OF PART (*b*) OF THEOREM 2.1. – Assume this time that $\gamma = \pi/2$. Fix $R > 2$ arbitrarily and set

$$K := R^2 H^2 * H^2 + (H^2)' * (H^2)' = \left(R^2 + \frac{d^2}{dx^2}\right)(H^2 H^2)\,.$$

It follows from the preceding lemma that K and its inverse Fourier transforms k are even real functions satisfying

$$K \in H_0^3(-\pi, \pi)\,, \tag{2.12}$$

$$k \in C^\infty(\mathbb{R})\,, \tag{2.13}$$

$$k \leqslant 0 \text{ outside } I := (-R, R)\,. \tag{2.14}$$

Furthermore, for $0 < x < \pi$ we have

$$K(x) = \frac{3R^2 - 4}{16} \sin 2x + \frac{R^2 - 4}{8} (\pi - x) \cos 2x + \frac{R^2}{4} (\pi - x),$$

$$K'(x) = -\frac{R^2}{4} (1 - \cos 2x) + \frac{R^2 - 4}{4} (x - \pi) \sin 2x,$$

$$K''(x) = -\frac{R^2 + 4}{4} \sin 2x + \frac{R^2 - 4}{2} (x - \pi) \cos 2x.$$

Hence

(2.15) $$K(0) > 0, \quad K'(0) = 0, \quad K''(0) < 0$$

and

$$K(\pi) = K'(\pi) = K''(\pi) = 0.$$

Applying Taylor's formula we obtain that

$$K(x) = K(0) - \frac{|K''(0)|}{2} x^2 + o(x^2), \qquad x \to 0,$$

and

$$K(y) = o((\pi - y)^2), \qquad y \to \pi.$$

Since we have also $K(y) = 0$ for $y \geqslant \pi$, there exists a constant $0 < \delta \leqslant \gamma$ such that

(2.16) $$K(x) > K(0)/2$$

and

(2.17) $$K(0) - K(x) - K(y) \geqslant \frac{|K''(0)|}{4} x^2$$

for all $0 < x \leqslant \delta$ and $y \geqslant \pi - x$.

Furthermore, observe that K is nonincreasing in $(0, \infty)$ because $K'(x) \leqslant 0$ in $(0, \pi)$ by the above formula and $K = 0$ in (π, ∞). Hence for all $x, y \geqslant \delta$ such that $x < \pi$ and $x + y \geqslant \pi$, we have

$$K(x) + K(y) \leqslant K(x) + K(\pi - x) = K(0) - \frac{(R^2 - 4)\pi}{8} (1 - \cos 2x)$$

and therefore

$$K(0)-K(x)-K(y)\geqslant \frac{(R^2-4)\pi}{8}(1-\cos 2\delta)=:\eta>0\,. \tag{2.18}$$

whenever $x,\, y\geqslant\delta$ and $x+y\geqslant\pi$. Let us change γ to δ in the definition of A and B.

As in part (a), (2.1) and (2.12) imply $K(\lambda_m-\lambda_n)=0$ whenever $|m-n|\geqslant 2$. Furthermore, $n\in A$ implies $n+1\notin A$. Therefore we have the following identity:

$$\sum_{m,\,n=-\infty}^{\infty} K(\lambda_m-\lambda_n)\, a_m\overline{a_n}=$$

$$\sum_{n=-\infty}^{\infty} K(0)\,|a_n|^2+K(\lambda_{n+1}-\lambda_n)(a_n\overline{a_{n+1}}+\overline{a_n}a_{n+1})=$$

$$\sum_{n=-\infty}^{\infty} K(0)\,|a_n|^2+K(\lambda_{n+1}-\lambda_n)|\,a_n+a_{n+1}|^2-K(\lambda_{n+1}-\lambda_n)(|a_n|^2+|a_{n+1}|^2)=$$

$$\sum_{n=-\infty}^{\infty}(K(0)-K(\lambda_{n+1}-\lambda_n)-K(\lambda_n-\lambda_{n-1}))|a_n|^2+K(\lambda_{n+1}-\lambda_n)\,|a_n+a_{n+1}|^2=:$$

$$\sum_{n=-\infty}^{\infty} S_n=\sum_{n\in A}(S_n+S_{n+1})+\sum_{n\in B}S_n\,.$$

Next we use (2.16), (2.17) and (2.18) with $x=\lambda_{n+1}-\lambda_n$ and $y=\lambda_n-\lambda_{n-1}$. If $n\in A$, then

$$S_n+S_{n+1}\geqslant\frac{|K''(0)|}{4}\,|\lambda_{n+1}-\lambda_n|^2(|a_n|^2+|a_{n+1}|^2)+\frac{K(0)}{2}\,|a_n+a_{n+1}|^2\,.$$

If $n\in B$, then

$$S_n\geqslant\eta\,|a_n|^2\,.$$

Using (2.15) and these inequalities we deduce from the above identity the estimate

$$\sum_{n\in A}|a_n+a_{n+1}|^2+|\lambda_{n+1}-\lambda_n|^2(|a_n|^2+|a_{n+1}|^2)+\sum_{n\in B}|a_n|^2\leqslant$$

$$c'\sum_{m,\,n=-\infty}^{\infty}K(\lambda_m-\lambda_n)\,a_m\overline{a_n}$$

with a suitable constant c'. We conclude by remarking that by (2.13), (2.14) k

has a finite maximum, and

$$\sum_{m,\,n=-\infty}^{\infty} K(\lambda_m-\lambda_n)\,a_m\overline{a_n}=\int_{-\infty}^{\infty} k(t)\,|f(t)|^2\,dt \leqslant (\max k)\int_I |f(t)|^2\,dt\,.$$

3. – On a theorem of Kahane.

We are going to generalize Ingham's theorem to several variables. Let (λ_n) be a sequence of vectors in $\mathbb{R}^N$, satisfying for some $\gamma>0$ the condition

$$\|\lambda_m-\lambda_n\|_2 \geqslant \gamma \quad \text{whenever } m\neq n\,, \tag{3.1}$$

where $\|\ \|_2$ stands for the usual Euclidean norm of $\mathbb{R}^N$. Let us denote by μ_N the smallest eigenvalue of $-\Delta$ in $H_0^1(B_1)$ where B_1 is the unit ball of $\mathbb{R}^N$.

THEOREM 3.1. – (*a*) *For every open ball B in $\mathbb{R}^N$ there exists a constant c_1, depending only on γ and on the radius of the ball, such that all finite sums*

$$f(t)=\sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n\cdot t} \qquad (a_n\in\mathbb{C}) \tag{3.2}$$

satisfy the estimate

$$\int_B |f(t)|^2\,dt \leqslant c_1 \sum_{n=-\infty}^{\infty} |a_n|^2\,. \tag{3.3}$$

(*b*) *For every open ball B of radius $R>2\sqrt{\mu_N}/\gamma$ there exists a constant c_2, depending only on γ and on R, such that all finite sums* (3.2) *satisfy the estimate*

$$\sum_{n=-\infty}^{\infty} |a_n|^2 \leqslant c_2 \int_B |f(t)|^2\,dt\,. \tag{3.4}$$

REMARK 3.2. – This result improves proposition III.1.2 of Kahane [9] by weakening his assumptions for the validity of (3.4). We do not know whether our condition $R>2\sqrt{\mu_N}/\gamma$ in part (*b*) is optimal. Note that this condition means that the smallest eigenvalue of $-\Delta$ in $H_0^1(B_R)$ is less than $\gamma^2/4$.

Let us recall, e.g., from [22] that the smallest eigenvalue of $-\Delta$ in $H_0^1(B_R)$ is equal to $(\varrho_N/R)^2$ where ϱ_N denotes the smallest positive zero of the Bessel function $J_{(N-2)/2}$.

REMARK 3.3. – As mentioned in Remark 2.4, both inequalities, once proved, extend to infinite sums with square summable coefficients. Furthermore, by an easy generalization of Remarks 2.6-2.8, in the proofs it will suffice to consider balls centered at the origin, and one particular value of γ.

Turning to the proof of Theorem 3.1, let us denote by B_r the open ball of radius r centered at the origin of $\mathbb{R}^N$. Fix a nonzero eigenfunction H of $-\Delta$ in $H_0^1(B_1)$, corresponding to the smallest eigenvalue μ_N of $-\Delta$ in $H_0^1(B_1)$, and extend it by zero outside B_1. We may assume that H is strictly positive in B_1. Then H is a real radial function in $H_0^1(B_1)$, therefore its inverse Fourier transform

$$h(t) := \int_{\mathbb{R}^N} e^{it\cdot x} H(x)\, dx$$

is a real radial function in $C^\infty(\mathbb{R}^N)$. (And h extends again to an entire analytic function.)

PROOF OF PART (a) OF THEOREM 3.1. – Assume that $\gamma = 2$. The function $K := H * H$ and its inverse Fourier transform $k = h^2$ are real radial functions having the following properties:

$$K \in H_0^1(B_2),$$

$$k \geqslant 0 \text{ on } \mathbb{R}^N,$$

$$k \geqslant \beta \text{ on some ball } B,$$

where β is some positive number. (The last property follows from the fact that $k \in C^\infty(\mathbb{R}^N)$ by the Paley-Wiener theorem and that k cannot be identically zero.)

Using (3.1) and these properties, (3.3) follows:

$$\beta \int_B |f(t)|^2\, dt \leqslant \int_{\mathbb{R}^N} k(t)\, |f(t)|^2\, dt =$$

$$(2\pi)^N \sum_{m,\, n = -\infty}^{\infty} K(\lambda_n - \lambda_m)\, a_m \overline{a_n} = (2\pi)^N K(0) \sum_{n=-\infty}^{\infty} |a_n|^2 .$$

PROOF OF PART (b) OF THEOREM 3.1. – Assume $\gamma = 2$ again. Choose $R > \sqrt{\mu_N}$ arbitrarily. The function

$$K = (R^2 + \Delta)(H * H) = R^2 H * H + \sum_{j=1}^{N} \partial_j H \partial_j H$$

and its inverse Fourier transform

$$k(t) = (R^2 - |t|^2)\, h(t)^2$$

are even radial functions satisfying the following conditions:

$$K \in H_0^1(B_2)\,,$$

$$K(0) > 0\,,$$

$$k \in C^\infty(\mathbb{R}^N)\,,$$

$$k \leqslant 0 \ \text{ outside } \ B := B_R\,.$$

The second property follows from the relation

$$K(0) = \int_{\mathbb{R}^N} R^2\,|H|^2 - |\nabla H|^2\, dx = (R^2 - \mu_N) \int_{\mathbb{R}^N} |H|^2\, dx\,.$$

Since $k \in C^\infty(\mathbb{R}^N)$ again by the Paley-Wiener theorem, it follows that k has a finite maximum α on $\mathbb{R}^N$, and (3.4) follows as in section 1:

$$(2\pi)^N K(0) \sum_{n=-\infty}^{\infty} |a_n|^2 = (2\pi)^N \sum_{m,\,n=-\infty}^{\infty} K(\lambda_n - \lambda_m)\, a_m \overline{a_n} =$$

$$\int_{\mathbb{R}^N} k(t)\,|f(t)|^2\, dt \leqslant (\max k) \int_B |f(t)|^2\, dt\,.$$

4. – On a theorem of Ullrich.

We are going to obtain an optimal variant of Kahane's theorem by changing the l_2-norm to the l_∞-norm in $\mathbb{R}^N$. Furthermore, more generally, we consider series with polynomial coefficients.

Let (λ_n) be a sequence of vectors in $\mathbb{R}^N$, satisfying for some $\gamma > 0$ the condition

$$\|\lambda_m - \lambda_n\|_\infty \geqslant \gamma \quad \text{ whenever } m \neq n\,. \tag{4.1}$$

Fix a positive integer M and consider all finite sums of the form

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{|j|_\infty < M} a_{jn}\, t^j e^{i\lambda_n \cdot t} \qquad (a_{jn} \in \mathbb{C})\,. \tag{4.2}$$

We apply here the usual multiindex notations: the components of $j =$

$(j_1, \ldots, j_N)$ are nonnegative integers and

$$|j|_\infty = \max\{j_1, \ldots, j_N\},$$

$$t^j = t_1^{j_1} \ldots t_N^{j_N} ;$$

$$|j| = j_1 + \ldots + j_N ,$$

$$\partial^j = \partial_1^{j_1} \ldots \partial_N^{j_N} ,$$

where

$$\partial_k = \partial/\partial x_k .$$

We recall that if

$$k(t) = \int_{\mathbb{R}^N} K(x)\, e^{it\cdot x}\, dx$$

is the inverse Fourier transform of K, then

$$K(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} k(t)\, e^{-ix\cdot t}\, dt ,$$

and more generally,

$$i^{|j|}\, \partial^j K(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} t^j\, k(t)\, e^{-ix\cdot t}\, dt$$

for all j. We are going to prove the

THEOREM 4.2. – (a) *For every open ball B in $\mathbb{R}^N$ there exists a constant c_1, depending on γ, M and on the radius of the ball B, such that all finite sums* (4.2) *satisfy the estimate*

$$\int_B |f(t)|^2\, dt \leq c_1 \sum_{n=-\infty}^{\infty} \sum_{|j|_\infty < M} |a_{jn}|^2 . \tag{4.3}$$

(b) *For every open ball B of radius $R > M\sqrt{N}\pi/\gamma$ in $\mathbb{R}^N$ there exists a constant c_2, depending on γ, M and on the radius of the ball B, such that all finite sums* (4.2) *satisfy the estimate*

$$\sum_{n=-\infty}^{\infty} \sum_{|j|_\infty < M} |a_{jn}|^2 \leq c_2 \int_B |f(t)|^2\, dt . \tag{4.4}$$

(c) *The estimate* (4.4) *can fail if $R < M\sqrt{N}\pi/\gamma$.*

REMARK 4.2. – For $N = 1$ the theorem reduces to an earlier result of Ullrich [21], proved by him in a different way. For $N = M = 1$ we get the original theorem of Ingham.

REMARK 4.3. – By an easy modification of Remarks 2.4 and 2.6-2.8, the estimates remain valid for infinite sums with square summable coefficients, and in the proof it is sufficient to consider balls centered at the origin and to consider the case $\gamma = \pi$.

PROOF OF PART (*a*) OF THEOREM 4.1. – By Remark 4.3 it suffices to consider balls centered at the origin and we may assume that $\gamma = \pi$. Set

$$B_\pi^\infty = \{\lambda \in \mathbb{R}^N : \|\lambda\|_\infty < \pi\}.$$

The function

$$K(x) := \prod_{p=1}^{N} (H^M H^M)(x_p), \qquad x = (x_1, \ldots, x_N) \in \mathbb{R}^N$$

and its inverse Fourier transform

$$k(t) := \int_{\mathbb{R}^N} e^{it\cdot x} \prod_{p=1}^{N} (H^M * H^M)(x_p)\, dx =$$

$$= \prod_{p=1}^{N} \int_{-\infty}^{\infty} e^{it_p x_p} (H^M * H^M)(x_p)\, dx_p = \prod_{p=1}^{N} |h_M(t_p)|^2$$

satisfy the conditions

$$K \in H_0^1(B_\pi^\infty),$$

$$k \geqslant 0 \text{ on } \mathbb{R}^N,$$

$$k \geqslant \beta \text{ on some ball } B,$$

where β is some positive number. Therefore

$$\beta(2\pi)^{-N} \int_B |f(t)|^2\, dt \leqslant (2\pi)^{-N} \int_{\mathbb{R}^N} k(t)\, |f(t)|^2\, dt =$$

$$\sum_{m,n=-\infty}^{\infty} \sum_{|j|_\infty, |k|_\infty < M} (2\pi)^{-N} a_{jm} \overline{a_{kn}} \int_{\mathbb{R}^N} t^{j+k} k(t)\, e^{i(\lambda_m - \lambda_n)\cdot t}\, dt =$$

$$\sum_{m,n=-\infty}^{\infty} \sum_{|j|_\infty, |k|_\infty < M} a_{jm} \overline{a_{kn}}\, i^{|j+k|}\, \partial^{j+k} K(\lambda_m - \lambda_n) =$$

$$\sum_{n=-\infty}^{\infty} \sum_{|j|_\infty, |k|_\infty < M} a_{jn} \overline{a_{kn}}\, i^{|j+k|}\, \partial^{j+k} K(0).$$

Hence

$$\beta(2\pi)^{-N}\int_B |f(t)|^2\,dt \leqslant c\sum_{n=-\infty}^{\infty}\sum_{|j|_\infty,\,|k|_\infty<M}|a_{jn}|\,|a_{kn}| \leqslant$$

$$\frac{c}{2}\sum_{n=-\infty}^{\infty}\sum_{|j|_\infty,\,|k|_\infty<M}|a_{jn}|^2+|a_{kn}|^2 = c\sum_{n=-\infty}^{\infty}\sum_{|k|_\infty<M}|a_{kn}|^2$$

with

$$c := \max\{|\partial^l K(0)| : |l| < 2M-1\}.$$

For the proof of part (b) we need the crucial

Lemma 4.4. – *If $R > M\sqrt{N}$ and*

$$K(x) = R^2\prod_{p=1}^{N}(H^M * H^M)(x_p) + \sum_{p=1}^{N}((H^M)' * (H^M)')(x_p)\prod_{q\neq p}(H^M * H^M)(x_q),$$

then the quadratic form

$$(a_j)_{|j|_\infty<M} \mapsto \sum_{|j|_\infty,\,|k|_\infty<M} i^{|j+k|}\,\partial^{j+k}(K)(0)\,a_j\overline{a_k}$$

is positive definite.

Proof. – We have

$$\sum_{|j|_\infty,\,|k|_\infty<M} i^{|j+k|}\,\partial^{j+k}K(0)a_j\overline{a_k} =$$

$$\sum_{|j|_\infty,\,|k|_\infty<M} a_j\overline{a_k}\,i^{|j+k|}\sum_{p=1}^{N}\int_{\mathbb{R}} R^2(H^M)^{(j_p)}(x_p)(H^M)^{(k_p)}(-x_p)+$$

$$(H^M)^{(j_p+1)}(x_p)(H^M)^{(k_p+1)}(-x_p)\,dx_p\prod_{q\neq p}\int_{\mathbb{R}}(H^M)^{(j_q)}(x_q)(H^M)^{(k_q)}(-x_q)\,dx_q =$$

$$\sum_{|j|_\infty,\,|k|_\infty<M} a_j\overline{a_k}\,i^{|j+k|}(-1)^{|k|}\sum_{p=1}^{N}\int_{\mathbb{R}} R^2(H^M)^{(j_p)}(x_p)(H^M)^{(k_p)}(x_p)-$$

$$(H^M)^{(j_p+1)}(x_p)(H^M)^{(k_p+1)}(x_p)\,dx_p\prod_{q\neq p}\int_{\mathbb{R}}(H^M)^{(j_q)}(x_q)(H^M)^{(k_q)}(x_q)\,dx_q\,.$$

Setting

$$\widetilde{H}(x) := \sum_{|j|_\infty<M} a_j\,i^{|j|}\prod_{q=1}^{N}(H^M)^{(j_q)}(x_q),$$

a simple computation shows that

$$\int_{[-\pi/2,\,\pi/2]^N} |\widetilde{H}|^2\,dx = \sum_{|j|_\infty,\,|k|_\infty<M} a_j\overline{a_k}\,i^{|j|}(-i)^{|k|}\prod_{q=1}^{N}\int_{-\pi/2}^{\pi/2}(H^M)^{(j_p)}(H^M)^{(k_p)}\,dx_q$$

and

$$\int_{[-\pi/2,\,\pi/2]^N} |\nabla\widetilde{H}|^2\,dx = \sum_{p=1}^{N}\sum_{|j|_\infty,\,|k|_\infty<M} a_j\overline{a_k}\,i^{|j|}(-i)^{|k|}\times$$

$$\int_{-\pi/2}^{\pi/2}(H^M)^{(j_p+1)}(H^M)^{(k_p+1)}\,dx_p\prod_{q\neq p}\int_{-\pi/2}^{\pi/2}(H^M)^{(j_q)}(H^M)^{(k_q)}\,dx_q\,.$$

Substituting them into the first identity we obtain that

$$\sum_{|j|_\infty,\,|k|_\infty<M} i^{|j|+|k|}\,\partial^{j+k}K(0)\,a_j\overline{a_k} = \int_{[-\pi/2,\,\pi/2]^N} R^2|\widetilde{H}|^2-|\nabla\widetilde{H}|^2\,dx\,.$$

We shall prove that the last integral is positive unless all coefficients a_j vanish.

Equivalently, setting $G(x)=\sin^M x$ and

$$H_0(x) := \sum_{|j|_\infty<M} a_j\,i^{|j|}\prod_{q=1}^{N}G^{(j_q)}(x_q)\,,$$

we have to prove that the integral

$$\int_{[0,\,\pi]^N} R^2|H_0|^2-|\nabla H_0|^2\,dx$$

is positive unless all a_j's vanish.

Observe that the function $G^{(m)}G^{(n)}$ is odd with respect to $\pi/2$ if $m-n$ is odd, and hence

$$\int_0^{\pi} G^{(m)}(x)\,G^{(n)}(x)\,dx = 0\,.$$

Hence, putting

$$H_1(x) := \sum_{|j|_\infty<M,\,|j|\ \text{odd}} a_j\,i^{|j|}\prod_{q=1}^{N}G^{(j_q)}(x_q)\,,$$

$$H_2(x) := \sum_{|j|_\infty<M,\,|j|\ \text{even}} a_j\,i^{|j|}\prod_{q=1}^{N}G^{(j_q)}(x_q)\,,$$

we have $H_0 = H_1 + H_2$ and

$$\int_{[0,\pi]^N} H_1 \overline{H_2}\, dx = 0\,.$$

Therefore

$$\int_{[0,\pi]^N} R^2 |H_0|^2 - |\nabla H_0|^2\, dx =$$

$$\int_{[0,\pi]^N} R^2 |H_1|^2 - |\nabla H_1|^2\, dx + \int_{[0,\pi]^N} R^2 |H_2|^2 - |\nabla H_2|^2\, dx\,.$$

Furthermore, $|H_1|$ and $|H_2|$ are even in each of their N variables, and therefore

$$2^N \int_{[0,\pi]^N} R^2 |H_0|^2 - |\nabla H_0|^2\, dx = \int_{[-\pi,\pi]^N} R^2 |H_1|^2 - |\nabla H_1|^2\, dx +$$

$$+ \int_{[-\pi,\pi]^N} R^2 |H_2|^2 - |\nabla H_2|^2\, dx = \int_{[-\pi,\pi]^N} R^2 |H_0|^2 - |\nabla H_0|^2\, dx\,.$$

Now observe that H_0 is a linear combination of the functions $e^{ij\cdot x}$ for $|j|_\infty \leqslant M$ and therefore

$$\int_{[-\pi,\pi]^N} R^2 |H_0|^2 - |\nabla H_0|^2\, dx \geqslant (R^2 - NM^2) \int_{[-\pi,\pi]^N} |H_0|^2\, dx\,.$$

Since $R^2 - NM^2 > 0$ by assumption and since the last integral is a positive definite quadratic form of the coefficients a_j by the linear independence of the functions $G, G', \ldots, G^{(M-1)}$, the lemma follows.

PROOF OF PART (*b*) OF THEOREM 4.1. – As in part (*a*), we consider balls centered at the origin, we assume that $\gamma = \pi$ and we introduce the set B_π^∞ as before.

Choose $R > M\sqrt{N}$ arbitrarily and set

$$K(x) = R^2 \prod_{p=1}^{N} (H^M * H^M)(x_p) + \sum_{p=1}^{N} ((H^M)' * (H^M)')(x_p) \prod_{q \neq p} (H^M * H^M)(x_q)\,.$$

Then K and its inverse Fourier transform

$$k(t) = (R^2 - |t|^2) \prod_{p=1}^{N} h_M(t_p)^2$$

are even real functions, satisfying the conditions

$$K \in W_0^{2M-1,\infty}(B_\pi^\infty)$$

and

$$k \leqslant 0 \quad \text{outside } B := \{\lambda \in \mathbb{R}^N : \|\lambda\|_2 < R\}.$$

In particular, k has a finite maximum α on $\mathbb{R}^N$.

We have

$$\alpha(2\pi)^{-N}\int_B |f(t)|^2\,dt \geqslant (2\pi)^{-N}\int_{\mathbb{R}^N} k(t)\,|f(t)|^2\,dt =$$

$$\sum_{m,n=-\infty}^{\infty}\ \sum_{|j|_\infty,\,|k|_\infty<M} a_{jm}\overline{a_{kn}}\, i^{|j+k|}\,\partial^{j+k}K(\lambda_n-\lambda_m) =$$

$$\sum_{n=-\infty}^{\infty}\ \sum_{|j|_\infty,\,|k|_\infty<M} a_{jn}\overline{a_{kn}}\, i^{|j+k|}\,\partial^{j+k}K(0),$$

and we conclude by recalling that the quadratic form

$$(a_j)_{|j|<M} \mapsto \sum_{|j|_\infty,\,|k|_\infty<M} i^{|j+k|}\,\partial^{j+k}K(0)\,a_j\overline{a_k}$$

is positive definite by Lemma 4.4.

PROOF OF PART (*c*) OF THEOREM 4.1. – According to Remark 2.4, if (4.4) holds for all finite sums, then it also holds for all sums with square summable coefficients.

Fix a small positive number $\varepsilon < 1$ and consider the function

$$f_\varepsilon(t) = \begin{cases} 1 & \text{if dist } (t, 2M\mathbb{Z}^N) < \varepsilon, \\ 0 & \text{otherwise}, \end{cases} \qquad (t \in \mathbb{R}).$$

For every $t \in [0, 2)^N$, the linear system

$$\sum_{|j|_\infty<M} (t+2k)^j f_j(t) = f_\varepsilon(t), \qquad |k| < M$$

has a unique solution $(f_j(t))_{|j|_\infty<M}$. Extending $f_0, \dots, f_{M-1}$ to $\mathbb{R}^N$ 2-periodically in each variable, we have

$$\sum_{|j|_\infty<M} t^j f_j = f_\varepsilon$$

on the set

$$\Omega := \bigcup_{|k|_\infty<M} (2k + [0, 2)^N).$$

Developing the functions f_j into N-fold trigonometric Fourier series, we obtain in Ω a development

$$f_\varepsilon(t) = \sum_{|j|_\infty < M} \sum_{n \in \mathbb{Z}^N} a_{jn} t^j e^{i\pi n \cdot t}$$

with square summable coefficients a_{jn}. Since f_ε does not vanish identically, there are nonzero coefficients, so that (4.4) cannot hold on the ball of center $(M, \dots, M)$ and radius $(M - \varepsilon)\sqrt{N}$, contained in Ω, where f_ε vanishes identically.

5. – Simultaneous observability of vibrating strings.

Fix a number $0 < a < 1$ arbitrarily and consider the following problem:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } (0, a) \times \mathbb{R}, \\ u_{tt} - u_{xx} = 0 & \text{in } (a, 1) \times \mathbb{R}, \\ u(0, \cdot) = u(a, \cdot) = u(1, \cdot) = 0 & \text{in } \mathbb{R}, \\ u(\cdot, 0) = u_0 \text{ and } u_t(\cdot, 0) = u_1 & \text{in } (0, 1). \end{cases} \tag{5.1}$$

We recall from Lions [14] that if

$$u_0 \in H_0^1(0, 1), \qquad u_1 \in L^2(0, 1) \text{ and } u_0(a) = 0,$$

then (5.1) has a unique solution

$$u \in C(\mathbb{R}; H_0^1(0, 1)) \cap C^1(\mathbb{R}; L^2(0, 1)),$$

and that this solution has the «hidden» regularity property

$$u_x(a - 0, \cdot), u_x(a + 0, \cdot) \in L^2_{\text{loc}}(\mathbb{R}).$$

We are going to study the following question. Assume we may observe the *sum* of the outward normal derivatives of the solutions of (5.1) at the common endpoint a during some time interval I. Does this observation allow us to identify the unknown initial data? Mathematically, we ask whether the linear map

$$H_0^1(0, 1) \times L^2(0, 1) \to L^2(I)$$

defined by the formula

$$(u_0, u_1) \mapsto u_x(a - 0, \cdot) - u_x(a + 0, \cdot)|_I$$

is injective or not.

The answer depends on the position of a and on the length of I:

THEOREM 5.1. – (a) *For almost every* $0 < a < 1$, *the solutions of* (5.1) *satisfy the inequality*

$$\|u_0\|^2_{H^{-\varepsilon}(0,\,1)} + \|u_1\|^2_{H^{-1-\varepsilon}(0,\,1)} \leq C \int_I |u_x(a-0,\,t) - u_x(a+0,\,t)|^2\, dt \tag{5.2}$$

for every bounded interval I *of length* $> 4\ \max\{a,\,1-a\}$ *and for every* $\varepsilon > 0$. *The constant* C *depends on* ε *and on* $|I|$ *but not on the particular choice of* u_0 *and* u_1.

(b) *The estimate* (5.2) *cannot hold for any* $0 < a < 1$ *if* $|I| < 2\ \max\{a,\,1-a\}$.

(c) *The estimate* (5.2) *cannot hold for any interval* I *if* a *is a rational number.*

REMARK 5.2. – This problem was first studied by Jaffard, Tucsnak and Zuazua [8]. They proved the estimate (5.2) under a stronger condition on the length of I. We follow their method but we apply Theorem 2.1 instead of their original result.

REMARK 5.3. – We hope to return to this problem in the near future and to determine the optimal condition on $|I|$ for the validity of the estimate (5.2).

PROOF OF PART (a). – By a density argument it suffices to consider initial data u_0, u_1 which are finite linear combinations of the eigenfunctions of $-\Delta$ in $H_0^1(0,\,a)$ and in $H_0^1(a,\,1)$. Then all sums in the sequel are finite, hence all convergence problems are avoided. Furthermore, assume that a is irrational; this excludes only a set of measure zero.

Applying the Fourier method, the solution of (5.1) is given by the formula

$$u(x,\,t) = \begin{cases} \sum_n b_n \sin(n\pi a^{-1} x)\, e^{in\pi a^{-1} t} & \text{if } 0 < x < a\,, \\ \sum_n c_n \sin(n\pi(1-a)^{-1}(1-x))\, e^{in\pi(1-a)^{-1} t} & \text{if } a < x < 1\,, \end{cases}$$

where n runs over the nonzero (positive or negative) integers, with suitable complex coefficients b_n and c_n depending on the initial data. A simple computation shows that

$$u_x(a-0,\,t) - u_x(a+0,\,t) = \sum_n (-1)^n \pi n \left(\frac{b_n}{a}\, e^{in\pi a^{-1} t} + \frac{c_n}{1-a}\, e^{in\pi(1-a)^{-1} t} \right).$$

Setting

$$\Lambda := \{ n\pi a^{-1}, n\pi(1-a)^{-1} : n \in \mathbb{Z} - \{0\} \}$$

and

$$a_\lambda = \begin{cases} (-1)^n \pi n b_n a^{-1} & \text{if } \lambda = n\pi a^{-1}, \\ (-1)^n \pi n c_n (1-a)^{-1} & \text{if } \lambda = n\pi(1-a)^{-1}, \end{cases}$$

we may rewrite it in the simpler form

$$u_x(a-0, t) - u_x(a+0, t) = \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t}.$$

Note that no $\lambda \in \Lambda$ has two different representations by the irrationality of a.

Next we obtain by another direct computation that

$$\|u_0\|^2_{H^{-\varepsilon}(0,1)} + \|u_1\|^2_{H^{-1-\varepsilon}(0,1)}$$

is equivalent to

$$\sum_n |n|^{-2\varepsilon} (|b_n|^2 + |c_n|^2),$$

which is in its turn equivalent to the sum

$$\sum_{\lambda \in \Lambda} |\lambda|^{-2-2\varepsilon} |a_\lambda|^2.$$

Indeed, for $\lambda = n\pi a^{-1}$ we have

$$|\lambda|^{-2-2\varepsilon} |a_\lambda|^2 = |n\pi a^{-1}|^{-2-2\varepsilon} |\pi n b_n a^{-1}|^2 = |n\pi a^{-1}|^{-2\varepsilon} |b_n|^2 \sim |n|^{-2\varepsilon} |b_n|^2,$$

while for $\lambda = n\pi(1-a)^{-1}$ we have similarly

$$|\lambda|^{-2-2\varepsilon} |a_\lambda|^2 = |n\pi(1-a)^{-1}|^{-2-2\varepsilon} |\pi n c_n (1-a)^{-1}|^2 =$$

$$|n\pi(1-a)^{-1}|^{-2\varepsilon} |c_n|^2 \sim |n|^{-2\varepsilon} |c_n|^2.$$

Hence the estimate (5.2) is equivalent to the following inequality:

$$\sum_{\lambda \in \Lambda} |\lambda|^{-2-2\varepsilon} |a_\lambda|^2 \leq c \int_I \left| \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda t} \right|^2 dt. \tag{5.3}$$

To prove (5.3) first we observe that, since a is assumed to be irrational, the numbers $n\pi a^{-1}$, $n\pi(1-a)^{-1}$, where n runs over the nonzero integers, are pairwise distinct. Furthermore, no interval of length $< \min\{\pi a^{-1}, \pi(1-a)^{-1}\}$ contains more than two elements of the set Λ. Therefore, applying theorem 2 with $\gamma = 1/2 \min\{\pi a^{-1}, \pi(1-a)^{-1}\}$ we obtain the inequality

$$(5.4)\qquad \sum_{|\lambda-\mu|<\gamma} |\lambda-\mu|^2(|a_\lambda|^2 + |a_\mu|^2) + \sum_{\lambda\in\Lambda'} |a_\lambda|^2 \leqslant c\int_I \Big|\sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda t}\Big|^2 dt$$

for every bounded interval I of length

$$|I| > 2\pi\gamma^{-1} = 4\ \max\{a, 1-a\},$$

where Λ' denotes the set of those $\lambda\in\Lambda$ for which $|\lambda-\mu| \geqslant \gamma$ for every other $\mu\in\Lambda$.

Next we recall from [2] a classical result from the theory of diophantine approximation: almost every real number a satisfies for all $\varepsilon > 0$ the inequalities

$$\operatorname{dist}(qa, \mathbb{Z}) \geqslant c_\varepsilon q^{-1-\varepsilon}, \qquad q = 1, 2, \ldots .$$

If $|\lambda-\mu| < \gamma$ in (5.4), then we have (changing, if necessary, the order of λ and μ) $\lambda = n\pi(1-a)^{-1}$ and $\mu = m\pi a^{-1}$ with suitable integers. Apart from a finite number of such pairs, the integers m and n have the same sign and have a sufficiently large absolute value. Hence

$$|\lambda-\mu| = \frac{\pi}{a(1-a)}\,|(n+m)a - m| \geqslant \frac{\pi}{a(1-a)}\, c_\varepsilon\, |n+m|^{-1-\varepsilon}.$$

Since the condition $|\lambda-\mu| < \gamma$ implies that

$$n+m \sim n \sim m \sim \lambda \sim \mu\,,$$

it follows that

$$|\lambda-\mu| \geqslant c_\varepsilon' \max\{|\lambda|, |\mu|\}^{-1-\varepsilon}$$

with a suitable positive constant c_ε'. (The right-hand side is well defined because $0\notin\Lambda$.) Hence the first sum on the left-hand side of (5.4) is minorized by

$$c_\varepsilon' \sum_{|\lambda-\mu|<\gamma} |\lambda|^{-2-2\varepsilon}|a_\lambda|^2 + |\mu|^{-2-2\varepsilon}|a_\mu|^2 .$$

Choosing, if necessary, a smaller c_ε', the second sum on the left-hand side of (5.4) can also be minorized as follows:

$$c_\varepsilon' \sum_{\lambda\in\Lambda'} |\lambda|^{-2-2\varepsilon}|a_\lambda|^2 \leqslant \sum_{\lambda\in\Lambda'} |a_\lambda|^2 .$$

This completes the proof of (5.3).

PROOF OF PART (b). – Assume that $a \geqslant 1/2$ (the other case is analogous) and fix $0 < T < a$ arbitrarily. We are going to show that the estimate (5.2) cannot hold for $I = (-T, T)$.

Choose *nonzero* initial data $u_0 \in H_0^1(0, 1)$ and $u_1 \in L^2(0, 1)$ satisfying

$$u_0 = u_1 = 0 \qquad \text{in } (a - T, 1).$$

Then the solution of (5.1) satisfies

$$u(x, t) = 0 \qquad \text{for } a - T + |t| < x < 1$$

for all t by the finite propagation property of the wave equation. Hence

$$u_x(a-0, t) = u_x(a+0, t) = 0 \qquad \text{for } -T < t < T,$$

so that the right-hand side of (5.2) vanishes for $I = (-T, T)$. On the other hand, the left-hand side of (5.2) is strictly positive because the initial data are not identically zero.

PROOF OF PART (c). – If a is a rational number, then there exist positive integers m and n such that

$$\frac{m\pi}{a} = \frac{n\pi}{1-a}.$$

Denoting this common value by λ, the formula

$$u(x, t) := \begin{cases} \sin \lambda x \, e^{i\lambda t} & \text{if } 0 < x < a, \\ -\sin \lambda(1-x) \, e^{i\lambda t} & \text{if } a < x < 1 \end{cases}$$

defines a nonzero solution of (5.1), so that the left-hand side of (5.1) is strictly positive. On the other hand, we have

$$u_x(a-0, t) = u_x(a+0, t) = 0$$

for all real t, so that the right-hand side of (5.2) vanishes for *every* bounded interval I. Hence (5.2) cannot hold.

6. – Simultaneous observability of beams.

As in the preceding section, fix $0 < a < 1$ arbitrarily. Now consider the following problem:

$$(6.1)\qquad \begin{cases} u_{tt} + u_{xxxx} = 0 & \text{in } (0, a) \times \mathbb{R}, \\ u_{tt} + u_{xxxx} = 0 & \text{in } (a, 1) \times \mathbb{R}, \\ u(0, \cdot) = u(a, \cdot) = u(1, \cdot) = 0 & \text{in } \mathbb{R}, \\ u_{xx}(0, \cdot) = u_{xx}(a, \cdot) = u_{xx}(1, \cdot) = 0 & \text{in } \mathbb{R}, \\ u(\cdot, 0) = u_0 \text{ and } u_t(\cdot, 0) = u_1 & \text{in } (0, 1). \end{cases}$$

This system models two vibrating beams with simply supported endpoints, one of which is common to both beams. We recall from [14] that if

$$u_0 \in H_0^1(0, 1), \qquad u_1 \in H^{-1}(0, 1) \text{ and } u_0(a) = 0,$$

then (6.1) has a unique solution

$$u \in C(\mathbb{R}; H_0^1(0, 1)) \cap C^1(\mathbb{R}; H^{-1}(0, 1)),$$

and this solution has the «hidden» regularity property

$$u_x(a - 0, \cdot), \qquad u_x(a + 0, \cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}).$$

Assume we may observe the *sum* of the outward normal derivatives of the solutions of (6.1) at the common endpoint a during some time interval I. Does it allow us to distinguish different sets of initial data? We are going to prove that the answer is affirmative for almost every point a, even if the observation time is arbitrarily small.

THEOREM 6.1. – (a) *For almost every* $0 < a < 1$, *the solutions of* (6.1) *satisfy the estimate*

$$(6.2)\qquad \|u_0\|^2_{H^{1-\varepsilon}(0,1)} + \|u_1\|^2_{H^{-1-\varepsilon}(0,1)} \le c \int_I |u_x(a-0, t) - u_x(a+0, t)|^2 \, dt$$

for every (*arbitrarily short*) *bounded interval* I, *and for every* $\varepsilon > 0$, *with a constant* $c = c(|I|, \varepsilon)$, *independent of the choice of* u_0 *and* u_1.

(b) *The estimate* (6.2) *cannot hold if* a *is a rational number.*

PROOF OF PART (a). – Applying the Fourier method as in the preceding section, the solution of (6.1) is given by the formula

$$u(x, t) = \begin{cases} \sum_n b_n \sin(n\pi a^{-1} x)\, e^{in|n|\pi^2 a^{-2} t} & \text{if } 0 < x < a, \\ \sum_n c_n \sin(n\pi(1-a)^{-1}(1-x))\, e^{in|n|\pi^2(1-a)^{-2} t} & \text{if } a < x < 1, \end{cases}$$

where n runs over the nonzero (positive or negative) integers, with suitable complex coefficients depending on the initial data. By a density argument it suffices to consider only finite sums.

It follows that

$$u_x(a-0,\,t)-u_x(a+0,\,t)=\sum_n(-1)^n\pi n\left\{\frac{b_n}{a}\,e^{in|n|\pi^2a^{-2}t}+\frac{c_n}{1-a}\,e^{in|n|\pi^2(1-a)^{-2}t}\right\}.$$

Assume that a is irrational, and assume by symmetry that $0<a<1/2$. Setting

$$\Lambda:=\{n|n|\pi^2a^{-2},\,n|n|\pi^2(1-a)^{-2}\colon n\in\mathbb{Z}-\{0\}\}$$

and

$$a_\lambda=\begin{cases}(-1)^n\pi nb_na^{-1} & \text{if } \lambda=n|n|\pi^2a^{-2},\\ (-1)^n\pi nc_n(1-a)^{-1} & \text{if } \lambda=n|n|\pi^2(1-a)^{-2},\end{cases}$$

the right-hand side of (6.2) takes the form

$$c\int_I\left|\sum_{\lambda\in\Lambda}a_\lambda e^{i\lambda t}\right|^2dt\,.$$

Next we obtain by a straightforward computation that the left-hand side of (6.2) is equal to

$$\sum|n|^{2-2\varepsilon}(|b_n|^2+|c_n|^2)$$

and that this sum is equivalent to

$$\sum_{\lambda\in\Lambda}|\lambda|^{-\varepsilon}|a_\lambda|^2.$$

Indeed, for $\lambda=n|n|\pi^2a^{-2}$ we have

$$|\lambda|^{-\varepsilon}|a_\lambda|^2=|n\pi a^{-1}|^{-2\varepsilon}|\pi nb_na^{-1}|^2\sim|n|^{2-2\varepsilon}|b_n|^2,$$

while for $\lambda=n|n|\pi^2(1-a)^{-2}$ we have

$$|\lambda|^{-\varepsilon}|a_\lambda|^2=|n\pi(1-a)^{-1}|^{-2\varepsilon}|\pi nb_n(1-a)^{-1}|^2\sim|n|^{2-2\varepsilon}|c_n|^2.$$

Hence the estimate (6.2) is equivalent to the following inequality:

$$\sum_{\lambda\in\Lambda}|\lambda|^{-\varepsilon}|a_\lambda|^2\le c\int_I\left|\sum_{\lambda\in\Lambda}a_\lambda e^{i\lambda t}\right|^2dt\,. \tag{6.3}$$

For the proof of (6.3) fix a bounded interval I and then fix a (sufficiently large) real number γ satisying

$$|I|>2\pi/\gamma\,.$$

Choose a sufficiently large positive integer N such that, setting

$$\Lambda_N := \{n|n|\pi^2 a^{-2},\ n|n|\pi^2(1-a)^{-2} : n \in \mathbb{Z} \text{ and } |n| \geq N\},$$

no interval of length $< 2\gamma$ contains more than two elements of Λ_N. Then, applying Theorem 2.1 we obtain the estimate

$$\text{(6.4)} \quad \sum_{|\lambda-\mu|<\gamma} |\lambda-\mu|^2(|a_\lambda|^2+|a_\mu|^2) + \sum_{\lambda\in\Lambda_N'} |a_\lambda|^2 \leq c\int_I \Big|\sum_{\lambda\in\Lambda_N} a_\lambda e^{i\lambda t}\Big|^2 dt$$

where the first sum is taken for all pairs of numbers in Λ_N whose distance is strictly between 0 and γ, while the second sum is taken for the remaining numbers in Λ_N.

We are going to deduce from (6.4) the inequality

$$\text{(6.5)} \quad \sum_{\lambda\in\Lambda_N} |\lambda|^{-\varepsilon} |a_\lambda|^2 \leq c\int_I \Big|\sum_{\lambda\in\Lambda_N} a_\lambda e^{i\lambda t}\Big|^2 dt .$$

(Compare to (6.3).) Since Λ_N has no finite accumulation points, for this it suffices to prove the estimate

$$\text{(6.6)} \quad |\lambda|^{-\varepsilon} \leq c|\lambda-\mu|^2$$

for all pairs in the first sum of (6.4). Moreover, it suffices to consider pairs with sufficiently large $|\lambda|$ and $|\mu|$. Now, for such a pair we have (exchanging λ and μ if needed)

$$\lambda = m|m|\pi^2 a^{-2} \quad \text{and } \mu = n|n|\pi^2(1-a)^{-2}$$

with suitable nonzero integers m, n of the same sign. Since $0 < a < 1/2$ by our choice at the beginning of the proof, we have

$$n+m \sim n-m \sim n \sim m$$

for $|\lambda| \to \infty$. Now let a be such that

$$\operatorname{dist}(qa, \mathbb{Z}) \geq c_\varepsilon q^{-1-\varepsilon}$$

for all $\varepsilon > 0$ and for all positive integers q. (We recall again from [2] that almost every a has this property.) Then we have

$$\text{(6.7)} \quad |\lambda-\mu| = \pi^2 a^{-2}(1-a)^{-2}\,|n^2a^2 - m^2(1-a)^2| =$$

$$\pi^2 a^{-2}(1-a)^{-2}\,|(n+m)a-m|\cdot|(n+m)a+m| .$$

Thanks to the choice of a we have

$$|(n+m)a-m| \geq c_\varepsilon\,|n+m|^{-1-\varepsilon} \geq c_\varepsilon'\,|n|^{-1-\varepsilon}$$

and

$$|(n+m)a+m| \geqslant c_\varepsilon\, |n-m|^{-1-\varepsilon} \geqslant c_\varepsilon'\, |n|^{-1-\varepsilon}.$$

Furthermore,

$$\big|\, |(n+m)a-m| - |(n+m)a+m|\, \big| \geqslant 2\,|m| \geqslant 2c|n|$$

for a suitable positive constant c, independent of m, n, and hence at least one of the numbers

$$(n+m)\,a-m \ \text{ and } \ (n+m)a+m$$

has an absolute value $\geqslant c|n|$. Therefore we have

$$|(n+m)a-m|\cdot|(n+m)a+m| \geqslant cc_\varepsilon'\, |n|^{-1-\varepsilon}\, |n| = cc_\varepsilon'\, |n|^{-\varepsilon}. \tag{6.8}$$

Using (6.8) we deduce from (6.7) the estimate

$$|\lambda-\mu| \geqslant cc_\varepsilon'\, \pi^2 a^{-2}(1-a)^{-2}\, |n|^{-\varepsilon}.$$

Since

$$|n| \sim |m| \sim |\lambda|^{1/2},$$

the desired estimate (6.6) follows.

We have thus proved (6.5). In other words, we have proved (6.3) for all (finite) sequences of complex numbers (a_λ) which satisfy the additional condition

$$a_\lambda = 0 \qquad \text{for all } \lambda \in \Lambda - \Lambda_N.$$

The proof of (a) is then completed by applying the

LEMMA 6.2. – *We are given a countable set Λ of real numbers without finite accumulation points, and for every $\lambda \in \Lambda$ two positive numbers $\alpha_\lambda < \beta_\lambda$. Assume that there exist a bounded interval I, a finite subset Λ_N of Λ and two positive constants c_1, c_2 such that*

$$c_1 \sum \alpha_\lambda\, |a_\lambda|^2 \leqslant \int_I \Big| \sum a_\lambda e^{i\lambda t} \Big|^2 dt \leqslant c_2 \sum \beta_\lambda\, |a_\lambda|^2$$

for all sequences of complex numbers a_λ where λ runs over some finite subset of $\Lambda - \Lambda_N$.
Then for every bounded interval J of length $> |I|$ there exist two positive constants c_3, c_4 such that

$$c_3 \sum \alpha_\lambda\, |a_\lambda|^2 \leqslant \int_J \Big| \sum a_\lambda e^{i\lambda t} \Big|^2 dt \leqslant c_4 \sum \beta_\lambda\, |a_\lambda|^2$$

for all sequences of complex numbers a_λ where λ runs over some finite subset of Λ.

In the special case where the numbers α_λ and β_λ do not depend on λ, this lemma was proved in [5]. His proof carries over easily to the proof of this general case. Alternatively, this lemma is a very particular case of Theorem 5.3 in [10] and of the more general Theorem 3.1 in [13].

PROOF OF PART (*b*). – If a is a rational number, then there exist positive integers m and n such that

$$\frac{m\pi}{a} = \frac{n\pi}{1-a} .$$

Denoting this common value by λ, the formula

$$u(x, t) := \begin{cases} \sin \lambda x \, e^{i\lambda^2 t} & \text{if } 0 < x < a , \\ -\sin\lambda(1-x)\, e^{i\lambda^2 t} & \text{if } a < x < 1 , \end{cases}$$

defines a nonzero solution of (6.1), so that the left-hand side of (6.1) is strictly positive. On the other hand, we have

$$u_x(a-0, t) = u_x(a+0, t) = 0$$

for all real t, so that the right-hand side of (6.2) vanishes for *every* bounded interval I. Hence (6.2) cannot hold.

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