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3-folds of general type with $K^3 = 4p_g - 14$


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Sunto. – In questo lavoro vengono costruite famiglie di 3-folds algebriche e non singolari X di tipo generale tali che l’invariante $K_X^3$ sia il minimo possibile rispetto al genere geometrico $p_g$, quando si suppone che il morfismo canonico sia birazionale. Per tali 3-folds vale la relazione lineare $K_X^3 = 4p_g - 14$, inoltre l’immagine del morfismo canonico è una varietà di Castelnuovo di $\mathbb{P}^{p_g-1}$.

1. – Introduction.

Let $X$ be a smooth minimal complex $n$-manifold such that $|K_X|$ has no base points and the canonical map $\phi_{K_X}: X \rightarrow F \subset \mathbb{P}^{p_g-1}$ is birational. It is well known from Castelnuovo that if $n = 2$ then $K_X^2 \geq 3p_g - 7$. The extremal case in which the equality holds is filled up by Castelnuovo surfaces (cf. [6]). In this article I find a lower bound for $K_X^n$ with respect to $p_g$ for any $n$. Such a bound is achieved by Castelnuovo varieties. The case $n = 3$ is studied in detail.

I use the standard notation of Algebraic Geometry.

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1.1. Canonical Castelnuovo varieties. Let $F$ be a complete irreducible and nondegenerate $n$-dimensional subvariety of $\mathbb{P}^N$, let $d$ be its degree and $p_g$ its geometric genus. Let $m$ and $\epsilon$ be integers such that

$$d - 1 = m(N - n) + \epsilon$$

where $\epsilon \in \{0, \ldots, N - n - 1\}$, then (cf. [5])

$$p_g \leq P(d, N, n) := \binom{m}{n+1} (N - n) + \binom{m}{n} \epsilon,$$

$P(d, N, n)$ is positive if and only if $d \geq n(N - n) + 2$. In this case, if $p_g = P(d, N, n)$, $F$ is said to be a Castelnuovo variety. It is known that:

– If $F$ is a Castelnuovo variety then such is its generic hyperplane section.
– If $F$ is a Castelnuovo variety then it is projectively normal.
It turns out that Castelnuovo varieties are divisors of $n + 1$-dimensional rational varieties of minimal degree. The class of linear equivalence of such divisors is also known (cf. [5], [3]).

We need some results concerning projective curves. We recall the following simple lemma (cf. [1], III, 2):

**Lemma 1.1.** – *If $p_1, \ldots, p_d$ is a set of points in $\mathbb{P}^{r-1}$ such that any $r$ among them are linearly indipendent then $p_1, \ldots, p_d$ impose at least $\min(d; k(r-1)+1)$ independent conditions on the hypersurfaces of degree $k$.*

We now expose a generalization of a classical theorem of the theory of curves due to Comessatti. The first proof of this theorem belongs to Jongmans (cf. [4]). Here we give a proof using Castelnuovo's method of hyperplane sections (see [1]).

**Theorem 1.2 (Comessatti).** – *Let $C$ be an irreducible curve of genus $g$ and $|L|$ be a special linear system of degree $d$ such that $\phi_L$ is a birational morphism. Let $a$ be the greatest integer less than or equal to $(2g-2)/d$. If $a \geq 1$, then*

$$\dim |L| = r \leq (d + a - 1)/(a + 1).$$

*If the equality holds than $\phi_L(C)$ is $a$-subcanonical, i.e. the canonical system of $\phi_L(C)$ is cut by the hypersurfaces of degree $a$. Moreover, $\phi_L(C)$ is projectively normal.*

**Proof.** – Let $\Gamma = \phi_L(C)$, let $h^0(\mathcal{O}_\Gamma(k)) \leq h^0(\mathcal{O}_{\mathbb{P}^N}(kL))$ be the Hilbert function of $\Gamma$, and $H$ a general hyperplane section. $H$ is a set of $d$ points in $\mathbb{P}^{r-1}$. We denote by $\Delta h^0(\mathcal{O}_Y(k))$ the difference $h^0(\mathcal{O}_Y(k)) - h^0(\mathcal{O}_Y(k-1))$ for a subvariety $Y$ of $\mathbb{P}^N$ (see [1]). Then $\Delta h^0(\mathcal{O}_\Gamma(k))$ is bounded below by $h^0(\mathcal{O}_H(k))$.

Therefore, applying Lemma 1.1 one has $\min \{d; k(r-1)+1\} \leq h^0(\mathcal{O}_H(k))$.

Thus

$$\Delta h^0(\mathcal{O}_\Gamma(k)) \geq \min \{d; k(r-1)+1\}$$

hence

(1.1) 

$$h^0(\mathcal{O}_\Gamma(a+1)) - 1 = \sum_{k=0}^{a+1} \Delta h^0(\mathcal{O}_\Gamma(k)) \geq \sum_{k=0}^{a+1} \min \{d; k(r-1)+1\}.$$

Let $m$ be such that $d - 1 = m(r-1) + \varepsilon$, where $\varepsilon \in \{0, \ldots, r-2\}$ then

(1.2) 

$$\min \{d; k(r-1)+1\} = k(r-1) + 1$$

for $k \leq m$. Comparing with Castelnuovo bound for the genus of a curve of de-
gred $d$ in $\mathbb{P}^r$, i.e. $g \leq P(d, r, 1) = m(m - 1)(r - 1)/2 + m\varepsilon$ we get

$$a \leq (2g - 2)/d \leq (2P(d, r, 1) - 2)/d =$$

$$= ((d - \varepsilon - r + 2\varepsilon)/2)m/d - 2/d = m(1 - (r - \varepsilon)/d) - 2/d$$

which implies $a < m$, since $1 < r - \varepsilon < r \leq d$. Hence $a + 1 \leq m$, therefore (1.2) holds for $k \leq a + 1$. Thus by (1.1) we obtain:

$$h^0(\mathcal{O}_C(a + 1)) - 1 \geq \sum_{k=0}^{a+1} k(r - 1) + 1 = \left(\frac{a + 2}{2}\right)(r - 1) + (a + 1).$$

Note that $|KL|$ is not special if $k \geq a + 1$, by definition of $a$. Then by Riemann-Roch $h^0(\mathcal{O}_C(a + 1)) - 1 = (a + 1) d - g$ and by (1.3) we get

$$(a + 1)(d - 1) - g \geq (a + 1)(a + 2)(r - 1)/2.$$  

Since $2g \geq ad + 2$ then:

$$(r - 1)(a + 1)(a + 2) \leq [2(a + 1)(d - 1) - ad - 2]$$

which is equivalent to

$$r \leq (d + a - 1)/(a + 1).$$

If the equality holds then it holds at each step, in particular one has that $2g = ad = +2$ and that the restriction map $\varphi_k: H^0(\mathcal{O}_C(k)) \to H^0(\mathcal{O}_C(k))$ is surjective for any $k \leq a + 1$. But since $|(k - 1)L|$ is not special if $k > a + 1$, $\varphi_k$ is surjective also for $k > a + 1$. Therefore $C$ is projectively normal.

If $C$ is not $a$-subcanonical, then $aL$ is not special, and the same computation of (1.3) can be done for $h^0(\mathcal{O}_C(a)) - 1$, obtaining for $r$ a value which is lower than the one allowed by the equality itself.

**Theorem 1.3.** Let $X$ be a $n$-fold whose canonical system $|K_X|$ is base point free and defines a birational morphism, then

$$K_X^n \geq (n + 1) p_g(X) - n^2 - 2n + 1.$$  

If equality holds the canonical image $F = \phi(X)$ is a Castelnuovo variety of $\mathbb{P}^{p_g - 1}$ with $m = n + 1$ and $\varepsilon = 1$. Moreover, $F$ is isomorphic to the canonical model of $X$.

**Proof.** Let $C$ be the intersection of $n - 1$ generic hypersurfaces in $|K_X|$, and let $|L| = |K_X|_C$. Then $K_C = nL$ and $g(C) = (n/2) K_C^n + 1$, thus $\phi_L(C)$ is $n$-subcanonical. By Theorem 1.2 we get $\dim |L| = r \leq (K_C^n + n - 1)/(n + 1)$. Moreover, restricting from $X$ to $C$ we find that $\dim |L| = h^0(C; \mathcal{O}_C(L)) - 1 \geq$
\[ p_g(X) - n , \text{ hence} \]
\[ K_X^n \geq (n + 1) p_g(X) - n^2 - 2n + 1 . \]

Consider the Castelnuovo bound \( P(d, N, n) \) for \( d = (n + 1)(N + 1) - n^2 - 2n + 1 \). Then one has
\[
(1.4) \quad \bar{d} - 1 = (n + 1)(N + 1) - n^2 - 2n = (n + 1)(N - n) + 1
\]
thus \( m = n + 1 \) and \( e = 3 \) and
\[ P(\bar{d}, N, n) = (N - n) + (n + 1) = N + 1 . \]

For \( N = p_g - 1 \) we get that \( F = \phi(X) \) is a Castelnuovo variety of degree \( \bar{d} = K_X^n = (n + 1) p_g(X) - n^2 - 2n + 1 \).

Moreover the projective normality of \( F \) implies that the hypersurfaces of degree \( n \) of \( \mathbb{P}^{p_g - 1} \) cut on \( F \) a complete system. Therefore the multiplication map
\[ \text{Sym}^n H^0(X; K_X) \to H^0(X; nK_X) \]
is surjective for every \( n \). This means that the canonical ring of \( X \) is generated in degree 1 and that \( F \) is isomorphic to the canonical model of \( X \).

By analogy with the case of surfaces we call the line \( K_X^2 = (n + 1) p_g(X) - n^2 - 2n + 1 \) of the \( \langle K_X^n, p_g(X) \rangle \)-plane the Castelnuovo line. It is easy to compare the result of this theorem with the classification of Castelnuovo varieties in [3], with the condition \( e = 1 \).

In the rest of the paper we treat the case \( n = 3 \). For \( n = 3 \) we have \( K_X^3 \geq 4 p_g(X) - 14 \).

We remark that a bound can also be obtained if we drop the base point freeness hypothesis. In this case the invariants \( h^1(X; \mathcal{O}_X) \) and \( h^2(X; \mathcal{O}_X) \) appear. The following lemma is well known (cf. [7]):

**Lemma 1.4.** – Let \( F \) be an irreducible nondegenerate \( n \)-subvariety of \( \mathbb{P}^N \), let \( c = N - n \), then
\[ q_0 := h^0(\mathbb{P}^N; \mathcal{O}(2)) \leq \binom{c + 2}{2} - \min \{ \deg F; 2c + 1 \} . \]

**Theorem 1.5.** – Let \( X \) be a minimal 3-fold whose canonical map is birational then
\[ K_X^3 + 6(h^1(X; \mathcal{O}_X) - h^2(X; \mathcal{O}_X)) \geq 4 p_g(X) - 14 . \]
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**Proof.** – Let $F = \phi_{K_X}(X) \subset \mathbb{P}^{p_g - 1}$ then $\deg F \leq K_X^3$, the canonical map being birational. By Lemma 1.4 we have

$$h^0(X; 2K_X) \geq \left( \frac{p_g + 1}{2} \right) - \left( \frac{c + 2}{2} \right) - \min(\deg F; 2c + 1)$$

(1.5)

where $c = p_g - 4$. Moreover, by Harris result in [5] one has

$$\deg F \geq 3(p_g(X) - 4) + 2 = 3p_g(X) - 10.$$ 

Thus, since $p_g \geq 5$, one has

$$\min\{\deg F; 2n + 1\} = \min\{\deg F; 2p_g(X) - 7\} = 2p_g(X) - 7.$$ 

Hence by (1.5) we have

$$h^0(X; 2K_X) \geq \left[ p_g(p_g + 1) - (p_g - 2)(p_g - 3) \right] / 2 + 2p_g - 7 = (6p_g - 6) / 2 + 2p_g - 7 = 5p_g - 10.$$ 

By Riemann–Roch, since $h^i(X; kK_X) = 0$ for $k > 0$ and $i = 1, 2$ (cf. [7], Theorem 5.5), one obtains

$$3(p_g - 1) + 3(h^1(X; \mathcal{O}_X) - h^2(X; \mathcal{O}_X)) + K_X^3 / 2 \geq 5p_g - 10$$

which is equivalent to the statement.

2. – 3-folds with $K_X^3 = 4p_g(X) - 14$.

The first case appearing on the Castelnuovo line $K_X^3 = 4p_g(X) - 14$ is the general hypersurface of degree 6 in $\mathbb{P}^4$. It has $p_g = 5$ and $K_X^3 = 6$. It’s easy to see, by looking at the Euler exact sequence restricted to $X$ and at the exact sequence of the normal bundle, that $h^2(X; \Theta_X) = h^3(X; \Theta_X) = 0$, so that an open subset of $\mathbb{P}H^9(\mathbb{P}^4; \mathcal{O}_{\mathbb{P}^1}(6))$ up to projectivities gives an open subset of the moduli space of $X$, which has dimension 185, and is clearly generically smooth.

If $p_g > 5$ then $F = \phi_{K_X}(X)$ is contained in a 4-dimensional scroll $W$. By Harris’ classification of Castelnuovo varieties (cf. [5]) we get:

i) $W$ is an irreducible quadric of $\mathbb{P}^5$, thus $p_g = 6$;

ii) $W$ is a cone over the Veronese surface with vertex a line, thus $p_g = 8$;

iii) $W$ is a rational normal scroll and $p_g \geq 8$.

2.1. The case $p_g = 6$ and $K_X^3 = 10$. Let us suppose that the quadric $W$ of $\mathbb{P}^5$ in which $F$ lies is nonsingular. Then $K_W = -4H|_W$ and from the adjunction formula we deduce that $F$ is cut on $W$ by a hypersurface of degree 5.
THEOREM 2.1. – The moduli space of 3-folds $X$ with $K_X^3 = 10$, $p_g = 6$ such that $|K_X|$ has no base points and the associated map $\phi_{K_X}: X \to F \subset \mathbb{P}^{p_g-1}$ is birational has a generically smooth component of dimension 180, whose generic point represents a 3-fold with canonical image a complete intersection of a nonsingular quadric and a quintic of $\mathbb{P}^5$.

PROOF. – Let $M(X)$ be the dimension of the family of the complete intersections of a quadric $W$ and a quintic of $\mathbb{P}^5$, up to isomorphisms. Then clearly

$$M(X) = [h^0(\mathbb{P}^5, \mathcal{O}(2)) - 1] + [h^0(W, \mathcal{O}(5)) - 1] - \dim(\text{PGL}(6)) =$$

$$[h^0(\mathbb{P}^5, \mathcal{O}(2)) - 1] + [h^0(\mathbb{P}^5, \mathcal{O}(5)) - h^0(\mathbb{P}^5, \mathcal{O}(3)) - 1] - 35 = 180.$$ 

The statement follows from being $h^2(\Theta_X) = 0$. This can be proved by standard computations involving the normal bundle sequence of $F \subset \mathbb{P}^5$, the Euler exact sequence and the restriction sequence of the tangent bundle; and recalling that $F$ is arithmetically Cohen-Macaulay.

The general $F$ specializes to a divisor of a singular quadric. More precisely, if $F_0$ is contained in a quadric $W$ of rank $q = 5$, $4$, $3$, then, by Theorem (2.6.1) in [3], point $(c_3)$, and point $(c_2, 1)$, $F_0$ is the complete intersection of $W$ and a quintic passing with multiplicity $m(W) \le q - 3$ through the vertex of $W$.

2.2. The Veronese case. We consider now the cone $W$ over the Veronese surface with vertex a line $r$. Let $\psi: \overline{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(2)) \to W$ be its desingularization and let $\pi$ be the projection of $\overline{W}$ on $\mathbb{P}^2$. Then $\text{Pic} \overline{W}$ is freely generated by $L \in |\pi^*(\mathcal{O}_{\mathbb{P}^5}(1))|$ and a section $H$ of the tautological bundle of $\overline{W}$. Moreover, $\psi$ is the morphism defined by $|H|$.

If $W$ contains a canonically embedded 3-fold $F$ with $p_g = 8$ and $K^3 = 18$ and $\overline{F}$ is its proper transform in $\overline{W}$, then the morphism $\phi_K$ lifts to a map $\overline{\phi}$ whose image is $\overline{F}$, such that $\phi_K = \psi \circ \phi_K$. Let $\overline{F} \sim aH + bL$, since $\deg \overline{F} = 18$ and $\psi$ is defined by the linear system $|H|$, then one has

$$18 = (aH + bL) \cdot H^3 = 4a + 2b, \quad b = 9 - 2a.$$ 

Moreover, since $K_{\overline{W}} \sim -3H - L$, one has $K_{\overline{F}} \sim (K_{\overline{W}} + \overline{F}) \sim (a+3)H + (8-2a) L$ and $K_F \sim H$. Thus $a = 4$ and $\overline{F} \in |4H + L|$. Hence, the expected linear class of $F$, if any $F \subset W$ canonically embedded exists, is $4H + L$. It remains to verify the existence of such 3-folds.

PROPOSITION 2.2. – The linear system $|4H + L|$ is not empty and base point free. Its generic element $\overline{F}$ is a minimal nonsingular 3-fold such that $p_g = 8$ and $K_F^3 = 18$. The morphism $\psi|_F: \overline{F} \to W$ for generic $\overline{F}$ is the canonical map.
PROOF. – The first part of the proposition is a consequence of the non emptiness and freeness of $|H|$ and $|L|$, thus it is possible to apply Bertini’ s theorem. Since $K_F = H|_F$, the canonical system is base point free and nef, i.e. $\bar{F}$ is minimal. Moreover one has: $K^3_F = H^3_F = H^3(4H + L) = 18$. Finally, the rationality of $\bar{W}$ implies that $h^3(\mathcal{O}_{\bar{W}}) = h^4(\mathcal{O}_{\bar{W}}) = 0$, thus from the exact sequence
\[
0 \to \mathcal{O}_{\bar{W}}(-4H - L) \to \mathcal{O}_{\bar{W}} \to \mathcal{O}_F \to 0
\]
we infer that
\[
h^3(\mathcal{O}_{\bar{F}}) = h^4(\bar{W}; \mathcal{O}_{\bar{W}}(-4H - L)) = h^0(\mathcal{O}_{\bar{W}}(H)) = h^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = 2 + h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 8.
\]
Note now that $h^1(\bar{W}; \mathcal{O}_{\bar{W}}(-3H - L)) = h^1(\bar{W}; \mathcal{O}(K_{\bar{W}})) = 0$, thus $|H|$ cut on $\bar{F}$ the complete canonical system. This shows that the restriction of $\psi$ to $\bar{F}$ is the canonical map. \[\blacksquare\]

We study now the number of moduli of the family of 3-folds $\bar{F}$.

**PROPOSITION 2.3.** – Let $\mathcal{N}$ be the moduli space of the minimal nonsingular 3-folds $X$ with $p_g = 8$ and $K^3 = 18$ such that the canonical map is a birational morphism on a divisor of the cone over a Veronese surface, then $\mathcal{N}$ has dimension $M(X) = 220$.

**PROOF.** – By what has been observed above, the linear class of the proper transform of the image $F$ of $X$ in $\bar{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ is $4H + L$. By projecting onto $\mathbb{P}^2$ we can compute
\[
h^0(\bar{W}, \mathcal{O}_{\bar{W}}(4H + L)) = h^0(\mathbb{P}^2, \pi_* \mathcal{O}_{\bar{W}}(4H) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = h^0(\mathbb{P}^2, \text{Sym}^4(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = 245.
\]
The group of the automorphism of $\bar{W}$ acts identifying the 3-folds in the family which are isomorphic. The dimension of the group can be computed from the following exact sequence
\[
1 \to \text{Aut}_{\mathbb{P}^2}(\bar{W}) \to \text{Aut}(\bar{W}) \to \text{Aut} \mathbb{P}^2 \to 1
\]
where $\text{Aut}_{\mathbb{P}^2}(\bar{W})$ is the subgroup of the automorphisms of $\bar{W}$ which fix the family of the fibers of $\pi: \bar{W} \to \mathbb{P}^2$. $\text{Aut}_{\mathbb{P}^2}(\bar{W})$ is formed by the $3 \times 3$ invertible
matrices with entries homogeneous polynomials as
\[
\begin{pmatrix}
[0] & [0] & [2] \\
[0] & [0] & [2] \\
0 & 0 & [0]
\end{pmatrix}
\]
where \([d]\) denotes a polynomial of degree \(d\), up to the \(C^*\)-action. Hence one has:
\[
\dim \text{Aut} \mathbb{P}^2 = \dim \mathbb{P}GL(\mathbb{C}, 3) = 8,
\]
\[
\dim \text{Aut}_{\mathbb{P}^2}(\mathbb{W}) = 5 + 2 \cdot 6 - 1 = 16.
\]
Thus:
\[
\dim \text{Aut}(\mathbb{W}) = \dim \text{Aut}_{\mathbb{P}^2}(\mathbb{W}) + \dim \mathbb{P}GL(\mathbb{C}, 3) = 16 + 8 = 24.
\]
The computation of the dimension of the coarse moduli space corresponding to the family is now easily obtained:
\[
M(X) = \dim |4H + L| - \dim \text{Aut}(\mathbb{W}) = 244 - 24 = 220.
\]

We want to compare \(M(X)\) with the dimension of the component \(\mathcal{M}\) of the moduli space of the 3-folds with \(K^3 = 18\) and \(p_g = 8\) to which our family \(\mathcal{N}\) belongs.

**Lemma 2.4.**
\[
\begin{align*}
\mathcal{H}^0(\Theta_{\mathbb{W}}) &= 24, \\
\mathcal{H}^i(\Theta_{\mathbb{W}}) &= 0 \quad \text{for} \quad i = 1, \ldots, 4.
\end{align*}
\]

**Proof.** Let \(\mathcal{E}\) be the sheaf \(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\). We can compute the cohomology of \(\Theta_{\mathbb{W}} / \mathbb{P}^2\) from the relative Euler exact sequence
\[
0 \rightarrow \mathcal{O}_{\mathbb{W}} \rightarrow \pi^{*}(\mathcal{E}^{*}(1)) \rightarrow \Theta_{\mathbb{W}} / \mathbb{P}^2 \rightarrow 0.
\]
We have
\[
\pi^{*}(\mathcal{E}^{*}(1)) = \pi^{*}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \mathcal{O}_{\mathbb{W}}(1).
\]
Since
\[
R^i\pi_{\#}(\pi^{*}(\mathcal{E}^{*}(1))) = R^i\pi_{\#}([\mathcal{O}_{\mathbb{W}}(1) \otimes \pi^{*}[\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)]]) =
\]
\[
R^i\pi_{\#}(\mathcal{O}_{\mathbb{W}}(1)) \otimes [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)]
\]
one has

\[ R^i \pi_* \pi^*(\mathcal{E}^*(1)) = 0 \quad \text{if } i > 0 \]

and

\[ R^0 \pi_* \pi^*(\mathcal{E}^*(1)) = [\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}] \oplus [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)]^{\oplus 2} = \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}. \]

Thus one has

(2.2) \hspace{1cm} h_i(\pi^*(\mathcal{E}^*(1))) = h_i(\pi_* \pi^*(\mathcal{E}^*(1))) = \begin{cases} 5 + 12 = 17 & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}

By (2.2) we find

(2.3) \begin{align*}
\begin{cases} h^0(\theta_{\mathbb{P}^2}) = 16, \\ h^i(\theta_{\mathbb{P}^2}) = 0 \quad \text{for } i = 1, \ldots, 4. \end{cases}
\end{align*}

We can now consider the exact sequence of the relative tangent bundle

(2.4) \hspace{1cm} 0 \to \theta_{\mathbb{P}^2} \to \theta \to \pi^* \pi_{\mathbb{P}^2} \to 0.

We note that \( h^i(\pi^* \theta_{\mathbb{P}^2}) = h^i(\theta_{\mathbb{P}^2}). \) In fact, from the Euler exact sequence for \( \mathbb{P}^2 \) we deduce

\[ R^i \pi_* \pi^*(\theta_{\mathbb{P}^2}) = \begin{cases} \theta_{\mathbb{P}^2} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases} \]

Thus from (2.3) and (2.4) it follows that

\[ h^i(\theta_{\mathbb{P}^2}) = 0 \quad \text{for } i = 1, \ldots, 4, \]

\[ h^0(\theta_{\mathbb{P}^2}) = \chi(\theta_{\mathbb{P}^2}) = \chi(\theta_{\mathbb{P}^2}) + \chi(\pi^*(\theta_{\mathbb{P}^2})) = h^0(\theta_{\mathbb{P}^2}) + \chi(\theta_{\mathbb{P}^2}) = 24. \]

**Theorem 2.5.** Let \( X \) be a minimal nonsingular 3-fold such that \( p_g = 8 \) and \( K_X^3 = 18. \) Suppose that the canonical map is a birational morphism on a divisor of the cone over a Veronese surface, then

\[ h^1(X; \theta_X) = 220, \quad h^2(X; \theta_X) = 2. \]

The local moduli space of \( X \) is smooth.
PROOF. – Let $\tilde{F}$ be the proper transform of the image $F$ of $X$ in $\tilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ by the canonical morphism, $\tilde{F}$ belongs to the linear system $|4H + L|$. We start from the sequence

$$0 \to \Theta_{\tilde{F}} \to \Theta_{\tilde{W}} |_{\tilde{F}} \to \mathcal{O}_{\tilde{F}}(\tilde{F}) \to 0.$$  

We know from the previous computation that $h^0(\mathcal{O}_{\tilde{W}}(\tilde{F})) = 245$, in the same way we can compute $h^i(\mathcal{O}_{\tilde{W}}(\tilde{F})) = 0$ for $i = 1, \ldots, 4$. From the sequence of restriction to $\tilde{F}$ we also find $h^i(\mathcal{O}_{\tilde{F}}(\tilde{F})) = 0$ for $i = 1, 2, 3$ and $h^0(\mathcal{O}_{\tilde{F}}(\tilde{F})) = 245 - 1 = 244$. Thus from (2.5) we deduce

$$h^i(\Theta_X) = h^i(\Theta_{\tilde{F}}) = h^i(\Theta_{\tilde{W}} |_{\tilde{F}}) \text{ for } i = 2, 3.$$  

To compute $h^i(\Theta_{\tilde{W}} |_{\tilde{F}})$ we combine the sequence

$$0 \to \Theta_{\tilde{W}}(-\tilde{F}) \to \Theta_{\tilde{W}} \to \Theta_{\tilde{W}} |_{\tilde{F}} \to 0$$

with Lemma 2.4. Thus we have

$$h^i(\Theta_{\tilde{W}} |_{\tilde{F}}) = h^{i+1}(\Theta_{\tilde{W}}(-\tilde{F})) \text{ for } i = 1, 2, 3$$

and

$$\chi(\Theta_{\tilde{W}} |_{\tilde{F}}) = \chi(\Theta_{\tilde{W}}) - \chi(\Theta_{\tilde{W}}(-\tilde{F})) = 24 - \chi(\Theta_{\tilde{W}}(-\tilde{F})).$$

To compute $\chi(\Theta_{\tilde{W}}(-\tilde{F}))$ we can tensor (2.4) and (2.1) with $\mathcal{O}(-\tilde{F})$

$$0 \to \Theta_{\tilde{W}/p^2}(-\tilde{F}) \to \Theta_{\tilde{W}}(-\tilde{F}) \to \pi^* \Theta_{p^2}(-\tilde{F}) \to 0,$$

$$0 \to \mathcal{O}_{\tilde{W}}(-\tilde{F}) \to \pi^*(\mathcal{O}(H - \tilde{F})) \to \Theta_{\tilde{W}/p^2}(-\tilde{F}) \to 0.$$

By Serre duality $H^j(\mathcal{O}_{\tilde{W}}(-\tilde{F}))$ is isomorphic to $H^{4-j}(\mathcal{O}_{\tilde{W}}(H))$. Moreover $H^i(\pi^*(\mathcal{O}(H - \tilde{F})))$ is isomorphic to $H^{4-i}(\pi^*(\mathcal{O}(K_{\tilde{W}} + F - H))) = H^{4-i}(\pi^*(\mathcal{O}(\varepsilon))).$ Since $i^j\pi_*\pi^*(\mathcal{O}(\varepsilon)) = 0$ for $i > 0$, one has $H^{4-i}(\pi^*(\mathcal{O}(\varepsilon))) = H^{4-i}(\mathcal{O}(\varepsilon))$. One also has $H^{4-i}(\mathcal{O}(\varepsilon)) = H^{4-i}(\mathcal{O}_{\tilde{W}}(H)) = H^{4-i}(\mathcal{O}(\varepsilon)).$ Thus the morphism $i$ is an isomorphism of cohomology. Then:

$$h^i(\Theta_{\tilde{W}/p^2}(-\tilde{F})) = 0 \text{ for } i = 0, \ldots, 4,$$

$$h^i(\Theta_{\tilde{W}}(-\tilde{F})) = h^i((\pi^* \Theta_{p^2})(-\tilde{F})) \text{ for } i = 0, \ldots, 4.$$

Thus, from (2.6), (2.8) and (2.9) we get

$$h^i(\Theta_X) = h^{i+1}((\pi^* \Theta_{p^2})(-\tilde{F})) \text{ for } i = 2, 3.$$
But then one has
\[(2.11) \quad h^i(\pi^* \Theta_{p^2}(-F)) = h^{4-i}((\pi^* \Theta_{p^2})^*(H)) = h^{4-i}((\pi^* \Omega_{p^2}^1)(H)) .\]
Since \(R^i \pi^* \pi^*(\Omega_{p^2}^1)(H) = 0\) for \(i > 0\), the Leray spectral sequence gives
\[(2.12) \quad h^i(\pi^* \Theta_{p^2}(-F)) = h^{4-i}(\mathbb{P}^2; \Omega_{p^2}^1 \otimes \pi^*(H)) =
\begin{align*}
2h^{4-i}(\mathbb{P}^2; \Omega_{p^2}^1) + h^{4-i}(\mathbb{P}^2; \Omega_{p^2}^1(2)) &= \begin{cases} 3 & \text{if } i = 4, \\ 2 & \text{if } i = 3, \\ 0 & \text{if } i = 0, 1, 2 . \end{cases}
\end{align*}\]
Finally from (2.10) and (2.12) we get \(h^3(\Theta_X) = h^4((\pi^* \Theta_{p^2})(-F)) = 3\) and \(h^2(\Theta_X) = h^3((\pi^* \Theta_{p^2})(-F)) = 2\), hence
\[
\chi(\Theta_X) = \chi(\Theta_{\bar{W}} | F) - \chi(\mathcal{O}_{\bar{W}}(\bar{F})) = (\chi(\Theta_{\bar{W}} - \chi(\Theta_{\bar{W}} (-\bar{F}))) - 244 =
(24 - \chi(\pi^* (\Theta_{p^2})(-\bar{F})) - 244 = (24 - (3 - 2)) - 244 = -221 .
\]
Moreover:
\[h^1(\Theta_X) = 221 + h^2(\Theta_X) - h^3(\Theta_X) = 221 - 1 = 220 .\]
The conclusion follows by comparing these numbers with the dimension of the family \(\mathcal{N}\) constructed above, which is parametrized by a smooth space.

**Remark 2.6.** We point out that the generic 3-fold \(X\) of the component of the moduli space here constructed contains a surface \(\Sigma\) such that \(\phi_{K_X}\) restricts on it to a morphism which is composed with a rational pencil of curves. In fact, \(\Sigma\) is the inverse image of the vertex of the cone over the Veronese. Such a line is a locus of compound Du Val singularities for \(\phi_{K_X}(X)\), and precisely, the generic hyperplane section meets the line in a point which is a double point of type \(A_1\) for the sectional surface of \(\phi_{K_X}(X)\).

Moreover, since the moduli space is nonsingular, any first order deformation of \(X\) deforms \(\Sigma\) to a surface which still contracts to a curve of double points by the canonical morphism.

**Remark 2.7.** It can be also observed that while the 3-folds of general type with birational canonical morphism into the cone over the Veronese belong, as has been shown, to a reduced component of their moduli space, the analogous 2-dimensional case gives a different situation. In this case the surfaces with \(p_g = p_a = 7\) and \(K^2 = 14\), which have canonical image in the cone with vertex a
point over the Veronese surface, are endowed with a \((-2)\)-curve and have a nonreduced moduli space (cf. [6]).

2.3. The rational normal scroll. Let \(X\) be a Castelnuovo 3-fold of general type with \(K_X^3 = 4p_g - 14\) and \(p_g \geq 6\). Let us suppose that the image of the canonical map lies in a rational normal scroll \(W\) of type \((a_1, a_2, a_3, a_4)\). \(W\) is the image of the morphism \(\psi\) defined by the tautological bundle \(H\) of the 4-fold

\[
\overline{W} := \mathbb{P}(\mathcal{O}_P(1) \oplus \mathcal{O}_P(2) \oplus \mathcal{O}_P(3) \oplus \mathcal{O}_P(4))
\]

where \(0 \leq a_1 \leq a_2 \leq a_3 \leq a_4\) and \(a_1 + a_2 + a_3 + a_4 = p_g - 4\). If \(a_1 > 0\) then \(\psi\) is an immersion of \(\overline{W}\) in the projective space, and \(W\) is nonsingular; if \(a_1 = 0\) then the morphism \(\psi\) contracts a subvariety of \(\overline{W}\), and \(W\) is a cone. In this case the group of the Weil divisors of \(W\) is isomorphic to the one of \(\overline{W}\). We will denote by the same symbols \(H\) and \(L\) the generators of both the groups.

**Lemma 2.8.** – Let \(X\) be a 3-fold of general type, with base point free canonical system \(\vert K_X \vert\), \(K_X^3 = 4p_g - 14\) and such that the canonical morphism is birational. Let us suppose that the canonical image \(F\) lies in a rational normal scroll \(W\). Then \(F\) is a Weil divisor which is linearly equivalent to \(5H - (p_g - 6)L\).

If \(W\) is singular with a \(\mathbb{P}^i\) of singular points, then \(F\) passes through \(\mathbb{P}^i\) with multiplicity \(m(W) \leq 2 - i\).

Vice versa, let \(\overline{F}\) be a divisor of class \(5H - (p_g - 6)L\) in \(\overline{W} = \mathbb{P}(\mathcal{O}_P(1) \oplus \mathcal{O}_P(2) \oplus \mathcal{O}_P(3) \oplus \mathcal{O}_P(4))\), where \(a_1 + a_2 + a_3 + a_4 = p_g - 4\). If \(\overline{F}\) has only canonical singularities then \(p_g(\overline{F}) = p_g\), \(K_{\overline{F}}^3 = 4p_g - 14\) and \(\psi\) restricts on \(\overline{F}\) to the canonical morphism.

**Proof.** – We apply the classification Theorem (2.6.1) in [3]. For \(n = 3\), \(N = p_g - 1\) and \(d = K_X^3 = 4p_g - 14\), if \(d - 1 = m(N - n) + e\), we have \(m = 4\) and \(e = 1\). Hence \(F\) is of class \(5H - (p_g - 6)L\), and the multiplicity \(m(W)\) is as in the statement.

Vice versa, we recall that \(K_{\overline{W}} \sim -4H + (p_g(X) - 6)L\); then by adjunction formula, \(H\) cuts on the proper transform \(\overline{F}\) of \(F\) the canonical system. Since \(h^1(\mathcal{O}_W(H - \overline{F})) = h^1(\mathcal{O}(K_W)) = 0\), the map \(H^0(\mathcal{O}_W(H)) \to H^0(\mathcal{O}_F(H))\) is surjective, and the hyperplanes cut on \(\overline{F}\) the complete canonical system. Let \(X\) be its desingularization, then its easy to verify by using the intersection formulas on \(\overline{W}\) that \(K_{\overline{W}}^3 = K_{\overline{F}}^3 = \overline{F}H^3 = 4p_g - 14\). \(\blacksquare\)

We now study the existence of such Castelnuovo varieties, by the previous lemma, it is sufficient to work on \(\overline{W}\).
Lemma 2.9. – Let \( p_g > 6 \) and
\[
\overline{W} = \mathbb{P}(\mathcal{O}_1(a_1) \oplus \mathcal{O}_2(a_2) \oplus \mathcal{O}_3(a_3) \oplus \mathcal{O}_4(a_4))
\]
where \( 0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \) and \( a_1 + a_2 + a_3 + a_4 = p_g - 4 \). If \(|5H - (p_g - 6)L|\) is not empty and its generic element is irreducible then
\[
5a_4 \leq 4p_g - 14 \quad \text{and} \quad a_3 > 0.
\]

Proof. – Let \( \overline{W}(a_1, a_2, a_3) \in |H - a_4L| \), let \( \overline{F} \in |5H - (p_g - 6)L| \) then \( \overline{F} \) does not contains \( \overline{W}(a_1, a_2, a_3) \) as component since it is irreducible. Hence \( H^2 \overline{W}(a_1, a_2, a_3) \geq 0 \), thus we have \( H^2(5H - (p_g - 6)L)(H - a_4L) \geq 0 \), i.e. \( 5(p_g - 4) - (p_g - 6) - 5a_4 \geq 0 \). If \( \psi(\overline{W}) \) is a cone of vertex a plane that is, \( \overline{W} \) is of type \((0, 0, 0, a_4)\) and \( a_4 = p_g - 4 \), then we have \( 5a_4 = 5(p_g - 4) \leq 4p_g - 14 \), hence \( p_g \leq 6 \).

Theorem 2.10. – The generic element \( \overline{F} \) in \(|5H - (p_g(X) - 6)L|\) has canonical singularities at most if and only if the following conditions are satisfied:

i) \( 5a_3 \geq p_g - 6 \),
ii) \( 3a_1 + 2a_4 \geq p_g - 6 \),
iii) if \( 5a_2 < p_g - 6 \) then \( 4a_1 + a_4 \geq p_g - 6 \).

Proof. – Let \( \overline{F} \) be in \(|5H - (p_g(X) - 6)L|\). Let \( x_i \) be sections of \( \mathcal{O}(H - a_iL) \) for \( i = 1, \ldots, 4 \) such that their restriction to each fiber of \( \pi \) gives a system of homogeneous coordinates. Let \([t_0; t_1]\) be homogeneous coordinates over \( \mathbb{P}^1 \). The equation of \( \overline{F} \) is of the following type:

\[
\overline{F} = \sum_{\substack{i, j, k \geq 0 \\small{i + j + k \leq 5}}} g_{ijk} x_1^{5-i-j-k} x_2^i x_3^j x_4^k
\]

where \( g_{ijk} \) are homogeneous polynomials of degree \((5 - i - j - k) a_1 + ia_2 + ja_3 + ka_4 - (p_g - 6)\) in the coordinates \([t_0; t_1]\) on \( \mathbb{P}^1 \). We recall that \( a_1 \leq a_2 \leq a_3 \leq a_4 \). Let \( l = i + j + k \), then one has:

\[
\deg g_{000} = (5 - l) a_1 + la_2 - (p_g - 6) \leq \deg g_{ijk} \leq \\
\deg g_{00l} = (5 - l) a_1 + la_4 - (p_g - 6).
\]

Let us suppose that \( \overline{F} \) has canonical singularities at most. If \( 5a_3 < p_g - 6 \), then

\[
\deg g_{ij0} = (5 - i - j) a_1 + ia_2 + ja_3 - (p_g - 6) \leq 5a_3 - (p_g - 6) < 0
\]

thus \( \{x_1 = 0\} \) would be a component of \( \overline{F} \). This proves the condition (i).
The condition (iii) is related to the surface $S$ of $\overline{W}$ defined by \( \{x_3 = x_4 = 0\} \): $S$ is contained in $\overline{F}$ if and only if $\deg g_{00} < 0$, hence if $(5 - l)a_1 + la_2 < (p_g - 6)$, for every $l \leq 5$, that is if $5a_2 < (p_g - 6)$. In this case $S$ is contained in the base locus of $|5H - (p_g(X) - 6)L|$.

Since $\overline{F}$ has at most canonical singularities, it does not contain any double surface, thus if $S$ is contained in the base locus of $|5H - (p_g(X) - 6)L|$, we have to exclude that every 3-fold of $|5H - (p_g(X) - 6)L|$ has $S$ as double surface. By computing the directional derivatives of $\overline{F}$ along $x_i$, for $i = 1, \ldots, 4$, and along $t_0$, $t_1$ it can be seen that they vanish on $S$ if $4a_1 + a_4 < p_g - 6$. In fact, since $g_{00} = 0$ if $l = 0, \ldots, 5$, then $\partial\overline{F}/\partial t_0$ and $\partial\overline{F}/\partial t_1$, $\partial\overline{F}/\partial x_1$ and $\partial\overline{F}/\partial x_2$ vanish on $S = \{x_3 = x_4 = 0\}$. One has $\partial\overline{F}/\partial x_3 |_{S} = \sum_{0 \leq i \leq 4} g_{i0} x_1^i x_2^i$ and $\partial\overline{F}/\partial x_4 |_{S} = \sum_{0 \leq i \leq 4} g_{i0} x_1^i x_2^i$, therefore they vanish on $S$ if $4a_1 + a_4 < p_g - 6$.

In the same way, we obtain condition (ii) imposing that the curve $C$ of equations $\{x_2 = x_3 = x_4 = 0\}$ is not a curve of triple points for the generic $\overline{F}$.

Now, let us suppose that (i), (ii), (iii), are verified, we distinguish two cases:

A) $5a_1 \geq (p_g - 6)$,

B) $5a_1 < (p_g - 6)$.

A) The system $|5H - (p_g(X) - 6)L|$ is base point free. In fact, the system can be described as sum of $|5(H - a_1 L)|$ and $|(p_g(X) - 6 + 5a_1)L|$. These are base point free systems, moreover $|5(H - a_1 L)|$ is not composed with a pencil. Thus we can apply Bertini theorem.

B) $g_{000}$ is zero, again we distinguish two cases:

$B_1)$ $5a_2 \geq (p_g - 6)$,

$B_2)$ $5a_2 < (p_g - 6)$.

It is sufficient to study the system $|5H - (p_g(X) - 6)L|$ around its base locus.

If $B_1)$ holds then $g_{000} \neq 0$, it’s easy to see by looking at the general equation (2.13) that the base locus is the curve $C = \{x_2 = x_3 = x_4 = 0\}$. Let $p$ be any point of $C$, we can suppose that the choice of $x_i$ is such that in $p$ one has $x_1 = 1$ and $x_2 = x_3 = x_4 = 0$.

If $4a_1 + a_4 \geq (p_g - 6)$, then $p$ is a simple point for $\overline{F}$, the tangent plane of a $\overline{F}$ in $p$ having equation

$$g_{100}(p) x_2 + g_{010}(p) x_3 + g_{001}(p) x_4 = 0 .$$

In other words, since $g_{000}$ is zero no term in the $[t_0; t_1]$ variables appears, thus if $g_{ijk}$ are generic the equation of the tangent plane is not identically zero.
If $4a_1 + a_4 < (p_g - 6)$, then $p$ is a double point for $F$, the tangent cone having equation
\[
\left[ \sum_{i+j+k=2} \max \{i; j; k\} g_{ijk}(p) x_i^j x_j^k \right] = 0.
\]
Such an equation is not identically zero by (ii), if $g_{ijk}$ are generic. It defines a quadric tangent cone.

If $B_2)$ holds then $g_{ijk} = 0$ if $j = k = 0$, hence the base locus of the system $|5H - (p_g - 6) L|$ is the surface $S = \{x_3 = x_4 = 0\}$. Let $p$ be any point of $S$, we can suppose that $x_1(p) = 1$ and $x_2(p) = x_3(p) = x_4(p) = 0$. Proceeding as in the previous case we find that $p$ is a simple point for $F$, the tangent plane to $F$ in $p$ having equation
\[
g_{100}(p) x_2 + g_{010}(p) x_3 + g_{001}(p) x_4 = 0.
\]
This equation is not identically zero by (iii) if $g_{100}$, $g_{010}$, and $g_{001}$ are generic.

In any case the singular locus of $F$ is of canonical type. 

2.4. Examples. If $p_g = 5$ and $K_X^3 = 6$ then the image of the canonical morphism of a Castelnuovo 3-fold with such invariants is a nonsingular hypersurface $F$ of $P^4$ of degree 6. The number of moduli is
\[
M(X) = h^0(P^4, \mathcal{O}(6)) - \dim (PGL(5)) - 1 = 210 - 25 = 185.
\]
It is easy to see, by using the normal bundle and the Euler sequences, that $h^2(\Theta_X) = 0$, thus the moduli space is generically smooth.

If $p_g = 6$ and $K_X^3 = 10$ then we find Castelnuovo 3-folds as divisors of quadrics of $P^5$, presented in section 2.1.

If $p_g = 7$ and $K_X^3 = 14$ the canonical image of the Castelnuovo 3-fold $X$ lies in a singular scroll. In fact the conditions pointed out in Theorem 2.10 and in Lemma 2.9 give as possible type for the scroll containing the canonical image $(0, 1, 1, 1)$ and $(0, 0, 1, 2)$.

A more interesting example is $p_g = 8$ and $K^3 = 18$. Apart from the case of the cone over the Veronese, the canonical image $F$ may be a divisor of the scroll $W(a_1, a_2, a_3, a_4)$ of $P^7$ of the following types:

(a) $(1, 1, 1, 1)$,  
(b) $(0, 1, 1, 2)$,  
(c) $(0, 0, 2, 2)$,  
(d) $(0, 0, 1, 3)$,  
(e) $(0, 0, 0, 4)$.  

The case (e) is excluded by the Lemma 2.9. We have the following result:

**Lemma 2.11.**

\[ h^0(\overline{W}, \mathcal{O}_{\overline{W}}(5H - 2L)) = \begin{cases} 
224 & \text{case (a),} \\
225 & \text{case (b),} \\
230 & \text{cases (c) and (d);} 
\end{cases} \]

\[ \dim \text{Aut} \overline{W} = \begin{cases} 
18 & \text{case (a),} \\
19 & \text{case (b),} \\
22 & \text{case (c),} \\
23 & \text{case (d).} 
\end{cases} \]

**Proof.** – The first part of the statement follows from

\[ h^0(\overline{W}, \mathcal{O}_{\overline{W}}(5H - 2L)) = h^0(\mathbb{P}^1; \pi_* (\mathcal{O}_{\overline{W}}(5H) \otimes \mathcal{O}_{\mathbb{P}^1}(-2))) =
\]

\[ h^0(\mathbb{P}^1; \text{Sym}^5(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4))) \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) \]

where \( \pi : \overline{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4)) \rightarrow \mathbb{P}^1 \). The second part of the statement follows from the exact sequence

\[ 1 \rightarrow \text{Aut}_{\mathbb{P}^1}(\overline{W}) \rightarrow \text{Aut}(\overline{W}) \rightarrow \text{Aut} \mathbb{P}^1 \rightarrow 1 \]

where \( \text{Aut}_{\mathbb{P}^1}(\overline{W}) \) is the group of the automorphisms of \( \overline{W} \) which fix the fibers of the projection \( \pi \), thus it is the projectivized group of the invertible \( 4 \times 4 \) matrices \( A = (P_{j, k}) \) with entries homogeneous polynomials \( P_{j, k} \) on \( \mathbb{P}^1 \) such that

\[ \deg(P_{j, k}) = a_h - a_j \text{ if } a_h - a_j \geq 0, \ P_{j, k} = 0 \text{ otherwise.} \]

For each type \( \alpha = (a_1, a_2, a_3, a_4) \), we obtain the number of moduli \( M(X)_\alpha \) by the formula

\[ M(X)_\alpha = \dim |5H - 2L| - \dim \text{Aut}(W) = h^0(W; 5H - 2L) - 1 - \dim \text{Aut}(W). \]

Thus:

\[ \begin{align*}
(a) & \quad \text{type (a) = (1, 1, 1, 1),} \quad M(X)_a = 223 - 18 = 205, \\
(b) & \quad \text{type (b) = (0, 1, 1, 2),} \quad M(X)_b = 223 - 19 = 204, \\
(c) & \quad \text{type (c) = (0, 0, 2, 2),} \quad M(X)_c = 229 - 22 = 207, \\
(d) & \quad \text{type (d) = (0, 0, 1, 3),} \quad M(X)_d = 229 - 23 = 206. 
\end{align*} \]

We recall that for \( \mu = a_1 + \ldots + a_4 = 4 \) the most general scroll is of type (a), be-
ing a type (b)-scroll a degeneration of a type (a); a type (c) a degeneration of a type (b); and a type (d) a degeneration of a type (c) (cf. [5]). Since \( M(X) > M(X) \), the generic 3-fold \( X \) which has canonical image in a scroll of type (c) cannot be a degeneration of a 3-fold which has canonical image in a scroll of type (b). Thus we have at least three components of the moduli space of the 3-folds with \( K^3_X = 18 \) and \( p_g(X) = 8 \) (counting the one relative to the Veronese cone). We point out that the same situation appears in the 2-dimensional case (cf. [2]).

More generally we have the following lemma

**Lemma 2.12.** Let \( a_1 + a_2 + a_3 + a_4 = \mu \), let \( W \) be the image of \( \overline{W}(a_1; a_2; a_3; a_4) \) in \( \mathbb{P}^{\mu + 3} \), then

\[
h^0(\mathcal{O}_W(5H - (\mu - 2)L)) - h^1(\mathcal{O}_W(5H - (\mu - 2)L)) =
\begin{cases}
14\mu + 168 & \text{if } W \text{ is non singular}, \\
15\mu + 165 & \text{if } W \text{ is a cone of vertex a point}, \\
20\mu + 150 & \text{if } W \text{ is a cone of vertex a line}.
\end{cases}
\]

**Proof.** Consider the exact sequence

\[
0 \to \mathcal{O}_W(5H - (\mu - 2)L) \to \mathcal{O}_W(5H) \to \mathcal{O}_{(\mu - 2)L}(5H) \to 0.
\]

The Hilbert function does not change by varying the type of the scrolls so that one can reduce himself to make computations only on an easy case. We compute \( h^0(\overline{W}; \mathcal{O}_W(5H)) \) on \( W' = W(0; 0; 0; \mu) \), by projecting on \( \mathbb{P}^1 \). One has

\[
h^0(W' , \mathcal{O}_{W'}(5H)) = h^0(\mathbb{P}^1; [\text{Sym}^5(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(\mu))]) =
\]

\[
h^0(\mathbb{P}^1; \bigoplus_{k=0}^{5} \mathcal{O}_{P^1}((5 - k)\mu) \oplus (\mathcal{O}_{P^1})^{\oplus(k + 2)(k + 1)/2}) =
\]

\[
\sum_{k=0}^{5} (k + 2)(k + 1)((5 - k)\mu + 1)/2 = 56 + 70\mu.
\]

Clearly if \( W \) is nonsingular, \( h^0(\mathcal{O}_{(\mu - 2)L}(5H)) = (\mu - 2) h^0(\mathcal{O}_{P^3}(5)) = (\mu - 2) 56 \). If \( W \) has a zero-dimensional vertex \( p \), then \( h^0(\mathcal{O}_{(\mu - 2)L}(5H)) = h^0(\mathcal{O}_{P^3}(5)) + (\mu - 3) h^0(\mathcal{O}_{P^3}; \mathcal{O}_p(5)) = 55\mu - 109 \). If \( W \) has a 1-dimensional vertex \( l \) then \( h^0(\mathcal{O}_{(\mu - 2)L}(5H)) = h^0(\mathcal{O}_{P^3}(5)) + (\mu - 3) h^0(\mathcal{O}_{P^3}; \mathcal{O}_l(5)) = 50\mu - 94 \). The conclusion follows from (2.16), being \( h^1(W; \mathcal{O}_W(5H)) = 0 \).}

It can be easily verified by projecting on \( \mathbb{P}^1 \) that \( h^1(\mathcal{O}_W(5H - (\mu - 2)L)) = 0 \) if \( 5a_1 > \mu - 4 \), since in this case one has a sum of line bundles on \( \mathbb{P}^1 \) of degree at least \( 5a_1 - \mu + 2 \). Analogously, \( h^1(\mathcal{O}_W(5H - (\mu - 2)L)) \geq h^1(\mathcal{O}_{P^1}(-\mu + 2)) = \mu - 3 \) if \( 0 = a_1 < a_2 \) and \( h^1(\mathcal{O}_W(5H - (\mu - 2)L)) \geq \).
6h^1(\mathcal{O}_{\mathbb{P}^1}(\mu - 2)) = 6(\mu - 3) if 0 = a_1 = a_2. We can conclude with the following theorem:

**Theorem 2.13.** – Let \( W_0 \subset \mathbb{P}^{\mu + 3} \) be a rational normal scroll of type \((a, a, a, a), (a, a, a, a + 1), (a, a, a + 1, a + 1), (a, a + 1, a + 1, a + 1), \) where \( \mu \equiv 0, \ldots, 3 \) modulo 4 respectively. Then there exists a family of isomorphism classes of nonsingular 3-folds \( X \) with \( K_X^3 = 4\mu + 2 \) and \( p_g = \mu + 4 \) whose canonical morphism is birational to a divisor of \( W_0 \), which is its canonical model. The linear class of such divisors is \( 5H - (\mu - 2)L \). Such a family is unirational of dimension \( M = 14\mu + 149 \).

**Proof.** – For a fixed \( \mu \), for any type of the scroll \( W \) verifying the conditions of Theorem 2.10 there exists a family of canonical Castelnuovo 3-folds with \( K^3 = 4\mu + 2 \) and \( p_g = \mu + 4 \) parametrized by an open set of \( \mathbb{P}^H_0(\mathcal{O}_{\mathcal{W}}(5H - (\mu - 2)L)) \). We then quotient the family by the group of the automorphisms of \( W \). We find a unirational family, represented by a subvariety of the moduli space. The computation of the dimension follows from

\[
M = h^0(\mathcal{W}_0, \mathcal{O}_{\mathcal{W}_0}(5H - (\mu - 2)L)) - \dim \text{Aut}(W_0),
\]

from Lemma 2.12 and from the exact sequence

\[
1 \to \text{Aut}_{\mathbb{P}^1}(\mathcal{W}) \to \text{Aut}(\mathcal{W}) \to \text{Aut}\mathbb{P}^1 \to 1
\]

where in the 4 cases one always gets \( \dim \text{Aut}_{\mathbb{P}^1}(\mathcal{W}) = 15 \).

Note that in the 4 cases above \( \dim \text{Aut}_{\mathbb{P}^1}(\mathcal{W}) \) is minimal in the family of scrolls \( \mathcal{W} \) with degree \( \mu \), so that \( \dim \text{Aut}(\mathcal{W}) \) is also minimal.

Moreover, \( h^1(\mathcal{O}_{\mathcal{W}}(5H - (\mu - 2)L)) = 0 \) since \( 5a_1 > \mu - 4 \), as it has been noticed above, thus by Lemma 2.12 \( \dim \mathbb{P}^H_0(\mathcal{O}_{\mathcal{W}}(5H - (\mu - 2)L)) \) assumes the minimal value for a scroll of fixed degree \( \mu \). But for families of scrolls of special type \( \dim \mathbb{P}^H_0(\mathcal{O}_{\mathcal{W}}(5H - (\mu - 2)L)) \) increases, and even if also \( \dim \text{Aut}_{\mathbb{P}^1}(\mathcal{W}) \) increases, one can find components of the moduli space of 3-folds \( X \) with higher dimension.

**References**

3-FOLDS OF GENERAL TYPE WITH $K^3 = 4p_g - 14$


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