Bollettino
Unione Matematica Italiana

Paola Supino

3-folds of general type with $K^3 = 4p_g - 14$

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **2-B** (1999), n.1, p. 169–187.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1999_8_2B_1_169_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1999.

Bollettino U. M. I.
(8) 2-B (1999), 169-187

3-Folds of General Type with $K^3 = 4p_g - 14$.

PAOLA SUPINO

Sunto. – *In questo lavoro vengono costruite famiglie di* 3*-folds algebriche e non singolari X di tipo generale tali che l' invariante* K_X^3 *sia il minimo possibile rispetto al genere geometrico* p_g*, quando si suppone che il morfismo canonico sia birazionale. Per tali* 3*-folds vale la relazione lineare* $K_X^3 = 4p_g - 14$*, inoltre l'immagine del morfismo canonico é una varietà di Castelnuovo di* $\mathbb{P}^{p_g - 1}$.

1. – Introduction.

Let X be a smooth minimal complex n-manifold such that $|K_X|$ has no base points and the canonical map $\phi_{K_X}\colon X \to F \subset \mathbb{P}^{p_g - 1}$ is birational. It is well known from Castelnuovo that if $n = 2$ then $K_X^2 \geq 3p_g - 7$. The extremal case in which the equality holds is filled up by Castelnuovo surfaces (cf. [6]). In this article I find a lower bound for K_X^n with respect to p_g for any n. Such a bound is achieved by Castelnuovo varieties. The case $n = 3$ is studied in detail.

I use the standard notation of Algebraic Geometry.

The results contained in this paper are part of the author's thesis, written under the direction of Professor C. Ciliberto. The author would like to express thanks to Professor C. Ciliberto for helpful conversations, good advise and careful reading.

1.1. *Canonical Castelnuovo varieties.* Let F be a complete irreducible and nondegenerate n-dimensional subvariety of $\mathbb{P}^N$, let d be its degree and p_g its geometric genus. Let m and ε be integers such that

$$d - 1 = m(N - n) + \varepsilon$$

where $\varepsilon \in \{0, \ldots, N - n - 1\}$, then (cf. [5])

$$p_g \leq P(d, N, n) := \binom{m}{n+1}(N - n) + \binom{m}{n}\varepsilon,$$

$P(d, N, n)$ is positive if and only if $d \geq n(N - n) + 2$. In this case, if $p_g = P(d, N, n)$, F is said to be a *Castelnuovo variety*. It is known that:

– If F is a Castelnuovo variety then such is its generic hyperplane section.

– If F is a Castelnuovo variety then it is projectively normal.

It turns out that Castelnuovo varieties are divisors of $n+1$-dimensional rational varieties of minimal degree. The class of linear equivalence of such divisors is also known (cf. [5], [3]).

We need some results concerning projective curves. We recall the following simple lemma (cf. [1], III, 2):

LEMMA 1.1. – *If $p_1, \ldots, p_d$ is a set of points in $\mathbb{P}^{r-1}$ such that any r among them are linearly indipendent then $p_1, \ldots, p_d$ impose at least* $\min(d;\, k(r-1)+1)$ *independent conditions on the hypersurfaces of degree k.*

We now expose a generalization of a classical theorem of the theory of curves due to Comessatti. The first proof of this theorem belongs to Jongmans (cf. [4]). Here we give a proof using Castelnuovo's method of hyperplane sections (see [1]).

THEOREM 1.2 (Comessatti). – *Let C be an irreducible curve of genus g and $|L|$ be a special linear system of degree d such that ϕ_L is a birational morphism. Let a be the greatest integer less than or equal to $(2g-2)/d$. If $a \geqslant 1$, then*

$$\dim|L| = r \leqslant (d+a-1)/(a+1)\,.$$

*If the equality holds than $\phi_L(C)$ is a-*subcanonical, *i.e. the canonical system of $\phi_L(C)$ is cut by the hypersurfaces of degree a. Moreover, $\phi_L(C)$ is projectively normal.*

PROOF. – Let $\Gamma = \phi_L(C)$, let $h^0(\mathcal{O}_\Gamma(k)) \leqslant h^0(\mathcal{O}_C(kL))$ be the Hilbert function of Γ, and H a general hyperplane section. H is a set of d points in $\mathbb{P}^{r-1}$. We denote by $\Delta h^0(\mathcal{O}_Y(k))$ the difference $h^0(\mathcal{O}_Y(k)) - h^0(\mathcal{O}_Y(k-1))$ for a subvariety Y of $\mathbb{P}^N$ (see [1]). Then $\Delta h^0(\mathcal{O}_\Gamma(k))$ is bounded below by $h^0(\mathcal{O}_H(k))$. Therefore, applying Lemma 1.1 one has $\min\{d;\, k(r-1)+1\} \leqslant h^0(\mathcal{O}_H(k))$. Thus

$$\Delta h^0(\mathcal{O}_\Gamma(k)) \geqslant \min\{d;\, k(r-1)+1\}$$

hence

$$h^0(\mathcal{O}_\Gamma(a+1)) - 1 = \sum_{k=0}^{a+1} \Delta h^0(\mathcal{O}_\Gamma(k)) \geqslant \sum_{k=0}^{a+1} \min\{d;\, k(r-1)+1\}\,. \tag{1.1}$$

Let m be such that $d-1 = m(r-1)+\varepsilon$, where $\varepsilon \in \{0, \ldots, r-2\}$ then

$$\min\{d;\, k(r-1)+1\} = k(r-1)+1 \tag{1.2}$$

for $k \leqslant m$. Comparing with Castelnuovo bound for the genus of a curve of degree d in $\mathbb{P}^r$, i.e. $g \leqslant P(d, r, 1) = m(m-1)(r-1)/2 + m\varepsilon$ we get

$$a \leqslant (2g-2)/d \leqslant (2P(d, r, 1) - 2)/d =$$

$$= ((d - \varepsilon - r + 2\varepsilon)/2)\, m/d - 2/d = m(1 - (r - \varepsilon)/d) - 2/d$$

which implies $a < m$, since $1 < r - \varepsilon < r \leqslant d$. Hence $a + 1 \leqslant m$, therefore (1.2) holds for $k \leqslant a + 1$. Thus by (1.1) we obtain:

$$\text{(1.3)} \quad h^0(\mathcal{O}_\Gamma(a+1)) - 1 \geqslant \sum_{k=0}^{a+1} k(r-1) + 1 = \binom{a+2}{2}(r-1) + (a+1).$$

Note that $|kL|$ is not special if $k \geqslant a+1$, by definition of a. Then by Riemann-Roch $h^0(\mathcal{O}_\Gamma(a+1)) - 1 = (a+1)\,d - g$ and by (1.3) we get

$$(a+1)(d-1) - g \geqslant (a+1)(a+2)(r-1)/2\,.$$

Since $2g \geqslant ad + 2$ then:

$$(r-1)(a+1)(a+2) \leqslant [2(a+1)(d-1) - ad - 2]$$

which is equivalent to

$$r \leqslant (d + a - 1)/(a+1)\,.$$

If the equality holds then it holds at each step, in particular one has that $2g = ad = +2$ and that the restriction map ϱ_k: $H^0(\mathcal{O}_C(k)) \to H^0((\mathcal{O}_\Gamma(k))$ is surjective for any $k \leqslant a+1$. But since $|(k-1)L|$ is not special if $k > a+1$, ϱ_k is surjective also for $k > a+1$. Therefore C is projectively normal.

If C is not a-subcanonical, then aL is not special, and the same computation of (1.3) can be done for $h^0(\mathcal{O}_\Gamma(a)) - 1$, obtaining for r a value which is lower than the one allowed by the equality itself. ∎

THEOREM 1.3. – *Let X be a n-fold whose canonical system $|K_X|$ is base point free and defines a birational morphism, then*

$$K_X^n \geqslant (n+1)\,p_g(X) - n^2 - 2n + 1\,.$$

If equality holds the canonical image $F = \phi(X)$ is a Castelnuovo variety of $\mathbb{P}^{p_g - 1}$ with $m = n+1$ and $\varepsilon = 1$. Moreover, F is isomorphic to the canonical model of X.

PROOF. – Let C be the intersection of $n-1$ generic hypersurfaces in $|K_X|$, and let $|L| = |K_X|_C$. Then $K_C = nL$ and $g(C) = (n/2)\,K_X^n + 1$, thus $\phi_L(C)$ is n-subcanonical. By Theorem 1.2 we get $\dim|L| = r \leqslant (K_X^n + n - 1)/(n+1)$. Moreover, restricting from X to C we find that $\dim|L| = h^0(C; \mathcal{O}_C(L)) - 1 \geqslant$

$p_g(X) - n$, hence

$$K_X^n \geqslant (n+1)\, p_g(X) - n^2 - 2n + 1\,.$$

Consider the Castelnuovo bound $P(d, N, n)$ for $\overline{d} = (n+1)(N+1) - n^2 - 2n + 1$. Then one has

$$\overline{d} - 1 = (n+1)(N+1) - n^2 - 2n = (n+1)(N-n) + 1 \tag{1.4}$$

thus $m = n+1$ and $\varepsilon = 3$ and

$$P(\overline{d}, N, n) = (N-n) + (n+1) = N+1\,.$$

For $N = p_g - 1$ we get that $F = \phi(X)$ is a Castelnuovo variety of degree $\overline{d} = K_X^n = (n+1)\, p_g(X) - n^2 - 2n + 1$.

Moreover the projective normality of F implies that the hypersurfaces of degree n of $\mathbb{P}^{p_g - 1}$ cut on F a complete system. Therefore the multiplication map

$$\mathrm{Sym}^n\, H^0(X; K_X) \to H^0(X; nK_X)$$

is surjective for every n. This means that the canonical ring of X is generated in degree 1 and that F is isomorphic to the canonical model of X. ■

By analogy with the case of surfaces we call the line $K_X^n = (n+1)\, p_g(X) - n^2 - 2n + 1$ of the $\langle K_X^n, p_g(X)\rangle$-plane the *Castelnuovo line*. It is easy to compare the result of this theorem with the classification of Castelnuovo varieties in [3], with the condition $\varepsilon = 1$.

In the rest of the paper we treat the case $n = 3$. For $n = 3$ we have $K_X^3 \geqslant 4p_g(X) - 14$.

We remark that a bound can also be obtained if we drop the base point freeness hypothesis. In this case the invariants $h^1(X; \mathcal{O}_X)$ and $h^2(X; \mathcal{O}_X)$ appear. The following lemma is well known (cf. [7]):

LEMMA 1.4. – *Let F be an irreducible nondegenerate n-subvariety of $\mathbb{P}^N$, let $c = N - n$, then*

$$\varrho_0 := h^0(\mathbb{P}^N; \mathcal{I}_F(2)) \leqslant \binom{c+2}{2} - \min\{\deg F; 2c+1\}\,.$$

THEOREM 1.5. – *Let X be a minimal 3-fold whose canonical map is birational then*

$$K_X^3 + 6(h^1(X; \mathcal{O}_X) - h^2(X; \mathcal{O}_X)) \geqslant 4p_g(X) - 14\,.$$

PROOF. – Let $F = \phi_{K_X}(X) \subset \mathbb{P}^{p_g - 1}$ then $\deg F \leqslant K_X^3$, the canonical map being birational. By Lemma 1.4 we have

$$h^0(X; 2K_X) \geqslant \binom{p_g + 1}{2} - \left[\binom{c+2}{2} - \min(\deg F; 2c+1)\right] \tag{1.5}$$

where $c = p_g - 4$. Moreover, by Harris result in [5] one has

$$\deg F \geqslant 3(p_g(X) - 4) + 2 = 3p_g(X) - 10 .$$

Thus, since $p_g \geqslant 5$, one has

$$\min\{\deg F; 2n+1\} = \min\{\deg F; 2p_g(X) - 7\} = 2p_g(X) - 7 .$$

Hence by (1.5) we have

$$h^0(X; 2K_X) \geqslant [p_g(p_g+1) - (p_g - 2)(p_g - 3)]/2 + 2p_g - 7 =$$

$$= (6p_g - 6)/2 + 2p_g - 7 = 5p_g - 10 .$$

By Riemann–Roch, since $h^i(X; kK_X) = 0$ for $k > 0$ and $i = 1, 2$ (cf. [7], Theorem 5.5), one obtains $h^0(X; 2K_X) = \chi(2K_X) = -3\chi(\mathcal{O}_X) + K_X^3/2$, thus

$$3(p_g - 1) + 3(h^1(X; \mathcal{O}_X) - h^2(X; \mathcal{O}_X)) + K_X^3/2 \geqslant 5p_g - 10$$

which is equivalent to the statement. ■

2. – 3-folds with $K_X^3 = 4p_g(X) - 14$.

The first case appearing on the Castelnuovo line $K_X^3 = 4p_g(X) - 14$ is the general hypersurface of degree 6 in $\mathbb{P}^4$. It has $p_g = 5$ and $K_X^3 = 6$. It's easy to see, by looking at the Euler exact sequence restricted to X and at the exact sequence of the normal bundle, that $h^2(X; \Theta_X) = h^3(X; \Theta_X) = 0$, so that an open subset of $\mathbb{P}H^0(\mathbb{P}^4; \mathcal{O}_{\mathbb{P}^4}(6))$ up to projectivities gives an open subset of the moduli space of X, which has dimension 185, and is clearly generically smooth.

If $p_g > 5$ then $F = \phi_{K_X}(X)$ is contained in a 4-dimensional scroll W. By Harris' classification of Castelnuovo varieties (cf. [5]) we get:

i) W is an irreducible quadric of $\mathbb{P}^5$, thus $p_g = 6$;

ii) W is a cone over the Veronese surface with vertex a line, thus $p_g = 8$;

iii) W is a rational normal scroll and $p_g \geqslant 8$.

2.1. *The case $p_g = 6$ and $K_X^3 = 10$.* Let us suppose that the quadric W of $\mathbb{P}^5$ in which F lies is nonsingular. Then $K_W = -4H|_W$ and from the adjunction formula we deduce that F is cut on W by a hypersurface of degree 5.

THEOREM 2.1. – *The moduli space of* 3*-folds* X *with* $K_X^3 = 10$, $p_g = 6$ *such that* $|K_X|$ *has no base points and the associated map* $\phi_{K_X}: X \to F \subset \mathbb{P}^{p_g - 1}$ *is birational has a generically smooth component of dimension* 180, *whose generic point represents a* 3*-fold with canonical image a complete intersection of a nonsingular quadric and a quintic of* $\mathbb{P}^5$.

PROOF. – Let $M(X)$ be the dimension of the family of the complete intersections of a quadric W and a quintic of $\mathbb{P}^5$, up to isomorphisms. Then clearly

$$M(X) = [h^0(\mathbb{P}^5, \mathcal{O}(2)) - 1] + [h^0(W, \mathcal{O}(5)) - 1] - \dim(\mathbb{P}GL(6)) =$$

$$[h^0(\mathbb{P}^5, \mathcal{O}(2)) - 1] + [h^0(\mathbb{P}^5, \mathcal{O}(5)) - h^0(\mathbb{P}^5, \mathcal{O}(3)) - 1] - 35 = 180 .$$

The statement follows from being $h^2(\Theta_X) = 0$. This can be proved by standard computations involving the normal bundle sequence of $F \subset \mathbb{P}^5$, the Euler exact sequence and the restriction sequence of the tangent bundle; and recalling that F is arithmetically Cohen-Macaulay. ∎

The general F specializes to a divisor of a singular quadric. More precisely, if F_0 is contained in a quadric W of rank $\varrho = 5, 4, 3$, then, by Theorem (2.6.1) in [3], point (c_3), and point $(c_{2,1})$, F_0 is the complete intersection of W and a quintic passing with multiplicity $m(W) \leq \varrho - 3$ through the vertex of W.

2.2. *The Veronese case.* We consider now the cone W over the Veronese surface with vertex a line r. Let $\psi: \widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \to W$ be its desingularization and let π be the projection of $\widetilde{W}$ on $\mathbb{P}^2$. Then Pic $\widetilde{W}$ is freely generated by $L \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and a section H of the tautological bundle of $\widetilde{W}$. Moreover, ψ is the morphism defined by $|H|$.

If W contains a canonically embedded 3-fold F with $p_g = 8$ and $K^3 = 18$ and $\widetilde{F}$ is its proper transform in $\widetilde{W}$, then the morphism ϕ_K lifts to a map $\widetilde{\phi}$ whose image is $\widetilde{F}$, such that $\phi_K = \psi \circ \phi_K$. Let $\widetilde{F} \sim aH + bL$, since deg $\widetilde{F} = 18$ and ψ is defined by the linear system $|H|$, then one has

$$18 = (aH + bL)\, H^3 = 4a + 2b\,, \qquad b = 9 - 2a\,.$$

Moreover, since $K_{\widetilde{W}} \sim -3H - L$, one has $K_{\widetilde{F}} \sim (K_{\widetilde{W}} + \widetilde{F})_{\widetilde{F}} \sim (a-3)\,H + (8 - 2a)\,L$ and $K_{\widetilde{F}} \sim H$. Thus $a = 4$ and $\widetilde{F} \in |4H + L|$. Hence, the expected linear class of F, if any $F \subset W$ canonically embedded exists, is $4H + L$. It remains to verify the existence of such 3-folds.

PROPOSITION 2.2. – *The linear system* $|4H + L|$ *is not empty and base point free. Its generic element* $\widetilde{F}$ *is a minimal nonsingular* 3*-fold such that* $p_g = 8$ *and* $K_{\widetilde{F}}^3 = 18$. *The morphism* $\psi|_{\widetilde{F}}: \widetilde{F} \to W$ *for generic* $\widetilde{F}$ *is the canonical map.*

PROOF. – The first part of the proposition is a consequence of the non emptiness and freeness of $|H|$ and $|L|$, thus it is possible to apply Bertini' s theorem. Since $K_{\widetilde{F}} = H|_{\widetilde{F}}$, the canonical system is base point free and nef, i.e. $\widetilde{F}$ is minimal. Moreover one has: $K^3_{\widetilde{F}} = H^3_{\widetilde{F}} = H^3(4H+L) = 18$. Finally, the rationality of $\widetilde{W}$ implies that $h^3(\mathcal{O}_{\widetilde{W}}) = h^4(\mathcal{O}_{\widetilde{W}}) = 0$, thus from the exact sequence

$$0 \to \mathcal{O}_{\widetilde{W}}(-4H-L) \to \mathcal{O}_{\widetilde{W}} \to \mathcal{O}_{\widetilde{F}} \to 0$$

we infer that

$$h^3(\mathcal{O}_{\widetilde{F}}) = h^4(\widetilde{W}; \mathcal{O}_{\widetilde{W}}(-4H-L)) = h^0(\mathcal{O}_{\widetilde{W}}(H)) =$$

$$h^0(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) = 2 + h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 8\,.$$

Note now that $h^1(\widetilde{W}; \mathcal{O}_{\widetilde{W}}(-3H-L)) = h^1(\widetilde{W}; \mathcal{O}(K_{\widetilde{W}})) = 0$, thus $|H|$ cut on $\widetilde{F}$ the complete canonical system. This shows that the restriction of ψ to $\widetilde{F}$ is the canonical map. ■

We study now the number of moduli of the family of 3-folds $\widetilde{F}$.

PROPOSITION 2.3. – *Let $\mathcal{N}$ be the moduli space of the minimal nonsingular 3-folds X with $p_g = 8$ and $K^3_X = 18$ such that the canonical map is a birational morphism on a divisor of the cone over a Veronese surface, then $\mathcal{N}$ has dimension $M(X) = 220$.*

PROOF. – By what has been observed above, the linear class of the proper transform of the image F of X in $\widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ is $4H + L$. By projecting onto $\mathbb{P}^2$ we can compute

$$h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(4H+L)) = h^0(\mathbb{P}^2, \pi_* \mathcal{O}_{\widetilde{W}}(4H) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) =$$

$$h^0(\mathbb{P}^2, \mathrm{Sym}^4(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)) \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = 245\,.$$

The group of the automorphism of $\widetilde{W}$ acts identifying the 3-folds in the family which are isomorphic. The dimension of the group can be computed from the following exact sequence

$$1 \to \mathrm{Aut}_{\mathbb{P}^2}(\widetilde{W}) \to \mathrm{Aut}(\widetilde{W}) \to \mathrm{Aut}\,\mathbb{P}^2 \to 1$$

where $\mathrm{Aut}_{\mathbb{P}^2}(\widetilde{W})$ is the subgroup of the automorphisms of $\widetilde{W}$ which fix the family of the fibers of $\pi\colon \widetilde{W} \to \mathbb{P}^2$. $\mathrm{Aut}_{\mathbb{P}^2}(\widetilde{W})$ is formed by the 3×3 invertible matrices with entries homogeneous polynomials as

$$\begin{pmatrix} [0] & [0] & [2] \\ [0] & [0] & [2] \\ 0 & 0 & [0] \end{pmatrix}$$

where $[d]$ denotes a polynomial of degree d, up to the $\boldsymbol{C}^*$-action. Hence one has:

$$\dim \operatorname{Aut} \mathbb{P}^2 = \dim \mathbb{P}GL(\boldsymbol{C}, 3) = 8 ,$$

$$\dim \operatorname{Aut}_{\mathbb{P}^2}(\widetilde{W}) = 5 + 2 \cdot 6 - 1 = 16 .$$

Thus:

$$\dim \operatorname{Aut}(\widetilde{W}) = \dim \operatorname{Aut}_{\mathbb{P}^2}(\widetilde{W}) + \dim \mathbb{P}GL(\boldsymbol{C}, 3) = 16 + 8 = 24 .$$

The computation of the dimension of the coarse moduli space corresponding to the family is now easily obtained:

$$M(X) = \dim |4H + L| - \dim \operatorname{Aut}(\widetilde{W}) = 244 - 24 = 220 . \qquad \blacksquare$$

We want to compare $M(X)$ with the dimension of the component $\mathcal{M}$ of the moduli space of the 3-folds with $K^3 = 18$ and $p_g = 8$ to which our family $\mathcal{N}$ belongs.

LEMMA 2.4.

$$h^0(\Theta_{\widetilde{W}}) = 24 ,$$
$$h^i(\Theta_{\widetilde{W}}) = 0 \qquad \textit{for } i = 1, \ldots, 4 .$$

PROOF. – Let $\mathcal{E}$ be the sheaf $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)$. We can compute the cohomology of $\Theta_{\widetilde{W}/\mathbb{P}^2}$ from the relative Euler exact sequence

$$0 \to \mathcal{O}_{\widetilde{W}} \to \pi^* \mathcal{E}^*(1) \to \Theta_{\widetilde{W}/\mathbb{P}^2} \to 0 . \tag{2.1}$$

We have

$$\pi^* \mathcal{E}^*(1) = \pi^*(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)) \otimes \mathcal{O}_{\widetilde{W}}(1).$$

Since

$$R^i \pi_* \pi^*(\mathcal{E}^*(1)) = R^i \pi_* \{\mathcal{O}_{\widetilde{W}}(1) \otimes \pi^*[\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)]\} =$$
$$R^i \pi_*(\mathcal{O}_{\widetilde{W}}(1)) \otimes [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)]$$

one has

$$R^i \pi_* \pi^*(\mathcal{E}^*(1)) = 0 \qquad \text{if } i > 0$$

and

$$R^0 \pi_* \pi^*(\mathcal{E}^*(1)) = [\mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}] \oplus [\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)]^{\oplus 2} =$$

$$\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2} .$$

Thus one has

(2.2) $$h^i(\pi^*(\mathcal{E}^*(1))) = h^i(\pi_* \pi^*(\mathcal{E}^*(1))) =$$

$$h^i(\mathcal{O}_{\mathbb{P}^2}(-2)^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^2}^{\otimes 5} \otimes \mathcal{O}_{\mathbb{P}^2}(2)^{\otimes 2}) = \begin{cases} 5 + 12 = 17 & \text{if } i = 0 , \\ 0 & \text{if } i > 0 . \end{cases}$$

By (2.2) we find

(2.3) $$\begin{cases} h^0(\Theta_{\widetilde{W}/\mathbb{P}^2}) = 16 , \\ h^i(\Theta_{\widetilde{W}/\mathbb{P}^2}) = 0 \qquad \text{for } i = 1, \ldots, 4 . \end{cases}$$

We can now consider the exact sequence of the relative tangent bundle

(2.4) $$0 \to \Theta_{\widetilde{W}/\mathbb{P}^2} \to \Theta_{\widetilde{W}} \to \pi^* \Theta_{\mathbb{P}^2} \to 0 .$$

We note that $h^i(\pi^* \Theta_{\mathbb{P}^2}) = h^i(\Theta_{\mathbb{P}^2})$. In fact from the Euler exact sequence for $\mathbb{P}^2$ we deduce

$$R^i \pi_* \pi^*(\Theta_{\mathbb{P}^2}) = \begin{cases} \Theta_{\mathbb{P}^2} & \text{if } i = 0 , \\ 0 & \text{if } i > 0 . \end{cases}$$

Thus from (2.3) and (2.4) it follows that

$$h^i(\Theta_{\widetilde{W}}) = 0 \qquad \text{for } i = 1, \ldots, 4 ,$$

$$h^0(\Theta_{\widetilde{W}}) = \chi(\Theta_{\widetilde{W}}) = \chi(\Theta_{\widetilde{W}/\mathbb{P}^2}) + \chi(\pi^*(\Theta_{\mathbb{P}^2})) = h^0(\Theta_{\widetilde{W}/\mathbb{P}^2}) + \chi(\Theta_{\mathbb{P}^2}) = 24 . \qquad \blacksquare$$

THEOREM 2.5. – *Let X be a minimal nonsingular 3-fold such that $p_g = 8$ and $K_X^3 = 18$. Suppose that the canonical map is a birational morphism on a divisor of the cone over a Veronese surface, then*

$$h^1(X;\, \Theta_X) = 220 , \qquad h^2(X;\, \Theta_X) = 2 .$$

The local moduli space of X is smooth.

PROOF. – Let $\widetilde{F}$ be the proper transform of the image F of X in $\widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ by the canonical morphism, $\widetilde{F}$ belongs to the linear system $|4H + L|$. We start from the sequence

$$0 \to \Theta_{\widetilde{F}} \to \Theta_{\widetilde{W}}|_{\widetilde{F}} \to \mathcal{O}_{\widetilde{F}}(\widetilde{F}) \to 0\,. \tag{2.5}$$

We know from the previous computation that $h^0(\mathcal{O}_{\widetilde{W}}(\widetilde{F})) = 245$, in the same way we can compute $h^i(\mathcal{O}_{\widetilde{W}}(\widetilde{F})) = 0$ for $i = 1, \ldots, 4$. From the sequence of restriction to $\widetilde{F}$ we also find $h^i(\mathcal{O}_{\widetilde{F}}(\widetilde{F})) = 0$ for $i = 1, 2, 3$ and $h^0(\mathcal{O}_{\widetilde{F}}(\widetilde{F})) = 245 - 1 = 244$. Thus from (2.5) we deduce

$$h^i(\Theta_X) = h^i(\Theta_{\widetilde{F}}) = h^i(\Theta_{\widetilde{W}}|_{\widetilde{F}}) \quad \text{for } i = 2, 3\,. \tag{2.6}$$

To compute $h^i(\Theta_{\widetilde{W}}|_{\widetilde{F}})$ we combine the sequence

$$0 \to \Theta_{\widetilde{W}}(-\widetilde{F}) \to \Theta_{\widetilde{W}} \to \Theta_{\widetilde{W}}|_{\widetilde{F}} \to 0 \tag{2.7}$$

with Lemma 2.4. Thus we have

$$h^i(\Theta_{\widetilde{W}}|_{\widetilde{F}}) = h^{i+1}(\Theta_{\widetilde{W}}(-\widetilde{F})) \quad \text{for } i = 1, 2, 3 \tag{2.8}$$

and

$$\chi(\Theta_{\widetilde{W}}|_{\widetilde{F}}) = \chi(\Theta_{\widetilde{W}}) - \chi(\Theta_{\widetilde{W}}(-\widetilde{F})) = 24 - \chi(\Theta_{\widetilde{W}}(-\widetilde{F}))\,.$$

To compute $\chi(\Theta_{\widetilde{W}}(-\widetilde{F}))$ we can tensor (2.4) and (2.1) with $\mathcal{O}(-\widetilde{F})$

$$0 \to \Theta_{\widetilde{W}/\mathbb{P}^2}(-\widetilde{F}) \to \Theta_{\widetilde{W}}(-\widetilde{F}) \to \pi^*\Theta_{\mathbb{P}^2}(-\widetilde{F}) \to 0\,,$$

$$0 \to \mathcal{O}_{\widetilde{W}}(-\widetilde{F}) \xrightarrow{i} \pi^*(\mathcal{E}^*(H - \widetilde{F})) \to \Theta_{\widetilde{W}/\mathbb{P}^2}(-\widetilde{F}) \to 0\,.$$

By Serre duality $H^i(\mathcal{O}_{\widetilde{W}}(-\widetilde{F}))$ is isomorphic to $H^{4-i}(\mathcal{O}_{\widetilde{W}}(H))$. Moreover $H^i(\pi^*(\mathcal{E}^*(H - \widetilde{F})))$ is isomorphic to $H^{4-i}(\pi^*(\mathcal{E}) \otimes \mathcal{O}_{\widetilde{W}}(K_{\widetilde{W}} + F - H)) = H^{4-i}(\pi^*(\mathcal{E}))$. Since $\mathcal{R}^i\pi_*\pi^*(\mathcal{E}) = 0$ for $i > 0$, one has $H^{4-i}(\pi^*(\mathcal{E})) \cong H^{4-i}(\pi_*\pi^*(\mathcal{E})) = H^{4-i}((\mathcal{E}))$. One also has $H^{4-i}(\mathcal{O}_{\widetilde{W}}(H)) \cong H^{4-i}(\pi_*\mathcal{O}_{\widetilde{W}}(H)) = H^{4-i}((\mathcal{E}))$. Thus the morphism i is an isomorphism of cohomology. Then:

$$\begin{cases} h^i(\Theta_{\widetilde{W}/\mathbb{P}^2}(-\widetilde{F})) = 0 & \text{for } i = 0, \ldots, 4\,, \\ h^i(\Theta_{\widetilde{W}}(-\widetilde{F})) = h^i((\pi^*\Theta_{\mathbb{P}^2})(-\widetilde{F})) & \text{for } i = 0, \ldots, 4\,. \end{cases} \tag{2.9}$$

Thus, from (2.6), (2.8) and (2.9) we get

$$h^i(\Theta_X) = h^{i+1}((\pi^*\Theta_{\mathbb{P}^2})(-\widetilde{F})) \quad \text{for } i = 2, 3\,. \tag{2.10}$$

But then one has

$$(2.11)\qquad h^i(\pi^*\Theta_{\mathbb{P}^2}(-\widetilde{F})) = h^{4-i}((\pi^*\Theta_{\mathbb{P}^2})^*(H)) = h^{4-i}((\pi^*\Omega^1_{\mathbb{P}^2})(H))\,.$$

Since $R^i\pi_*\pi^*(\Omega^1_{\mathbb{P}^2})(H) = 0$ for $i > 0$, the Leray spectral sequence gives

$$(2.12)\qquad h^i(\pi^*\Theta_{\mathbb{P}^2}(-\widetilde{F})) = h^{4-i}(\mathbb{P}^2;\,\Omega^1_{\mathbb{P}^2}\otimes\pi_*(H)) =$$

$$h^{4-i}(\mathbb{P}^2;\,\Omega^1_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}(2)) =$$

$$2h^{4-i}(\mathbb{P}^2;\,\Omega^1_{\mathbb{P}^2}) + h^{4-i}(\mathbb{P}^2;\,\Omega^1_{\mathbb{P}^2}(2)) = \begin{cases} 3 & \text{if } i = 4\,,\\ 2 & \text{if } i = 3\,,\\ 0 & \text{if } i = 0, 1, 2\,.\end{cases}$$

Finally from (2.10) and (2.12) we get $h^3(\Theta_X) = h^4((\pi^*\Theta_{\mathbb{P}^2})(-\widetilde{F})) = 3$ and $h^2(\Theta_X) = h^3((\pi^*\Theta_{\mathbb{P}^2})(-\widetilde{F})) = 2$, hence

$$\chi(\Theta_X) = \chi(\Theta_{\widetilde{W}}\,|_{\widetilde{F}}) - \chi(\mathcal{O}_{\widetilde{F}}(\widetilde{F})) = \big(\chi(\Theta_{\widetilde{W}} - \chi(\Theta_{\widetilde{W}}(-\widetilde{F})) - 244 =$$

$$\big(24 - \chi(\pi^*(\Theta_{\mathbb{P}^2})(-\widetilde{F})) - 244 = (24 - (3-2)) - 244 = -221\,.$$

Moreover:

$$h^1(\Theta_X) = 221 + h^2(\Theta_X) - h^3(\Theta_X) = 221 - 1 = 220\,.$$

The conclusion follows by comparing these numbers with the dimension of the family $\mathcal{N}$ constructed above, which is parametrized by a smooth space. ■

REMARK 2.6. – We point out that the generic 3-fold X of the component of the moduli space here constructed contains a surface Σ such that ϕ_{K_X} restricts on it to a morphism which is composed with a rational pencil of curves. In fact, Σ is the inverse image of the vertex of the cone over the Veronese. Such a line is a locus of compound Du Val singularities for $\phi_{K_X}(X)$, and precisely, the generic hyperplane section meets the line in a point which is a double point of type A_1 for the sectional surface of $\phi_{K_X}(X)$.

Moreover, since the moduli space is nonsingular, any first order deformation of X deforms Σ to a surface which still contracts to a curve of double points by the canonical morphism.

REMARK 2.7. – It can be also observed that while the 3-folds of general type with birational canonical morphism into the cone over the Veronese belong, as has been shown, to a reduced component of their moduli space, the analogous 2-dimensional case gives a different situation. In this case the surfaces with $p_g = p_a = 7$ and $K^2 = 14$, which have canonical image in the cone with vertex a point over the Veronese surface, are endowed with a (-2)-curve and have a nonreduced moduli space (cf. [6]).

2.3. *The rational normal scroll.* Let X be a Castelnuovo 3-fold of general type with $K_X^3 = 4p_g - 14$ and $p_g \geqslant 6$. Let us suppose that the image of the canonical map lies in a rational normal scroll W of type (a_1, a_2, a_3, a_4). W is the image of the morphism ψ defined by the tautological bundle H of the 4-fold

$$\widetilde{W} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4))$$

where $0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $a_1 + a_2 + a_3 + a_4 = p_g - 4$. If $a_1 > 0$ then ψ is an immersion of $\widetilde{W}$ in the projective space, and W is nonsingular; if $a_1 = 0$ then the morphism ψ contracts a subvariety of $\widetilde{W}$, and W is a cone. In this case the group of the Weil divisors of W is isomorphic to the one of $\widetilde{W}$. We will denote by the same symbols H and L the generators of both the groups.

LEMMA 2.8. – *Let X be a 3-fold of general type, with base point free canonical system $|K_X|$, $K_X^3 = 4p_g - 14$ and such that the canonical morphism is birational. Let us suppose that the canonical image F lies in a rational normal scroll W. Then F is a Weil divisor which is linearly equivalent to $5H - (p_g - 6)L$.*

If W is singular with a $\mathbb{P}^i$ of singular points, then F passes through $\mathbb{P}^i$ with multiplicity $m(W) \leqslant 2 - i$.

Vice versa, let $\widetilde{F}$ be a divisor of class $5H - (p_g - 6)L$ in $\widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4))$, where $a_1 + a_2 + a_3 + a_4 = p_g - 4$. If $\widetilde{F}$ has only canonical singularities then $p_g(\widetilde{F}) = p_g$, $K_{\widetilde{F}}^3 = 4p_g - 14$ and ψ restricts on $\widetilde{F}$ to the canonical morphism.

PROOF. – We apply the classification Theorem (2.6.1) in [3]. For $n = 3$, $N = p_g - 1$ and $d = K_X^3 = 4p_g - 14$, if $d - 1 = m(N - n) + \varepsilon$, we have $m = 4$ and $\varepsilon = 1$. Hence F is of class $5H - (p_g - 6)L$, and the multiplicity $m(W)$ is as in the statement.

Vice versa, we recall that $K_{\widetilde{W}} \sim -4H + (p_g(X) - 6)L$; then by adjunction formula, H cuts on the proper transform $\widetilde{F}$ of F the canonical system. Since $h^1(\mathcal{O}_{\widetilde{W}}(H - \widetilde{F})) = h^1(\mathcal{O}(K_{\widetilde{W}})) = 0$, the map $H^0(\mathcal{O}_W(H)) \to H^0(\mathcal{O}_{\widetilde{F}}(H))$ is surjective, and the hyperplanes cut on $\widetilde{F}$ the complete canonical system. Let X be its desingularization, then its easy to verify by using the intersection formulas on $\widetilde{W}$ that $K_X^3 = K_{\widetilde{F}}^3 = \widetilde{F} H^3 = 4p_g - 14$. ■

We now study the existence of such Castelnuovo varieties, by the previous lemma, it is sufficient to work on $\widetilde{W}$.

LEMMA 2.9. – *Let $p_g > 6$ and*

$$\widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3) \oplus \mathcal{O}_{\mathbb{P}^1}(a_4))$$

where $0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ and $a_1 + a_2 + a_3 + a_4 = p_g - 4$. If $|5H - (p_g - 6) L|$ is not empty and its generic element is irreducible then

$$5a_4 \leqslant 4p_g - 14 \quad \text{and} \quad a_3 > 0 .$$

PROOF. – Let $\widetilde{W}(a_1, a_2, a_3) \in |H - a_4 L|$, let $\widetilde{F} \in |5H - (p_g - 6) L|$ then $\widetilde{F}$ does not contains $\widetilde{W}(a_1, a_2, a_3)$ as component since it is irreducible. Hence $H^2 \widetilde{F}\, \widetilde{W}(a_1, a_2, a_3) \geqslant 0$, thus we have $H^2(5H - (p_g - 6) L)(H - a_4 L) \geqslant 0$, i.e. $5(p_g - 4) - (p_g - 6) - 5a_4 \geqslant 0$. If $\psi(\widetilde{W})$ is a cone of vertex a plane that is, $\widetilde{W}$ is of type $(0, 0, 0, a_4)$ and $a_4 = p_g - 4$, then we have $5a_4 = 5(p_g - 4) \leqslant 4p_g - 14$, hence $p_g \leqslant 6$. ■

THEOREM 2.10. – *The generic element $\widetilde{F}$ in $|5H - (p_g(X) - 6) L|$ has canonical singularities at most if and only if the following conditions are satisfied*:

i) $5a_3 \geqslant p_g - 6$,

ii) $3a_1 + 2a_4 \geqslant p_g - 6$,

iii) *if* $5a_2 < p_g - 6$ *then* $4a_1 + a_4 \geqslant p_g - 6$.

PROOF. – Let $\widetilde{F}$ be in $|5H - (p_g(X) - 6) L|$. Let x_i be sections of $\mathcal{O}(H - a_i L)$ for $i = 1, \ldots, 4$ such that their restriction to each fiber of π gives a system of homogeneous coordinates. Let $[t_0; t_1]$ be homogeneous coordinates over $\mathbb{P}^1$. The equation of $\widetilde{F}$ is of the following type:

$$\widetilde{F} = \sum_{\substack{i, j, k \geqslant 0 \\ i + j + k \leqslant 5}} g_{ijk}\, x_1^{5 - i - j - k} x_2^i x_3^j x_4^k \tag{2.13}$$

where g_{ijk} are homogeneous polynomials of degree $(5 - i - j - k) a_1 + ia_2 + ja_3 + ka_4 - (p_g - 6)$ in the coordinates $[t_0; t_1]$ on $\mathbb{P}^1$. We recall that $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Let $l = i + j + k$, then one has:

$$\deg g_{l00} = (5 - l) a_1 + la_2 - (p_g - 6) \leqslant \deg g_{ijk} \leqslant \deg g_{00l} = (5 - l) a_1 + la_4 - (p_g - 6). \tag{2.14}$$

Let us suppose that $\widetilde{F}$ has canonical singularities at most. If $5a_3 < p_g - 6$, then

$$\deg g_{ij0} = (5 - i - j) a_1 + ia_2 + ja_3 - (p_g - 6) \leqslant 5a_3 - (p_g - 6) < 0$$

thus $\{x_4 = 0\}$ would be a component of $\widetilde{F}$. This proves the condition (i).

The condition (iii) is related to the surface S of $\widetilde{W}$ defined by $\{x_3 = x_4 = 0\}$: S is contained in $\widetilde{F}$ if and only if $\deg g_{l00} < 0$, hence if $(5-l)a_1 + la_2 < (p_g - 6)$, for every $l \leqslant 5$, that is if $5a_2 < (p_g - 6)$. In this case S is contained in the base locus of $|5H - (p_g(X) - 6)L|$.

Since $\widetilde{F}$ has at most canonical singularities, it does not contain any double surface, thus if S is contained in the base locus of $|5H - (p_g(X) - 6)\,L|$, we have to exclude that every 3-fold of $|5H - (p_g(X) - 6)\,L|$ has S as double surface. By computing the directional derivatives of $\widetilde{F}$ along x_i, for $i = 1, \ldots, 4$, and along t_0, t_1 it can be seen that they vanish on S if $4a_1 + a_4 < p_g - 6$. In fact, since $g_{l00} = 0$ if $l = 0, \ldots, 5$, then $\partial \widetilde{F} / \partial t_0$ and $\partial \widetilde{F} / \partial t_1$, $\partial \widetilde{F} / \partial x_1$ and $\partial \widetilde{F} / \partial x_2$ vanish on $S = \{x_3 = x_4 = 0\}$. One has $\partial \widetilde{F} / \partial x_3 |_S = \sum_{0 \leqslant i \leqslant 4} g_{i10} x_1^{4-i} x_2^i$ and $\partial \widetilde{F} / \partial x_4 |_S = \sum_{0 \leqslant i \leqslant 4} g_{i01} x_1^{4-i} x_2^i$, therefore they vanish on S if $4a_1 + a_4 < p_g - 6$.

In the same way, we obtain condition (ii) imposing that the curve C of equations $\{x_2 = x_3 = x_4 = 0\}$ is not a curve of triple points for the generic $\widetilde{F}$.

Now, let us suppose that (i), (ii), (iii), are verified, we distinguish two cases:

A) $5a_1 \geqslant (p_g - 6)$,

B) $5a_1 < (p_g - 6)$.

A) The system $|5H - (p_g(X) - 6)\,L|$ is base point free. In fact, the system can be described as sum of $|5(H - a_1 L)|$ and $|(p_g(X) - 6 + 5a_1)\,L|$. These are base point free systems, moreover $|5(H - a_1 L)|$ is not composed with a pencil. Thus we can apply Bertini theorem.

B) g_{000} is zero, again we distinguish two cases:

B_1) $5a_2 \geqslant (p_g - 6)$,

B_2) $5a_2 < (p_g - 6)$.

It is sufficient to study the system $|5H - (p_g(X) - 6)L|$ around its base locus.

If B_1) holds then $g_{500} \neq 0$, it's easy to see by looking at the general equation (2.13) that the base locus is the curve $C = \{x_2 = x_3 = x_4 = 0\}$. Let p be any point of C, we can suppose that the choice of x_i is such that in p one has $x_1 = 1$ and $x_2 = x_3 = x_4 = 0$.

If $4a_1 + a_4 \geqslant (p_g - 6)$, then p is a simple point for $\widetilde{F}$, the tangent plane of a $\widetilde{F}$ in p having equation

$$g_{100}(p)\,x_2 + g_{010}(p)\,x_3 + g_{001}(p)\,x_4 = 0\,.$$

In other words, since g_{000} is zero no term in the $[t_0;\, t_1]$ variables appears, thus if g_{ijk} are generic the equation of the tangent plane is not identically zero.

If $4a_1 + a_4 < (p_g - 6)$, then p is a double point for $\widetilde{F}$, the tangent cone having equation

$$\Big[\sum_{i+j+k=2} \max\{i;\, j;\, k\}\, g_{ijk}(p)\, x_2^i x_3^j x_4^k\Big] = 0\,.$$

Such an equation is not identically zero by (ii), if g_{ijk} are generic. It defines a quadric tangent cone.

If B_2) holds then $g_{ijk} = 0$ if $j = k = 0$, hence the base locus of the system $|5H - (p_g - 6)\,L|$ is the surface $S = \{x_3 = x_4 = 0\}$. Let p be any point of S, we can suppose that $x_1(p) = 1$ and $x_2(p) = x_3(p) = x_4(p) = 0$. Proceeding as in the previous case we find that p is a simple point for $\widetilde{F}$, the tangent plane to $\widetilde{F}$ in p having equation

$$g_{100}(p)\, x_2 + g_{010}(p)\, x_3 + g_{001}(p)\, x_4 = 0\,.$$

This equation is not identically zero by (iii) if g_{100}, g_{010}, and g_{001} are generic.

In any case the singular locus of $\widetilde{F}$ is of canonical type. ■

2.4. *Examples*. If $p_g = 5$ and $K_X^3 = 6$ then the image of the canonical morphism of a Castelnuovo 3-fold with such invariants is a nonsingular hypersurface F of $\mathbb{P}^4$ of degree 6. The number of moduli is

$$M(X) = h^0(\mathbb{P}^4, \mathcal{O}(6)) - \dim(\mathbb{P}GL(5)) - 1 = 210 - 25 = 185\,.$$

It is easy to see, by using the normal bundle and the Euler sequences, that $h^2(\Theta_X) = 0$, thus the moduli space is generically smooth.

If $p_g = 6$ and $K_X^3 = 10$ then we find Castelnuovo 3-folds as divisors of quadrics of $\mathbb{P}^5$, presented in section 2.1.

If $p_g = 7$ and $K_X^3 = 14$ the canonical image of the Castelnuovo 3-fold X lies in a singular scroll. In fact the conditions pointed out in Theorem 2.10 and in Lemma 2.9 give as possible type for the scroll containing the canonical image $(0, 1, 1, 1)$ and $(0, 0, 1, 2)$.

A more interesting example is $p_g = 8$ and $K^3 = 18$. Apart from the case of the cone over the Veronese, the canonical image F may be a divisor of the scroll $W(a_1, a_2, a_3, a_4)$ of $\mathbb{P}^7$ of the following types:

(*a*) $(1, 1, 1, 1)$,

(*b*) $(0, 1, 1, 2)$,

(*c*) $(0, 0, 2, 2)$,

(*d*) $(0, 0, 1, 3)$,

(*e*) $(0, 0, 0, 4)$.

The case (e) is excluded by the Lemma 2.9. We have the following result:

LEMMA 2.11.

$$h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(5H-2L)) = \begin{cases} 224 & \textit{case } (a), \\ 225 & \textit{case } (b), \\ 230 & \textit{cases } (c) \textit{ and } (d); \end{cases}$$

$$\dim \operatorname{Aut} \widetilde{W} = \begin{cases} 18 & \textit{case } (a), \\ 19 & \textit{case } (b), \\ 22 & \textit{case } (c), \\ 23 & \textit{case } (d). \end{cases}$$

PROOF. – The first part of the statement follows from

$$h^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(5H-2L)) = h^0(\mathbb{P}^1; \pi_*(\mathcal{O}_{\widetilde{W}}(5H)\otimes \mathcal{O}_{\mathbb{P}^1}(-2))) =$$

$$h^0(\mathbb{P}^1; [\mathrm{Sym}^5(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus \mathcal{O}_{\mathbb{P}^1}(a_2)\oplus \mathcal{O}_{\mathbb{P}^1}(a_3)\oplus \mathcal{O}_{\mathbb{P}^1}(a_4))]\otimes \mathcal{O}_{\mathbb{P}^1}(-2))$$

where $\pi\colon \widetilde{W} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus \mathcal{O}_{\mathbb{P}^1}(a_2)\oplus \mathcal{O}_{\mathbb{P}^1}(a_3)\oplus \mathcal{O}_{\mathbb{P}^1}(a_4)) \to \mathbb{P}^1$. The second part of the statement follows from the exact sequence

$$1 \to \operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W}) \to \operatorname{Aut}(\widetilde{W}) \to \operatorname{Aut}\mathbb{P}^1 \to 1 \tag{2.15}$$

where $\operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W})$ is the group of the automorphisms of $\widetilde{W}$ which fix the fibers of the projection π, thus it is the projectivized group of the invertible 4×4 matrices $A = (P_{j,h})$ with entries homogeneous polynomials $P_{j,h}$ on $\mathbb{P}^1$ such that $\deg(P_{j,h}) = a_h - a_j$ if $a_h - a_j \geq 0$, $P_{j,h} = 0$ otherwise. ■

For each type $\alpha = (a_1, a_2, a_3, a_4)$, we obtain the number of moduli $M(X)_\alpha$ by the formula

$$M(X)_\alpha = \dim|5H-2L| - \dim \operatorname{Aut}(W) = h^0(W; 5H-2L) - 1 - \dim \operatorname{Aut}(W).$$

Thus:

(a) type $(a) = (1, 1, 1, 1)$, $M(X)_a = 223 - 18 = 205$,

(b) type $(b) = (0, 1, 1, 2)$, $M(X)_b = 223 - 19 = 204$,

(c) type $(c) = (0, 0, 2, 2)$, $M(X)_c = 229 - 22 = 207$,

(d) type $(d) = (0, 0, 1, 3)$, $M(X)_d = 229 - 23 = 206$.

We recall that for $\mu = a_1 + \ldots + a_4 = 4$ the most general scroll is of type (a), being a type (b)-scroll a degeneration of a type (a); a type (c) a degeneration of a type (b); and a type (d) a degeneration of a type (c) (cf. [5]). Since $M(X)_c > M(X)_b$, the generic 3-fold X which has canonical image in a scroll of type (c) cannot be a degeneration of a 3-fold which has canonical image in a scroll of type (b). Thus we have at least three components of the moduli space of the 3-folds with $K_X^3 = 18$ and $p_g(X) = 8$ (counting the one relative to the Veronese cone). We point out that the same situation appears in the 2-dimensional case (cf. [2]).

More generally we have the following lemma

LEMMA 2.12. – *Let* $a_1 + a_2 + a_3 + a_4 = \mu$, *let* W *be the image of* $\widetilde{W}(a_1; a_2; a_3; a_4)$ *in* $\mathbb{P}^{\mu+3}$, *then*

$$h^0(\mathcal{O}_W(5H - (\mu-2)L)) - h^1(\mathcal{O}_W(5H - (\mu-2)L)) =$$

$$\begin{cases} 14\mu + 168 & \textit{if } W \textit{ is non singular}, \\ 15\mu + 165 & \textit{if } W \textit{ is a cone of vertex a point}, \\ 20\mu + 150 & \textit{if } W \textit{ is a cone of vertex a line}. \end{cases}$$

PROOF. – Consider the exact sequence

$$0 \to \mathcal{O}_W(5H - (\mu-2)L) \to \mathcal{O}_W(5H) \xrightarrow{\varrho} \mathcal{O}_{(\mu-2)L}(5H) \to 0 . \tag{2.16}$$

The Hilbert function does not changes by varying the type of the scrolls so that one can reduce himself to make computations only on an easy case. We compute $h^0(\widetilde{W}; \mathcal{O}_{\widetilde{W}}(5H))$ on $W' = W(0; 0; 0; \mu)$, by projecting on $\mathbb{P}^1$. One has

$$h^0(W', \mathcal{O}_{W'}(5H)) = h^0(\mathbb{P}^1; [\mathrm{Sym}^5(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\mu))]) =$$

$$h^0(\mathbb{P}^1; \bigoplus_{k=0}^{5} \mathcal{O}_{\mathbb{P}^1}((5-k)\mu) \oplus (\mathcal{O}_{\mathbb{P}^1})^{\oplus (k+2)(k+1)/2}]) =$$

$$\sum_{k=0}^{5} (k+2)(k+1)((5-k)\mu + 1))/2 = 56 + 70\mu .$$

Clearly if W is nonsingular, $h^0(\mathcal{O}_{(\mu-2)L}(5H)) = (\mu-2)\, h^0(\mathcal{O}_{\mathbb{P}^3}(5)) = (\mu-2)56$. If W has a zero-dimensional vertex p, then $h^0(\mathcal{O}_{(\mu-2)L}(5H)) = h^0(\mathcal{O}_{\mathbb{P}^3}(5)) + (\mu-3)\, h^0(\mathbb{P}^3; \mathfrak{I}_p(5)) = 55\mu - 109$. If W has a 1-dimensional vertex l then $h^0(\mathcal{O}_{(\mu-2)L}(5H)) = h^0(\mathcal{O}_{\mathbb{P}^3}(5)) + (\mu-3)\, h^0(\mathbb{P}^3; \mathfrak{I}_l(5)) = 50\mu - 94$. The conclusion follows from (2.16), being $h^1(W; \mathcal{O}_W(5H)) = 0$. ■

It can be easily verified by projecting on $\mathbb{P}^1$ that $h^1(\mathcal{O}_W(5H - (\mu - 2)L)) = 0$ if $5a_1 > \mu - 4$, since in this case one has a sum of line bundles on $\mathbb{P}^1$ of degree at least $5a_1 - \mu + 2$. Analogously, $h^1(\mathcal{O}_W(5H - (\mu-2)L)) \geq h^1(\mathcal{O}_{\mathbb{P}^1}(-\mu+2)) = \mu - 3$ if $0 = a_1 < a_2$ and $h^1(\mathcal{O}_W(5H - (\mu-2)L)) \geq$

$6h^1(\mathcal{O}_{\mathbb{P}^1}(-\mu+2)) = 6(\mu-3)$ if $0 = a_1 = a_2$. We can conclude with the following theorem:

THEOREM 2.13. – *Let $W_0 \subset \mathbb{P}^{\mu+3}$ be a rational normal scroll of type (a, a, a, a), $(a, a, a, a+1)$, $(a, a, a+1, a+1)$, $(a, a+1, a+1, a+1)$, where $\mu \equiv 0, \ldots, 3$ modulo 4 respectively. Then there exists a family of isomorphism classes of nonsingular 3-folds X with $K_X^3 = 4\mu+2$ and $p_g = \mu+4$ whose canonical morphism is birational to a divisor of W_0, which is its canonical model. The linear class of such divisors is $5H-(\mu-2)L$. Such a family is unirational of dimension $M = 14\mu+149$.*

PROOF. – For a fixed μ, for any type of the scroll W verifying the conditions of Theorem 2.10 there exists a family of canonical Castelnuovo 3-folds with $K^3 = 4\mu+2$ and $p_g = \mu+4$ parametrized by an open set of $\mathbb{P}H^0(\widetilde{W}, \mathcal{O}_{\widetilde{W}}(5H-(\mu-2)L))$. We then quotient the family by the group of the automorphisms of W. We find a unirational family, represented by a subvariety of the moduli space. The computation of the dimension follows from

$$M = h^0(\widetilde{W}_0, \mathcal{O}_{\widetilde{W}_0}(5H-(\mu-2)L)) - \dim \operatorname{Aut}(W_0),$$

from Lemma 2.12 and from the exact sequence

$$1 \to \operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W}) \to \operatorname{Aut}(\widetilde{W}) \to \operatorname{Aut}\mathbb{P}^1 \to 1 \qquad (2.17)$$

where in the 4 cases one always gets $\dim \operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W}) = 15$. ■

Note that in the 4 cases above $\dim \operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W})$ is minimal in the family of scrolls $\widetilde{W}$ with degree μ, so that $\dim \operatorname{Aut}(\widetilde{W})$ is also minimal.

Moreover, $h^1(\mathcal{O}_W(5H-(\mu-2)L)) = 0$ since $5a_1 > \mu-4$, as it has been noticed above, thus by Lemma 2.12 $\dim \mathbb{P}H^0(\mathcal{O}_W(5H-(\mu-2)L))$ assumes the minimal value for a scroll of fixed degree μ. But for families of scrolls of special type $\dim \mathbb{P}H^0(\mathcal{O}_W(5H-(\mu-2)L))$ increases, and even if also $\dim \operatorname{Aut}_{\mathbb{P}^1}(\widetilde{W})$ increases, one can find components of the moduli space of 3-folds X with higher dimension.

REFERENCES

[1] E. ARBARELLO - M. CORNALBA - P. GRIFFITHS - J. HARRIS, *Geometry of Algebraic Curves*, Springer-Verlag, Berlin, Heidelberg, New York (1985).

[2] T. ASHIKAGA - K. KONNO, *Algebraic surfaces of general type with $c_1^2 = 3P_g - 7$*, Tohoku Math. J., **42** (1990), 517-536.

[3] C. CILIBERTO, *On a property of Castelnuovo varieties*, Trans. Amer. Math. Soc., **303** (1987), 201-210.
[4] F. JONGMANS, *Contribution à la théorie des variétés algébriques*, Thése d'Agrégation de l'enseignement supérior présentée à la Faculté des Science de l'Université de Liege Bruxelles (1947).
[5] J. HARRIS, *A bound on the geometric genus of projective varieties*, Ann. Sc. Norm. Sup. Pisa, 8 (1981), 35-68.
[6] R. MIRANDA, *On canonical surfaces of general type with* $K_1^2 = 3\chi - 10$, Math. Z., **198** (1988), 83-93.
[7] M. REID, *Canonical 3-folds*, in Géométrie algébrique d'Angers, A. Beauville ed. (1979), 273-310.

Dipartimento di Matematica «G. Castelnuovo»
Università di Roma «La Sapienza», P.le Aldo Moro 2 - 00185 Roma
e-mail: Supino@axp.mat.utovrm.it

Pervenuta in Redazione
il 29 marzo 1997