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Normally Constrained p-Groups.

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Sunto. – *In questo lavoro si studiano i gruppi finiti di ordine una potenza di un numero primo in cui i sottogruppi normali sono compresi tra due termini successivi della serie centrale discendente. Si ottengono numerose proprietà generali di questi gruppi, e una loro dettagliata descrizione in classe di nilpotenza* 2.

1. – Introduction and examples.

In this paper G will denote a finite p-group, where p is a prime. With the exception of the last example in section 2, p will be odd. Let $\{G_i\}$ be the lower central series of G.

Definition. – We say that G is *normally constrained* (NC for short) if for every i, $1 \leq i \leq c$, G_i satisfies the following equivalent conditions:

(i) G_i is the only normal subgroup of G of order $|G_i|$,

(ii) if $N \lhd G$, we have $N \leq G_i$ or $N \geq G_i$,

(iii) if $x \in G - G_i$ then $G_i \leq \langle x \rangle^G$.

Note that factor groups of NC-p-groups are NC. Other elementary properties of NC-p-groups are found in [Bo]. We now list some examples, to show that the class of NC-p-groups is rather rich. This list is by no means complete. However we will characterize below the NC-p-groups of class 2, showing that they are like one of those listed in examples 1, 2, or 4. Furthermore, we will show that the associated Lie algebra (as described *e.g.* on [HB, VIII.9]) of the central factor of a NC-p-group of sufficiently large nilpotency class can be obtained as a factor of a tensor product over $GF(p)$ of a 2-generated Lie algebra with a larger finite field. Therefore the groups described in example 3 seem really to be the crucial examples of NC-p-groups of large class, at least under the point of view of the associated Lie algebra. It is well known that the associated Lie algebra of a p-group does not capture all the structural features of

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the group itself, but our results here, with those of [CMNS]), suggest that in the case of NC-p-groups a classification up to *isomorphism of the associated Lie algebras* should be both possible and interesting, while a classification up to group isomorphism seems to be completely out of reach.

EXAMPLES. – 1. If G is of class 2, and G is monolithic with monolith G', then clearly G is normally constrained: *e.g.*, the split extension of a cyclic p-group of order larger than p with the group generated by its automorphism of order p.

2. If G is a p-group of class 2 or 3 such that every nontrivial coset of G' consists of conjugate elements, then, by the results of [McD], G is normally constrained. Furthermore, we have that $|G: G'|$ is a square and that G/G_3 is special. (We say, in this case, that (G, G') is a *Camina pair*, or , if G has class 2, that G is a *semi-extraspecial* p-group; see also [Be], [MS]). An example here is given by a Sylow p-subgroup of $SL(3, q)$, where $q = p^n$.

3. The class of NC-p-groups includes all the p-groups of maximal class and thin p-groups (see [Bl], [BCS], [CMNS]). Among these, we note in particular the finite quotients of some p-adic analytic groups, like $M_{0,1,1}$, in the notation of [H, III.17], and the finite quotients of the well-known «Nottingham group» (see [Y]).

4. Let $Z = C_{p^2} \times C_{p^2}$ (additive notation) and let α act on Z as

$$\begin{pmatrix} 1 & p\nu \\ p & 1 \end{pmatrix},$$

where ν is not a square mod p. Now α is an automorphism of Z, and we construct the semidirect product $\langle \alpha \rangle Z = G$. We note that G has class 2, $G' = G^p = pZ$, and that for $x \in G - Z$ we have $[x, G] = G'$, while for $x \in Z - G'$ we have $G' = \langle px, [x, G] \rangle$. Then G is an NC-p-group.

2. – NC-p-groups of class 2.

We give here an elementary account of the structure of the normally constrained p-groups of class 2.

PROPOSITION 2.1. – *Let p be an odd prime. Then G is a NC p-group of class* 2 *if and only if either G is one of the groups described in examples* 1, 2, *or G is special, $G' = \langle x^p \rangle [x, G]$ for every $x \in G - G'$, $|G: G'| = p^{2n+1}$, $G^p = G'$, and $|G'| \leqslant p^{n+1}$.*

PROOF. – If G is a group like those described in the statement, and $x \in G - G'$, we have $\langle x \rangle^G \geq G'$, and G is NC. Note that examples 1, 2, 4 show that all the classes listed in the statement are non-empty.

To show the converse, assume first that $G' < Z(G)$. Then $Z(G)$ is cyclic, because it has only one subgroup of order G'. Since G is noncyclic, it has a normal elementary subgroup N of order p^2, by [H, III.7.5(a)]. We have $N \not\leq G'$, since $G' < Z(G)$ is cyclic. Then $N > G'$, $|G'| = p$ and G is like in example 1.

Then we may assume $G' = Z(G)$, $|G'| > p$. Set $\overline{G} = G/(G')^p$. Note that $\overline{G}$ is NC. If $Z(\overline{G}) > \overline{G}'$ we would get, as above, $|\overline{G}'| = p$, and G' would be cyclic. By [H, III.7.5(a)] again, we would have $|G'| = p$, contradiction. Hence $Z(\overline{G}) = \overline{G}'$, and $G/G' \simeq \overline{G}/Z(\overline{G})$ has exponent p, by [H, III.2.13(a)]. Then $G^p \leq G'$, and G is special.

If $|G : G'|$ is a square, G is like in example 2, because, for every maximal subgroup K of G', G/K is NC, $Z(G/K)$ is cyclic, and G/K is extraspecial, by [H, III.13.7]. If $|G : G'| = p^{2n+1}$, we apply again [H, III.13.7] to G/K, for every maximal subgroup K of G', and we get that $G' = \langle x \rangle^p [x, G]$ for every $x \in G - G'$, and that $G^p = G'$. Thus we are only left to show that, in this case, $|G'| \leq p^{n+1}$.

Let $x_1, \ldots, x_{2n+1}$ be generators of G, $z_1, \ldots, z_l$ be generators of $Z(G) = G'$, where $|Z(G)| = p^l$. We now switch to additive notation, for G/G' and G', and we describe the p-th power map and the commutation in G in terms of matrices, as follows:

$$x_i^p = \sum_{k=1}^{l} \alpha_{i0k} z_k ,$$

$$[x_i, x_j] = \sum_{k=1}^{l} \alpha_{ijk} z_k .$$

Our Condition NC can now be rephrased in any of the following, obviously equivalent, ways (for $1 \leq i \leq 2n+1$, $0 \leq j \leq 2n+1$, $1 \leq k \leq l$):

(1) $\forall x \in G - G'$, x^p and $\{[x, x_j]\} (j = 1, \ldots, 2n+1)$ generate G';

(2) $\forall (\beta) = (\beta_1, \ldots, \beta_{2n+1}) \neq (0, \ldots, 0)$ we have $G' = \langle \sum_{i,k} \beta_i \alpha_{ijk} z_k \rangle$;

(3) $\forall (\beta) \neq (0, \ldots, 0)$ the matrix B, defined by $(B)_{j,k} = \sum_i \beta_i \alpha_{ijk}$ has rank l;

(4) $\forall (\beta) \neq (0, \ldots, 0)$ the matrix B has independent columns;

(5) $\forall (\beta) \neq (0, \ldots, 0)$, $\sum_{i,k} \beta_i \alpha_{ijk} \gamma_k = 0$ implies $(\gamma) = (0, \ldots, 0)$;

(6) $\forall (\gamma) = (\gamma_1, \ldots, \gamma_l) \neq (0, \ldots, 0)$ $\sum_{i,k} \beta_i \alpha_{ijk} \gamma_k = 0$ implies $(\beta) = (0, \ldots, 0)$;

(7) $\forall(\gamma) \neq (0, \ldots, 0)$ the matrix Γ, defined by $(\Gamma)_{i,j} = \sum_k \alpha_{ijk}\gamma_k$ has independent rows;

(8) Γ has rank $2n+1$, for every $(\gamma) \neq (0, \ldots, 0)$.

Note now that, if we define $\alpha_{0jk} = -\alpha_{j0k}\delta_{ij}$, and $\alpha_{00k} = 0$, we can complete Γ to a skew-symmetric matrix $\tilde{\Gamma}$, $(\tilde{\Gamma})_{i,j} = \sum_k \alpha_{ijk}\gamma_k$.

The rank of $\tilde{\Gamma}$ is at least $2n+1$, but, by [H, III.9.6(a)], that rank is even; therefore $\tilde{\Gamma}$ is nonsingular, for every $(\gamma) \neq (0, \ldots, 0)$.

We now proceed as in [McD], [Be]: $\det(\tilde{\Gamma}) = (Pf(\tilde{\Gamma}))^2$ can be seen as a polynomial in the indeterminates γ_k, and, by Chevalley-Warning's theorem, $\det(\tilde{\Gamma}) = 0$ has nontrivial solutions if $l > n+1$. Thus $l \leqslant n+1$.

REMARK. – A. Caranti and A. Mann have independently noticed that we can equivalently obtain the last part of the proof of Proposition 2.1 extending our group G by the automorphism of G that sends $x \in G$ into x^{p+1}. This extension is easily seen to form a Camina pair with its commutator subgroup, and as in [McD], [Be] we can conclude $l \leqslant n+1$.

EXAMPLE. – The hypothesis $p \neq 2$ cannot be removed from the statement of Proposition 2.1, as shown by the following construction, due to H. Heineken.

Let $F = GF(8)$, and set

$$L = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b \in F \right\}.$$

Clearly $|L| = 2^6$. Furthermore

$$L' = L^2 = Z(L) = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a \in F \right\}$$

and $L/L' \simeq L'$ is elementary abelian of order 8. To show that L is NC, we will show that, if $x \in L - L'$, then $\langle x \rangle^L = \langle x, L' \rangle$. Let

$$x = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & r & s \\ 0 & 1 & r^2 \\ 0 & 0 & 1 \end{pmatrix},$$

so that

$$x^2 = \begin{pmatrix} 1 & 0 & a^3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [x, y] = \begin{pmatrix} 1 & 0 & ar^2 + a^2 r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and therefore if $a \neq 0$ we have $C_L(x) = \langle x, L' \rangle$. Thus we have that $\langle x \rangle^L \neq \langle x, L' \rangle$ if and only if there exists a value of r such that $ar^2 + a^2 r = r^3$. But, setting $u = ra^{-1}$, our condition becomes $u^2 + u + 1 = 0$, and this last equation does not have any solution in F.

3. – Associated Lie algebras of NC-p-groups.

From now on, we assume that p is an odd prime, and we study some properties of the lower central sections of NC-p-groups.

PROPOSITION 3.1. – *Let G be an* NC-*p-group, with* $\mathrm{cl}(G) \geq 3$. *Then G/G_3 is special of exponent p, and* $|G'/G_3|^2 = |G/G'|$.

PROOF. – We may assume that $|G_3| = p$. Let $\overline{G} = G/G_3$. We first show that $\overline{G}$ is special. Otherwise, by Prop. 2.1, we have that $|\overline{G}'| = p$, and that $Z(\overline{G})$ is cyclic. Again, if $Z(\overline{G}) = \overline{G}'$, we get easily that $\overline{G}/\overline{G}'$ is of exponent p, $\overline{G}^p \leq \overline{G}'$, and $\overline{G}$ is (extra-)special. Hence $Z(\overline{G}) > \overline{G}'$, and

$$(*) \qquad |Z(\overline{G})| \geq p^2 .$$

Now $|Z(G)| > p$, or we would have a contradiction by $(*)$ and [H, III.2.13(a)]. But since G' is the only subgroup of order p^2 of G, we get $Z(G) \geq G'$, a contradiction with $\mathrm{cl}(G) = 3$. Thus $\overline{G}$ is special.

We now show that G/G' has even rank. By way of contradiction, assume $|G : G'| = p^{2n+1}$. By proposition 2.1, we have that $\overline{G}$ has exponent p^2. Note first that G' does not have exponent p. In fact, let $x \in G - G'$ such that $x^p \notin G_3$. Then $H = \langle x \rangle G'$ is a normal subgroup of class at most 2, and, if G' is of exponent p, $|H^p| = p$, but $H^p \neq G_3$, contradiction. Now $\overline{G}$ is regular, because p is odd, and, by Prop. 2.1, there exists $x \in G - G'$ such that $x^p \in G_3$. Set $H = G'\langle x \rangle$. H has class at most 2, $H \lhd G$, and thus $\Omega_1(H) \lhd G$. Now $|\Omega_1(H)| = |H : H^p| = |H : G_3| = |G'|$, because H is regular, and $\Omega_1(H)$ has exponent p. But we have seen that G' does not have exponent p, and thus $G' \neq \Omega_1(H)$, contradiction. Then G/G' has even rank. By Proposition 2.1, G/G_3 is like in example 2, and by [M, 1.1] we have that G/G_3 has exponent p and $|G'/G_3|^2 = |G/G'|$.

COROLLARY 3.2. – *Let G be a* NC-*p-group of class at least* 3. *Then G_i/G_{i+2} is elementary abelian for every* $i \geq 2$.

We can now state one more condition which is equivalent to any of those defining NC-p-groups:

PROPOSITION 3.3. – *Let G be a p-group of class $c \geq 3$. The following are equivalent*:

(a) *G is a* NC-*p-group*;

(b) $\forall i \geqslant 1$, $\forall x \in G_i - G_{i+1}$, *we have* $[x, G]\, G_{i+2} = G_{i+1}$.

PROOF. – By Corollary 3.2, if $x \in G_i - G_{i+1}$,we have that $[x, G]\, G_{i+2}$ is maximal in $\langle x \rangle^G$. But $[x, G]\, G_{i+2} \leqslant G_{i+1}$, and by ($a$) we get that G_{i+1} is properly contained in $\langle x \rangle^G$, whence (b).

Conversely, we show that, for any $c \geqslant 1$, if a p-group G of class c satisfies (b) then G is NC. We proceed by induction on c, starting with $c = 1$: in this case the result is trivial. Assume it true for groups of class less than c. It is then enough to show that, if $N \lhd G$, then either $N \leqslant G_c$ or $G_c \leqslant N$. Let $N \nleqslant G_c$, and $x \in N - G_c$. By our hypothesis, we may assume that $x \in G_{c-1} - G_c$, and therefore $G_c \leqslant N$.

REMARK. – Note that (b) is equivalent to the apparently stronger:

(b') $\forall i \geqslant 1$, $\forall x \in G_i - G_{i+1}$, we have $[x, G] = G_{i+1}$.

In fact, (b) inductively implies that $\forall i \geqslant 1$, $\forall x \in G_i - G_{i+1}$ we have $[x, G]\, G_{i+j+1} \geqslant G_{i+j}$. However, the equivalence holds for finite p-groups only, while (b) can be extended to pro-p-groups with open lower central subgroups. Furthermore, (b) stresses the Lie theoretical nature of our condition. In fact, we can define a NC-Lie algebra as a finitely generated Lie algebra L over the field $GF(p)$, graded by its lower central factors L_i, such that the following condition holds:

$\forall i \geqslant 1$, $\forall x \in L_i$, $x \neq 0$, we have $[x, L_1] = L_{i+1}$.

COROLLARY 3.4. – *Let G be a* NC-*p-group of class at least* 3. *Then the upper and lower central series of G coincide.*

PROOF. – Since G/G_3 is special, it is enough to show, by induction, that $Z(G) = G_c$. If $x \in Z(G) - G_c$ we have $1 = [x, G] \geqslant G_c$, contradiction.

We now adapt some techniques from [DS]. We begin extending [M, 1.1]:

THEOREM 3.5. – *Let G be a* NC-*p-group of class $c \geqslant 3$ such that $|G : G'| = p^{2n}$. Then for $2 \leqslant i < c$ we have $p^n \leqslant |G_i : G_{i+1}| \leqslant p^{2n}$.*

PROOF. – The second inequality is an immediate consequence of Proposition 3.3.

We already know $|G_2 : G_3| = p^n$, if $\mathrm{cl}(G) = 3$, and we prove the first inequality by way of contradiction. Let G be a a NC-p-group of class $c \geqslant 4$ such that $|G : G'| = p^{2n}$, $p^n \leqslant |G_i : G_{i+1}|$ for $2 \leqslant i < c - 1$, and $|G_{c-1} : G_c| = p^r < p^n$. We may assume, without loss of generality, $|G_c| = p$. Set $C = C_G(G_{c-1})$, $Z = G_{c-2} \cap Z(G')$. Since the centralizers of the elements of $G_{c-1} - G_c$ are maximal subgroups, we have $|G: C| \leq p^r$. Note here that, in fact, we have $|G: C| = p^r$, as G_{c-1}/G_c can be identified with a subspace of the $GF(p)$-dual space of G/C. Let $x \in G_{c-2} - Z$. Setting $D = C_G(x)$ we have easily $|G: D| \leq p^{r+1}$. As $r < n$, there exists $b \in C \cap D - G'$. Let $g \in G$. We have $[x, b, g] = 1$, since $b \in D$, and $[g, x, b] = 1$, since $b \in C$. Then Witt identity gives $[b, g, x] = 1$, and by the remark after Proposition 3.3, $[G', x] = 1$, as g is arbitrary. This contradicts with our choice of x. Then $G_{c-2} = Z$. Note that $[Z, G, C] = [G, C, Z] = 1$, and, by the three-subgroup lemma, also $[C, Z, G] = 1$. Therefore $[C, G_{c-2}] \leq G_c$. Let now M be a maximal subgroup of G_{c-1} containing G_c. We apply our inductive hypothesis to G/M, and conclude that $|G: C_G(G_{c-2}/M)| = |G_{c-2}/G_{c-1}| \geq p^n$. But $C \leq C_G(G_{c-2}/G_c) \leq C_G(G_{c-2}/M)$, and $p^r = |G: C| \geq |G: C_G(G_{c-2}/M)| \geq p^n$.

THEOREM 3.6. – *Let G be a* NC-*p-group of class $c \geq 3$ such that $|G: G'| = p^{2n}$. If $|G_3| \geq p^n$ we have that G/G_3 is isomorphic to a Sylow p-subgroup of $SL(3, p^n)$.*

PROOF. – By Theorem 3.5, we may assume that G has class 3, and that $|G_3| = p^n$. Let $x \in G - G_2$. We will show that $C_G(xG_3)/G_3$ is abelian, and by Lemma 1.2 of [MS], we will get the result. Assume first that $[x, G'] < G_3$, and let M be a maximal subgroup of G_3 that contains $[x, G']$. Then xM centralizes G'/M in G/M. As in [MS, 1.3(v)] we conclude easily that $C_G(xG_3)/G_3$ is abelian. Assume then that $[x, G'] = G_3$. We have $|[x, G']| = p^n$, and $|C_{G'}(x)| = p^n$, then $G_3 = C_{G'}(x)$. We have $[C_G(x): C_{G'}(x)] = p^n$, since $|C_G(x)| = p^{2n}$; but $C_G(x)/C_{G'}(x)$ is abelian, thus $C_G(xG_3)/G_3 = G'/G_3 \times C_G(x)/G_3$ is abelian.

We now fix some notation: if F is a field, we denote the free Lie algebra on d generators over F by $L_F(d)$. We can look at $L_F(d)$ as a graded Lie algebra, with graduation $\bigoplus_i L_F(d)_i$ associated with the filtration given by the lower central series. If L is a Lie algebra over F, and F is an extension of the field K, we denote by L_K the algebra L viewed as a Lie algebra over K.

LEMMA 3.7 [DS]. – *Let $F = GF(p^n)$, $K = GF(p)$. Then the kernel of the canonical homomorphism from $L_K(2n)$ onto $L_F(2)_K$ is generated, as a Lie ideal, by elements of degree 2.*

THEOREM 3.8. – *Let G be an* NC-*p-group of class $c \geq 4$. Let L be the associated graded Lie algebra of the group G/G_c. Let F and K be as above. Then there exists an F-Lie ideal J of $L_F(2)$ such that L is isomorphic to $(L_F(2)/J)_K$.*

PROOF. – By Theorem 3.6, $(L_F(2)_K)_1 \oplus (L_F(2)_K)_2$ is isomorphic, as a K-Lie algebra, to $L_1 \oplus L_2$. Thus, by Lemma 3.7, $L_F(2)_K$ projects canonically onto L.

Consider $J = \mathrm{Ker}\,\varphi$, where φ is the canonical projection of $L_F(2)_K$ onto L, as K-Lie algebras. We show that J is closed under field multiplication by elements of F. Let $x \in J$. We may assume $x \in J_r$, for some $r < c$. If $\vartheta \in F$, we show that $x\vartheta \in J_r$. By our assumption on G, and Proposition 3.3, we would have otherwise $[x\vartheta, L_F(2)_1] \not\subseteq J_{r+1}$. But $[x\vartheta, L_F(2)_1] = [x, (L_F(2)_1)\vartheta] = [x, L_F(2)_1] \subseteq J_{r+1}$, contradiction.

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