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Divisible Designs Admitting a Suzuki Group as an Automorphism Group.

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Sunto. – *Si costruiscono, facendo uso delle rette dei piani di Lüneburg e degli ovali di Tits, due classi di disegni divisibili ipersemplici che ammettono il gruppo di Suzuki $S(q)$ ($q = 2^{2t+1}$ con $t \geqslant 1$) come gruppo di automorfismi. Inoltre si studiano le strutture ottenute determinandone le orbite di $S(q)$.*

1. – Introduction.

The Suzuki group are among the finite simple groups which allow a representation as a permutation group on interesting geometrical objects, for instance on the Tits ovoids and the Lüneburg planes (see for instance [Lu_1]). Starting from these geometries we construct divisible designs (see the next section and [BJL] for the notation and definitions) admitting a Suzuki group as an automorphism group.

In a previus paper, one of the authors has defined the concept of a 2-R-homogeneous permutation group (for an equivalence relation R on a set X) by the property that the group respects the equivalence classes of R and is transitive on the transversal sets of size two of X (see Spera [Sp_2]). Such a 2-R-homogeneous group G allows the construction of an (s, k, λ)-divisible design (X, B^G) (see [Sp_2]) where B denotes an R-transversal k-subset chosen as starter block. This generalizes the construction of 2-designs by 2-homogeneous permutation groups.

The idea of using the set of all lines of a finite translation plane π as the set X, the parallelism relation as R and an automorphism group G which is 2-R-transitive on the set X (and therefore 2-transitive on the line at infinity) restricts the possibilities for π to the desarguesian planes and the Lüneburg planes (Schulz [Sch], Czerwinski [Cz], see also [Ka] or [Lu_1]). The desarguesian case has been considered in [Sp_3] where a class of divisible designs has been constructed choosing a subovoid of the Tits ovoid as base block.

So in the present paper, the set X is taken to be the set of lines of the Lüneburg plane $\pi(L)$ of order q^2 with $q = 2^{2t+1}$, $t > 1$, the equivalence relation R as the parallelism relation on X and the group G as the product of the translation group of $\pi(L)$ and the Suzuki group $S(q)$ in the representation on the Lüneburg plane. We consider three possibilities for the starter block. Besides the divisible design obtained in a standard way from the dual structure of the Lüneburg plane, one construction gives a $(q^2, q, q-1)$-divisible design with $q^2(q^2+1)$ points and $q^5(q^2+1)$ blocks, and one a $(q^2, q^2+1-q, (q^2+1-q)(q-1))$-divisible design on the same set of points and, in some way, complementary blocks. Both types admit $S(q)$ as an automorphism group and are hypersimple.

2. – Basic definitions.

Let X be a set and R an equivalence relation on X. If x is an element of X, we shall denote by $[x]$ the equivalence class containing x and by $\mathcal{R}$ the set of all equivalence classes. A subset B of X is said to be an *R-transversal k-subset* of X if $|B| = k$ and B meets each equivalence class in at most one element of X. If $Y \subset X$, we will denote by $[Y]$ the union of all equivalence classes which meet Y. Suppose now that s, k, λ and v are positive integers with $1 < k < v$ and $s < v$. Let X be a finite set of cardinality v endowed with an equivalence relation R and $\mathcal{B}$ a family of R-transversal k-subsets of X. Then $D = (X, \mathcal{B})$ is said to be an (s, k, λ)-*divisible design* (in short an (s, k, λ)-DD) if:

i) $[x] = s$ for every $x \in X$;

ii) for every $x, y \in X$ with $[x] \neq [y]$ there are exactly λ elements of $\mathcal{B}$ containing x and y.

The elements of X are called *points*, the elements of $\mathcal{B}$ *blocks* and those of $\mathcal{R}$ *point classes*. In the case where every block meets every points class, D is called *transversal*. It is well known that, for an (s, k, λ)-DD with v points and b blocks,each point belongs to exactly r blocks and

$$r(k-1) = (v-s)\lambda \quad \text{and} \quad bk = vr\,.$$

A DD is called *μ-near-symmetric* if $\mu = b/(sv)$ is a positive integer which divides λ, whereas it is said to be *hypersimple* if, for every $B \in \mathcal{B}$ and $x, y \in [B]$ with $[x] \neq [y]$ there exists exactly one block B' containing x and y and such that $[B'] = [B]$. Notice that the notion of hypersimple DD contains the one of simple DD (that is without repeated blocks).

Let G be a permutation group on the set X and R an equivalence relation on X which is G-admissible (that is $x, y \in X$ and xRy imply $(x^g)\,R(y^g)$ for every $g \in G$). Then the triple (G, X, R) is said to be an *R-permutation group* (see [Sp_1]). If t is a positive integer and $\Omega = (G, X, R)$ is an R-permutation group, then Ω will be called *t-R-homogeneous* (*t-R-transitive*) if for every two R-transversal t-subsets $S = \{x_1, x_2, \ldots, x_t\}$ and $S' = \{y_1, y_2, \ldots, y_t\}$ of X there exists $g \in G$ such that $S' = S^g$ ($y_i = (x_i)^g$ for all $i = 1, 2, \ldots, t$). The fol-

Moreover we put

$$\mu(k) = k^{-2^t-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k^{\sigma+2} & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k^{\sigma+1} \end{bmatrix}$$

and denote with $\mu^*(k)$ the projectivity associated with $\mu(k)$. It is well-known that $T = \{\delta^*(a, b) \mid a, b \in K\}$ and $Z = \{\mu^*(k) \mid k \in K - \{0\}\}$ are subgroup of $S(q)$.

THEOREM 1 ([Su]). – i) *$S(q)$ acts 2-transitively on the ovoid $\mathcal{O}$.*

ii) *ZT is a group of order $q^2(q-1)$ and it is the stabilizer of U in $S(q)$, that is $ZT = (S(q))_U$.*

iii) *If $\gamma \notin (S(q))_U$, then γ can be written uniquely as $\gamma = \mu^*(k)\,\delta^*(a, b)\cdot \omega^*\delta^*(c, d)$ where $k \in K - 0$, $a, b, c, d \in K$ and ω^* is the projectivity associated with*

$$w = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Notice that $\delta(a, b)$, $\mu(k)$ and ω belong to $SL(4, q)$ fot every $a, b \in K$ and $k \in K^*$.

Now, let $H(\infty)$ be the cone over the line $I = \{P \in PG(3, q) \mid P$ is a zero of $x_0 = 0 = x_2\}$, that is $H(\infty) = \{(0, x_1, 0, x_3) \in K^4 \mid x_1, x_3 \in K\}$, and $H(a, b)$ the one over $I^{\omega^*\delta^*(a, b)}$. Then $L = \{H(a, b) \mid a, b \in K\} \cup \{H(\infty)\}$ is a spread of K^4 whose associated plane $\pi(L)$ is the Lüneburg plane. The line at infinity of $\pi(L)$ is (or can be thought as) $\mathcal{O}$.

REMARK 1. – Remember that T acts regularly on $\mathcal{O} - \{U\}$. From which we have that, if $P \in \mathcal{O} - \{U\}$ then there exists exacly one $\delta^*(a, b) \in T$ such that $P = \langle(1, 0, 0, 0)\rangle^{\delta^*(a, b)} = \langle(1, 0, 0, 0)^{\delta(a, b)}\rangle$. If we set $v(a, b) := (1, 0, 0, 0)^{\delta(a, b)}$ and define Ψ: $\mathcal{O} \to L$ by $\Psi(U) := H(\infty)$ and $\Psi(P) := H(a, b)$ if $P \in \mathcal{O} - \{U\}$ and $P = \langle v(a, b)\rangle$, then it is easy to see that Ψ is a one-to-one correspondence. Thus we obtain that the action of $S(q)$ on L is «equivalent» at its action on $\mathcal{O}$. So we get that if $\gamma \in S(q)$, then $H(a, b)^\gamma = \Psi(\langle v(a, b)\rangle^\gamma)$ and being $\Psi(\langle v(a, b)\rangle^\gamma) = \Psi(\langle v(a, b)^\gamma\rangle)$ we have that $H(a, b)^\gamma = H(c, d)$ when $\langle v(a, b)^\gamma\rangle = \langle v(c, d)\rangle$ and $H(a, b)^\gamma = H(\infty)$ when $\langle v(a, b)^\gamma\rangle = U$; while $H(\infty)^\gamma = \Psi(U^\gamma) = H(c, d)$ if $U^\gamma = \langle v(c, d)\rangle$ and $H(\infty)^\gamma = H(\infty)$ if γ fixes U.

REMARK 2. – As $\mu(k)$, ω ,$\delta(a, b) \in SL(4, q)$ for every $k \in K^*$ and $a, b \in K$, we obtain (see [Lu$_1$]) that $S(q)$ is, up to isomorphism, a subgroup of $PSL(4, q)$. But $PSL(4, q) \simeq SL(4, q)$ since for $q = 2^{2t+1}$ the center of $SL(4, q)$ is trivial. Thus $S(q)$ is, up to isimorphism, a subgroup of $SL(4, q)$ and so it acts on the set of vectors of K^4 too. Precisaly $v^{\alpha^*} := v^{\alpha}$ for every $v \in K^4$ if α^* is a generical element of $S(q)$.

3. – The constructions.

Essential for this article is the following behaviour of the automorphism group of the Lüneburg planes.

PROPOSITION 2. – *Let $\mathcal{L}$ be the set of lines of the Lüneburg plane $\pi(L)$. If R is the parallelism relation on $\mathcal{L}$ and if $\mathcal{T}$ is the translation group of $\pi(L)$, then* $(\mathcal{T}S(q), \mathcal{L}, R)$ is an R-permutation group which is *2-R-transitive.*

PROOF. – Choose L as a representative system of the R-classes in $\mathcal{L}$. Since $\mathcal{T}$ is transitiven on each R-class and $S(q)$ is transitive on L we get $\mathcal{T}S(q)$ is transitive on $\mathcal{L}$. So, it is enough (see [Sp$_1$]) to show that the stabilizer $(\mathcal{T}S(q))_{H(\infty)}$ of the line $H(\infty)$ in $\mathcal{T}S(q)$ is transitive on $\mathcal{L} - [H(\infty)]$, being $[H(\infty)]$ the R-class represented by $H(\infty)$. By Remark 1 and ii) of Theorem 1, we have that $ZT = S(q)_{H(\infty)}$. Thus, if $\mathcal{T}(\infty)$ denotes all translations which fix $H(\infty)$, this is $\mathcal{T}(\infty) = \{\tau_v \in \mathcal{T} | v \in H(\infty)\}$, we get that $\mathcal{T}(\infty)\, ZT \subseteq (\mathcal{T}S(q))_{H(\infty)}$. But $\mathcal{T}(\infty)$ is transitive on each R-class which is different to $[H(\infty)]$ since it is a component of a spread of K^4. Moreover ZT is transitive on $\mathcal{O}' = \mathcal{O} - \{U\}$ and so also on $L - \{H(\infty)\}$. Therefore $(\mathcal{T}S(q))_{H(\infty)}$ is transitive on $\mathcal{L} - \{[H(\infty)]\}$, and the proposition is proved. ■

By Proposition 1, we now are able to construct (s, k, λ)-divisible designs from the $2-R-$permutation group described above. Of course L is a R-transversal set in $\mathcal{L}$. So, if we choose L as a base block, it is an easy computation to show that the obtained divisible design $D(\Omega, L)$ is a $(q^2, q^2+1, 1)$-transversal design with $q^2(q^2+1)$ points. Here Ω denotes the R-permutation group $(\mathcal{T}S(q), \mathcal{L}, R)$. But $D(\Omega, L)$, as it is known, can be obtained in a standard way considering the dual structure of the Lüneburg plane.

That there are no other translation planes exept the Lüneburg planes and the desarguesian planes which allow a construction similar to that of $D(\Omega, L)$ shows the following

PROPOSITION 3. – *Let π be a finite non-desarguesian translation plane and T its translation group. Suppose D is a divisible design constructed by a representative system L of the parallel classes of the lines of π as a base block and by a group G, with $T \subseteq G \subseteq \operatorname{Aut}\pi$, fixing no parallel class and possessing a flag*

(P, h) such that $G_{(P, h)}$ is transitive on the set of lines through P different from h. Then D contains a substructure consisting of all points and a subset of blocks of D which is isomorphic to a design $D(\Omega, L)$ of the form constructed above.

PROOF. – Since $T \subseteq G$ we have that L^G is the set of all lines of π. Now $G_{(P, h)}$, being transitive on the lines unequal h through P, operates transitively on the points at infinity different from the parallel class $[h]$ of h. Since G does not fix a parallel class it operates 2-transitively on the line of infinity of π. By a theorem of Schulz and Czerwinski (see Kallaher [Ka] 4.3 (16) or Lüneburg [Lu_1]), π is either desarguesian or a Lüneburg plane, the first case of which is excluded by our assumptions. By Theorem 39.2 of Lüneburg [Lu_1] and the proof of 39.3, G_P contains a subgroup isomorphic to $S(q)$. All groups $S(q)$ contained in $P\Gamma L(4, q)$ are conjugate (see 27.3 of [Lu_1]), each possesses an ovoid as an orbit and acts in its natural representation on the set of lines through P (Dembowski, see [Lu_1] 28.4). Hence, up to isomorphism, the design constructed by the group $TS(q)$ consists of the set of all points and a subset of the blocks of D. ■

To get more interesting divisible designs, we choose a proper subset of L. In the following we shall consider the base block

$$B = \{H(0, b) \mid b \in K\}. \tag{1}$$

THEOREM 2. – *Let $q = 2^{2t+1}$ where t is a positive integer. Then there exists an $(q^2, q, q-1)$-divisible design D with $q^2(q^2+1)$ points and $q^5(q^2+1)$ blocks. Moreover D admits the Suzuki group $S(q)$ as an automorphism group which is 2-transitive on the set of point classes.*

PROOF. – Let B be defined as in (1). Our first goal is to determine G_B where G denotes the group $\mathcal{T}S(q)$. Let $f \in G_B$ and suppose that $f = \tau_v \gamma$ where τ_v is the translation given by the vector v and $\gamma \in S(q)$. Then, for every $b \in K$, $H(0, b)^{\tau_v \gamma} = H(0, b')$ for some $b' \in K$. But $H(0, b)^{\tau_v \gamma} = H(0, b)^\gamma + v^\gamma$. So $f \in G_B$ if and only if $H(0, b)^\gamma + v^\gamma = H(0, b')$. It follows that $H(0, b)^\gamma = H(0, b')$ and $v^\gamma \in H(0, b)^\gamma$ for every $b \in K$. Hence $\gamma \in S(q)_B$ and $v \in H(0, b)$ for every $b \in K$. From this we get $v = 0$ since L is a spread of K^4. Therefore $f \in G_B$ if and only if $f = \gamma \in S(q)_B$.

Case 1: $\gamma = \mu^*(k)\delta^*(c, d)$ for some $k \in K^*$ and $c, d \in K$.

For every $b \in K$ (see the Remark 1 above), $H(0, b)^\gamma = \Psi(\langle v(0, b)\rangle^\gamma) = \Psi(\langle (1, 0, 0, 0)\rangle^{\delta^*(0, b)\mu^*(k)\delta^*(c, d)})$ and, (see 21.5 and 21.4 in [Lu_1]) since $\delta^*(0, b)\mu^*(k)\delta^*(c, d) = \mu^*(k)\delta^*(0, k^{\sigma+1}b)\delta^*(c, d) = \mu^*(k)\delta^*(c, k^{\sigma+1}b + d)$, we obtain that

$$H(0, b)^\gamma = \Psi(\langle (1, 0, 0, 0)\rangle^{\mu^*(k)\delta^*(c, k^{\sigma+1}b + d)}) =$$

$$\Psi(\langle (1, 0, 0, 0)^{\delta(c, k^{\sigma+1}b+d)}\rangle) = \Psi(\langle v(c, k^{\sigma+1}b + d)\rangle) = H(c, k^{\sigma+1}b + d).$$

Therefore, in the case 1, $\gamma \in S(q)_B$ if and only if $c = 0$ and so, if and only if $\gamma = \mu^*(k)\delta^*(0, d) \in ZC(T)$ where $C(T) = \{\delta^*(0, d) \mid d \in K\}$ is the center of T.

Case 2: $\gamma = \mu^*(k)\delta^*(c, d)\omega^*\delta^*(e, h)$ where $k \in K^*$, and $c, d, e, h \in K$.

Since ZT fixes U we have $H(\infty)^\gamma = \Psi(U^\gamma) = \Psi(U^{\mu^*(k)\delta^*(c, d)\omega^*\delta^*(e, h)}) = \Psi(U^{\omega^*\delta^*(e, h)}) = \Psi(\langle(1, 0, 0, 0)^{\delta(e, h)}\rangle) = \Psi(\langle v(e, h)\rangle) = H(e, h)$. So if $e = 0$ we get $H(\infty)^\gamma \in B$ against the assumption $\gamma \in S(q)_B$ and $H(\infty) \notin B$. It follows that $e \neq 0$. In the same way, considering $H(\infty)^{\gamma^{-1}}$, we also obtain that $c \neq 0$ since

$$\gamma^{-1} = \delta^*(e, h)^{-1}\omega^*\delta^*(c, d)^{-1}\mu^*(k^{-1}) =$$

$$\delta^*(e, h + ee^\sigma)\,\omega^*\delta^*(c, d + cc^\sigma)\,\mu^*(k^{-1}).$$

Now, since $\gamma \in S(q)_B$, we have that $H(0, b)^\gamma = \Psi(\langle v(0, b)\rangle^\gamma) \in B$ for every $b \in K$. So, also for $b = dk^{-\sigma-1}$, we have that $\Psi(\langle v(0, dk^{-\sigma-1})\rangle^\gamma) \in B$. But

$$\langle v(0, dk^{-\sigma-1})\rangle^\gamma = \langle(1, 0, 0, 0)\rangle^{\delta^*(0, dk^{-\sigma-1})\mu^*(k)\delta^*!(c, d)\omega^*\delta^*(e, h)} =$$

$$\langle(1, 0, 0, 0)\rangle^{\mu^*(k)\delta^*(c, 0)\omega^*\delta^*(e, h)} = \langle(1, 0, 0, 0)^{\delta(c, 0)\omega\delta(e, h)}\rangle =$$

$$\langle(1, c^{\sigma+2}, c, 0)^{\omega\delta(e, h)}\rangle = \langle(c^{\sigma+2}, 1, 0, c)^{\delta(e, h)}\rangle =$$

$$\langle(c^{\sigma+2}, [eh + e^{\sigma+2} + h^\sigma]\,c^{\sigma+2} + 1 + ec, ec^{\sigma+2}, hc^{\sigma+2} + c)\rangle.$$

Thus necessarily $ec^{\sigma+2} = 0$, a contradiction.

Therefore we have proved that $G_B = ZC(T)$ and so $|G_B| = (q-1)q$. Now we are able to determine the parameters (see Proposition 1) of the regular (s, k, λ)-divisible design $D(\Omega, B)$ whose set of points is $\mathcal{L}$ and the one of bloks is B^G. Of course it has $q^2(q^2+1)$ points and each point class holds $s = q^2$ points. Moreover $k = |B| = q$ and if b denotes the number of blocks, we have $b = |G|/|G_B| = |\mathcal{TS}(q)|/|ZC(T)| = [q^4(q^2+1)q^2(q-1)]/[(q-1)q] = q^5(q^2+1)$ where as $\lambda = [|G|k(k-1)]/[|G_B|v(v-s)] = q^5(q^2+1)q(q-1)/[q^2(q^2+1)q^4] = q-1$. Clearly G is an automorphism group of $D(\Omega, B)$, so also $S(q)$ is an automorphism group of $D(\Omega, B)$ being $S(q)$ a subgroup of G. Moreover, since $S(q)$ is 2-transitive on L (being 2-transitive on $\mathcal{O}$) and L is a representative system for point classes, we get that $S(q)$ is 2-transitive on the set of point classes because of the G-admissibility of the relation R. This completes the proof. ∎

COROLLARY. – *Let q and $S(q)$ be as in Theorem 2. Then there exists a (s', k', λ')-divisible design admitting $S(q)$ as an automorphism group and having the following parameters: $v' = q^2(q^2+1)$, $b' = q^5(q^2+1)$, $s' = q^2$, $k' = q^2+1-q$ and $\lambda' = (q^2+1-q)(q-1)$.*

PROOF. – Let $G = \mathcal{C}S(q)$, B be as in (1) and put $B' := L - B$. Since, as seen in the proof of Theorem 2, $G_B = ZC(T)$ and $ZC(T)$ fixes L, we obtain that $G_B \subseteq G_{B'}$. Set

$$I = \{(a, b) \mid a, b \in K \text{ and } a \neq 0\} \cup \{\infty\}.$$

Of course we get $B' = \{H(x)/x \in I\}$. If $f \in G_{B'}$, we have that $f = \tau_v \gamma$ where $\tau_v \in \mathcal{T}$ and $\gamma \in S(q)$. For every $x \in I$ such that $H(x)^f = H(x)^\gamma + v^\gamma = H(y)$. It follows that $H(x)^\gamma = H(y)$ and $v^\gamma \in H(x)^\gamma$ for every $x \in I$. So $v = 0$ and $f \in S(q)_{B'}$. Therefore $G_{B'} = S(q)_{B'}$ and, since $S(q)$ fixes L, we obtain that $G_{B'} = S(q)_{B'} = S(q)_B = ZC(T)$. Thus $b' = |G|/|G_{B'}| = [q^4(q^2+1)\, q^2(q-1)]/[(q-1)\, q] = q^5(q^2+1)$. Moreover $k' = |B'| = |L - B| = (q^2+1) - q$ and, being $v' = q^2(q^2+1)$ and $s' = q^2$, we get that $\lambda' = [\,|G| k'(k'-1)]/[\,|G_{B'}|\, v'(v'-s')] = [q^4(q^2+1)q^2(q-1)(q^2+1-q)(q^2-q)]/[(q-1)\, qq^2(q^2+1)\, q^4] = (q^2+1-q)(q-1)$. Thus the corollary is shown. ■

In the following proposition we give the orbits of the divisible designs constructed in Theorem 2. (Clearly an analogous proposition can be shown for the ones of corollary).

PROPOSITION 4. – *Let $S(q)$ be the Suzuki group, where $q = 2^{2t+1}$ with $t > 0$, and D the $(q^2, q, q-1)$-divisible designs constructed in the above Theorem 2, then:*

i) *The set of points of D is split by $S(q)$ into one orbit of size q^2+1, one orbit of size $(q-1)(q^2+1)$ and one orbit having size $(q-1)q(q^2+1)$. Each of this orbit meets every point class in the same number of points.*

ii) *$S(q)$ splits the set of blocks of D into q^4 orbits of size $q(q^2+1)$ each.*

PROOF. – Let $H(x) + v$ be a point of D where $x \in \{(a, b) \mid a, b \in K\} \cup \{\infty\}$ and $v \in K^4$. For every $H(x') \in L \subseteq \mathcal{L}$ there exists some $\gamma \in S(q)$ such that $H(x)^\gamma = H(x')$. So $(H(x)+v)^\gamma = H(x)^\gamma + v^\gamma = H(x') + v^\gamma \in [H(x')]$. Thus $(H(x) + v)^{S(q)} \cap [H(x')] \neq \emptyset$, and since L is a representative system of the point classes, we get that any orbit meets any point class. Hence it follows that every orbit has some representatives on $[H(\infty)]$. Moreover, $(H(x)+v)^{S(q)}$ meets every point class in the same number of points since $S(q)$ is a R-permutation group on $\mathcal{L}$ (see Prop. 2). Of course the orbit $H(\infty)^{S(q)}$ is L because of the transitivity of $S(q)$ on L; so we have $|H(\infty)^{S(q)}| = q^2+1$.

Let $e = (0, 0, 1, 0)$ and consider the orbit $(H(\infty)+e)^{S(q)}$. It has size $|S(q)|/|S(q)_{(H(\infty)+e)}|$. But $S(q)_{(H(\infty)+e)} = T$. In fact if $\delta(a, b) \in T$, then $(H(\infty)+e)^{\delta(a,b)} = H(\infty)^{\delta(a,b)} + e^{\delta(a,b)} = H(\infty) + (0, a^{\sigma+1}+b, 1, a^\sigma) = H(\infty) + e$ since $(0, a^{\sigma+1}+b, 1, a^\sigma) - e \in H(\infty)$. Thus $T \subseteq S(q)_{(H(\infty)+e)}$. Vice versa, if $\gamma \in S(q)_{(H(\infty)+e)}$ then $(H(\infty)+e)^\gamma = H(\infty)+e$ iff $H(\infty)^\gamma = H(\infty)$ and $e^\gamma - e \in H(\infty)$ iff $\gamma \in ZT$ and $e^\gamma - e \in H(\infty)$. But $\gamma \in ZT$ implies that $\gamma = \mu(k)\,\delta(a, b)$ for some $k \in K^*$ and $a, b \in K$. Thus $e^\gamma - e = e^{\mu(k)\,\delta(a,b)} - e = (0, k^{-2^t}(a^{\sigma+1}+b), k^{-2^t} - 1, k^{-2^t} a^\sigma)$. Hence we deduce that $k = 1$ is necessary for $e^\gamma - e \in H(\infty)$ and so $\gamma \in T$. Therefore we have that $|(H(\infty)+e)^{S(q)}| = |S(q)|/|T| = (q^2+1)\,q^2(q-1)/q^2 = (q^2+1)(q-1)$ and so we get an orbit with $(q^2+1)(q-1)$ elements. Now, consider the orbit $(H(\infty)+e')^{S(q)}$, where $e' = (1, 0, 0, 0)$. By the same method as above we get that $S(q)_{H(\infty)+e'} = C(T)$ and so $|(H(\infty)+e')^{S(q)}| = |S(q)|/|C(T)| = (q^2+1)\,q^2(q-1)/q = (q^2+1)\,q(q-1)$. There are no other orbits. In fact the considered orbits are distict, being of different size, and the total number of elements of their union is equal to $|\mathcal{L}|$. Thus i) is proved. (Note that the existence of the three orbits and their sizes can be deduced from [Lu$_2$] or [Lu$_1$] page 139).

Now, we consider the block orbit $B^{S(q)}$. In the proof of Theorem 2 was shown that $S(q)_B = ZC(T)$. Thus $|B^{S(q)}| = |S(q)|/|S(q)_B| = (q^2+1)\,q^2(q-1)/[(q-1)q] = (q^2+1)q$. Of course $B^{\tau_v} \in B^G$ does not belong to $B^{S(q)}$ for every $v \in K^4 - \{0\}$ and, being $S(q)_{B^{\tau_v}} = \tau_v^{-1} S(q)_B \tau_v$, we also have that $|(B^{\tau_v})^{S(q)}| = (q^2+1)\,q$. Therefore necessarily there are exactly q^4 orbits and so ii) is shown too. ∎

Note that the divisible designs of Theorem 2 and the ones of its corollary are not μ-near-symmetric although $b/(sv)$ is an integer in both cases (but does not divides λ). However we can state the following

PROPOSITION 5. – *The $(q^2, q, q-1)$-divisible designs constructed in Theorem 2 and the $(q^2, q^2+1-q, (q^2+1-q)(q-1))$-divisible designs given in the corollary are both hypersimple.*

PROOF. – Let $B = \{H(0, b)\,|\,b \in K\}$ be the base block of a $(q^2, q, q-1)$-divisible design D constructed in Teorem 2. Clearly $\mathcal{C}ZC(T) \subseteq G_{[B]}$ where G, as before, denotes $\mathcal{C}S(q)$. If $f = \tau_v\gamma \in G_{[B]}$, since we have $(H(0, b)+w)^{\tau_v\gamma} = H(0, b)^\gamma + (w+v)^\gamma$ for every $b \in K$ and $w \in K^4$, we obtain that $H(0, b)^\gamma + (w+v)^\gamma \in [B]$. This implies that $H(0, b)^\gamma \in B$ and so $\gamma \in ZC(T)$. Therefore $\mathcal{C}ZC(T) = G_{[B]}$. Of course $(G_{[B]})_B = G_B$ because $G_B \subseteq G_{[B]}$. Now it is an easy exercise to see that G_B is 2-transitive on B and that therefore $G_{[B]}$ is 2-R-transitive in its action on $[B]$. So, being B a transversal subset of $[B]$ of maximal size, we obtain that $([B], B^{G_{[B]}})$ is a transversal (s, k, λ)-divisible design where $s = q^2$, $k = q$, $v = sk = q^3$, $b = |G_{[B]}|/|G_B| = [q^4 q(q-1)]/[q(q-1)] = q^4$ and $\lambda = [|G_{[B]}|k(k-1)]/[|G_B|v(v-s)] = [q^4 q(q-1)]/[q^3(q^3-q^2)] = 1$. Thus, being G transitive on block set of D, we infer that D is hypersimple. Let now D' be a divisible design constructed in the corollary by the base block B' and suppose that $x, y \in [B']$ with $[x] \neq [y]$. As noticed at the beginning of this section, $D(\Omega, L)$ is a transversal $(q^2, q^2+1, 1)$-divisible design. So there exists exactly one block $L^{\tau_v\gamma}$ containing x and y, where $\tau_v\gamma \in \mathcal{C}S(q)$. Let $z, u \in L^{\tau_v\gamma}$ with $z \neq u$ and $z, u \in [B]$. Since, as see above, D is hypersimple there is exactly one block B^ξ containing z and u where $\xi \in \mathcal{C}ZC(T)$. But $L^\xi = L^{\tau_v\gamma}$ since they are both blocks of $D(\Omega, L)$ through the same points z and u. Therefore B'^ξ is a block of D' with $[B'^\xi] = [B']$. If B'^ζ is an other block of D' through x and y with $[B'^\zeta] = [B']$, then we have that

$B'^{\xi} = B'^{\zeta}$ since $L^{\xi} = L^{\zeta}$ being $D(\Omega, L)$ a $(q^2, q^2+1, 1)$-divisible design. Therefore D' also is hypersimple because of the transitivity of G on the block set of D'. ■

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