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Relatively Maximal Convergences.

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Sunto. – *Topologie, pretopologie, paratopologie e pseudotopologie sono importanti classi di convergenze, chiuse per estremi superiori (superiormente chiuse) ed inoltre caratterizzabili mediante le aderenze di certi filtri. Convergenze J-massimali in una classe superiormente chiusa $D \supset J$, cioè massimali fra le D-convergenze aventi la stessa imagine per la proiezione su J, svolgono un ruolo importante nella teoria dei quozienti; infatti, una mappa J-quoziente sulla convergenza J-massimale in D è automaticamente D-quoziente; d'altro lato, per $D \supset J$, una mappa D-quoziente conserva più proprietà topologiche che una mappa J-quoziente. Si stabilisce una caratterizzazione generale della J-massimalità in D quando J ed D appartengono alle classi di topologie, pretopologie, paratopologie e pseudotopologie. In casi particolari si ritrova le topologie di accessibilità di Whyburn e di forte accessibilità di Siwiec.*

By a *convergence* ξ on a set X we understand here a relation between X and the set of all filters on X, denoted

$$x \in \lim_\xi \mathcal{F},$$

such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ and such that the principal filter (x) of x converges to x for every $x \in X$. Given a convergence ξ on X and a convergence τ on Y, a mapping $f: X \rightarrow Y$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$ for every filter $\mathcal{F}$ on X; a convergence ξ on X is *finer* than a convergence ζ on X (in symbols, $\xi \geqslant \zeta$) if the identity i_X is continuous from ξ to ζ.

Several important classes of convergences, like topologies, pretopologies, paratopologies and pseudotopologies, are closed for arbitrary suprema in the class of convergences. To every sup-closed class of convergences there corresponds a projection J (i.e., an isotone decreasing idempotent map) such that this class (of *J-convergences*) is equal to $\{\xi: J\xi \geqslant \xi\}$.

If J and D are projections with $J \leqslant D$, then a D-convergence τ is *J-maximal at x_0 with respect to the class of D-convergences* (shortly, *J-maximal at x_0 in D*) if for every D-convergence $\xi \geqslant \tau$ such that $J\xi = J\tau$, one has $x_0 \in \lim_\tau \mathcal{F}$ implies $x_0 \in \lim_\xi \mathcal{F}$.

Thanks to the characterizations of various classes of quotient maps in terms of projections on classes of convergences [2, 6] ([1]) , relative maximality of an image space automatically improves the quotient class of the corresponding map.

V. Kannan in [5] characterizes implicitly (i.e., without introducing the notion of maximality) topologies that are topologically maximal with respect to the class of pretopologies; one of his characterizations amounts to accessibility (a notion due to G. T. Whyburn [8]). Unaware of [5] S. Dolecki and G. Greco gave in [3] a characterization of pretopologies that are topologically maximal with respect to the class of pretopologies extending the result of Kannan. In [4] we provided a unified characterization of topological maximality with respect to pretopologies, paratopologies and pseudotopologies. In particular we showed that a topology is topologically maximal with respect to paratopologies if and only if it is strong accessibility (the notion is due to F. Siwiec [7]).

In this paper, we succeed to give a single general criterion of J-maximality in D (in terms of adherences for special classes of filters dependent on J and D), where J and D are projections on topologies, pretopologies, paratopologies or pseudotopologies (Theorem 2); it specializes to the known criteria mentioned above.

Quotient maps onto maximal convergences.

If $f\colon X \to Y$ is surjective and if ξ is a convergence on X, then $f\xi$ stands for the finest convergence on Y making f into a continuous mapping. If τ is a convergence on Y, then f is a *J-quotient* map if and only if $\tau = J(f\xi)$. We consider a broader notion of *J-map*, i.e., such that $\tau \geqslant J(f\xi)$. A J-map is J-quotient if and only if it is continuous. It turns out that J-quotient maps onto J-maximal convergences in D are D-quotient maps.

THEOREM 1 [2], [4]. – *A D-convergence τ is J-maximal in D if and only if every J-quotient map f from a D-convergence ξ to τ is a D-quotient map.*

PROOF. – Let f be a J-quotient map from (X, ξ) onto (Y, τ) and let $D(f\xi) > \tau$. As $J(D(f\xi)) = J(f\xi) = J\tau$, the D-convergence τ is not J-maximal in D.

Conversely, if τ is not J-maximal in D, then there exists a D-convergence ξ such that $\xi > \tau$ and $J\xi = J\tau = \tau$. Therefore the identity map $\iota\colon \xi \to \tau$ is J-quotient but not D-quotient. ■

([1]) Bi-quotient maps coincide with pseudotopologically quotient maps, hereditarily quotient with pretopologically quotient [6], almost open with convergence quotient, countably bi-quotient with paratopologically quotient, and quotient with topologically quotient [2]; here the first term corresponds to classical terminology and the latter to convergence-theoretic terminology.

In the terms of J-maps, classical quotient maps are continuous topological quotient maps. It was pointed out in [6] that *pseudo-open* (i.e., *hereditarily quotient*) maps coincide with pretopological quotients ($\tau \geq P(f\xi)$), while *bi-quotient* maps coincide with pseudotopological quotients ($\tau \geq S(f\xi)$), and in [2] that *countably bi-quotient* maps coincide with paratopological quotients ($\tau \geq P_\omega(f\xi)$) and *almost open* maps coincide with convergence quotients ($\tau \geq f\xi$).

Actually, for the discussed four classes of convergences the sufficiency part of the above theorem can be improved on replacing «a convergence ξ» by «a J-convergence ξ» (see [4, Theorem 4.4]). In particular, one recovers the results of G. Whyburn [4, Theorem 4.4] and [9, Theorem 2] that a T_1 topology is accessibility if and only if every quotient map onto it is pseudo-open, and of F. Siwiec [7, Theorem 4.3] that a topology is strong accessibility if and only if every quotient map onto it is countably bi-quotient.

Adherence-representable convergences.

Let $\mathcal{F}$ be a filter on a convergence space X. The *adherence* of $\mathcal{F}$ is the union of the limits of all filters that are finer than $\mathcal{F}$:

$$\operatorname{adh} \mathcal{F} = \bigcup_{\mathcal{G} \supset \mathcal{F}} \lim \mathcal{G} . \tag{1}$$

The *closure* of a subset A of X is equal to the adherence of the principal filter of A. A subset A of X is *ξ-closed* whenever for every filter $\mathcal{F}$ with $A \in \mathcal{F}$, one has $\lim_\xi \mathcal{F} \subset A$. A set is *$\xi$-open* if its complement is ξ-closed. A convergence with unicity of limits is called *Hausdorff*.

A convergence is a *pseudotopology* (G. Choquet [1]) if $x \in \lim \mathcal{F}$ whenever $x \in \lim \mathcal{U}$ for every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$. A convergence is a *pretopology* (G. Choquet [1]) if for every point x, its neighborhood filter

$$\mathcal{N}(x) = \bigcap_{x \in \lim \mathcal{F}} \mathcal{F}$$

converges to x. A pretopology is a *topology* if each neighborhood filter admits a base of open sets.

As we have already mentioned, the classes of pseudotopologies, pretopologies and topologies are closed for suprema. The corresponding projections are denoted by S (pseudotopologization) P (pretopologization) and T (topologization).

The projections S, P and T can be expressed in terms of adherences. Recall that the *grill* $\mathcal{H}^\#$ of a family $\mathcal{H}$ is the set of all G such that $G \cap H \neq \emptyset$. Families $\mathcal{F}$ and $\mathcal{H}$ *mesh* ($\mathcal{F} \# \mathcal{H}$) whenever $\mathcal{F} \subset \mathcal{H}^\#$ (equivalently $\mathcal{H} \subset \mathcal{F}^\#$).

$$\lim_{D\xi} \mathcal{F} = \bigcap_{\mathcal{F} \# \mathcal{H} \in \mathfrak{D}} \operatorname{adh}_\xi \mathcal{H} , \tag{2}$$

where $\mathfrak{D} = \mathfrak{D}(\xi)$ is respectively the class of all filters (case of S), of the principal filters (case of P) and of the principal filters of ξ-closed sets (case of T).

A convergence ξ is a *paratopology* [2] if (2) holds with $\mathfrak{D}$ being the class of countably based filters. It follows that the class of paratopologies is sup-closed; we denote the corresponding projection P_ω.

Characterization of maximality.

We study the cases where J and D correspond to one of the usual classes: pseudotopologies, paratopologies and pretopologies. The remaining case where $J = T$ is the class of topologies has been characterized in [4].

The maximality characterization theorem that we present in this section is restricted to convergences classes D and J corresponding to fixed families $\mathfrak{D}$ and $\mathfrak{J}$ in (2). The characterizing condition is the same as in [4, Theorem 3.1], but the argument although very similar to that used in the proof of that theorem, at one point uses a different property, namely that

$$\mathcal{H} \in \mathfrak{D}, \quad A \notin \mathcal{H} \Rightarrow \mathcal{H} \backslash A \in \mathfrak{D}\,. \tag{3}$$

This property is satisfied in all the considered cases ($\mathfrak{J} = \mathfrak{S}$, $\mathfrak{P}_\omega$, $\mathfrak{P}$). We were unable to find a unified proof for both the mentioned theorems.

THEOREM 2. – *Let $J \subset D$ be convergence classes defined by* (2) *with respect to families $\mathfrak{J} \subset \mathfrak{D}$ so that $\mathfrak{D}$ fulfills* (3) *and $\mathfrak{J}$ is independent of convergence. A D-convergence τ is J-maximal at x_0 in D if and only if for each $\mathcal{H} \in \mathfrak{D}$ with $x_0 \in \mathrm{adh}_\tau(\mathcal{H} \backslash x_0)$, there exists $\mathcal{G} \in \mathfrak{J}$ such that $x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash x_0)$ and*

$$\forall_{H \in \mathcal{H}} \; x_0 \notin \mathrm{adh}_\tau(\mathcal{G} \backslash H \backslash x_0)\,. \tag{4}$$

In [4, Theorem 3.1], for the topological maximality, the family of τ-closed subsets has the role of $\mathfrak{J}$.

PROOF. – ($\Rightarrow$) Let $\mathcal{H} \in \mathfrak{D}$ be such that $x_0 \in \mathrm{adh}_\tau(\mathcal{H} \backslash x_0)$ and such that for every $\mathcal{G} \in \mathfrak{J}$,

$$x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash x_0) \Rightarrow \exists_{H \in \mathcal{H}} \; x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash H \backslash x_0)\,. \tag{5}$$

Let $\mathcal{H}_0 = \mathcal{H} \backslash x_0$. Then $\mathcal{H}_0 \in \mathfrak{D}$ by our assumption. We define the following convergence θ:

$$\lim_\theta \mathcal{F} = \begin{cases} \lim_\tau \mathcal{F} \backslash \{x_0\}, & \text{if } \mathcal{H}_0 \# \mathcal{F}, \\ \lim_\tau \mathcal{F}, & \text{otherwise}\,. \end{cases} \tag{6}$$

It is a D-convergence strictly finer than τ at x_0. Let us show that $J\tau = J\theta$. To this end by (2) used for $\mathfrak{J}$ independent of convergences, it is enough to prove that for each $\mathcal{G} \in \mathfrak{J}$, the difference $\mathrm{adh}_\tau \mathcal{G} \backslash \mathrm{adh}_\theta \mathcal{G}$ is empty. Since this difference is included in $\{x_0\}$, we need only assume $x_0 \in \mathrm{adh}_\tau\, \mathcal{G} \backslash \mathrm{adh}_\theta \mathcal{G}$ and get a contradiction. We infer that $x_0 \notin \bigcap_{G \in \mathcal{G}} G$, hence $x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash x_0)$, thus by (5), there is $H \in \mathcal{H} \subset \mathcal{H}_0$ such that $x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash H \backslash x_0) = \mathrm{adh}_\theta(\mathcal{G} \backslash H \backslash x_0)$ by (6).

($\Leftarrow$) Let $\tau = D\tau$ be not J-maximal at x_0 in D: there exists $\xi = D\xi \geqslant \tau$ with $J\xi = J\tau$ and such that for some filter $\mathcal{F}$, $x_0 \in \lim_\tau \mathcal{F} \backslash \lim_\xi \mathcal{F}$. Consequently by (2), there exists a filter $\mathcal{H} \in \mathfrak{D}$ such that $x_0 \in \mathrm{adh}_\tau\, \mathcal{H} \backslash \mathrm{adh}_\xi\, \mathcal{H}$. Therefore, $x_0 \in \mathrm{adh}_\tau(\mathcal{H} \backslash x_0)$ and thus with the aid of $\mathcal{H}_0 = \mathcal{H} \backslash x_0$ we can define by (6) the convergence θ.

Let now $\mathcal{G} \in \mathfrak{J}$ be such that $x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash x_0)$. As $\xi \geqslant \theta > \tau$, one has $J\theta = J\tau$ and $\mathcal{G} \backslash x_0 \in \mathfrak{J}$, therefore by (2) for $\mathfrak{J}$ fixed, $x_0 \in \mathrm{adh}_\theta(\mathcal{G} \backslash x_0)$. By (6), there exists $H \in \mathcal{H}$ such that $H^c \cup \{x_0\} \in (\mathcal{G} \backslash x_0)^{\#}$, hence $x_0 \in \mathrm{adh}_\theta(\mathcal{G} \backslash H \backslash x_0)$ and thus (5) holds. ■

COROLLARY 3. – *A paratopology τ (resp. a pseudotopology τ), on a set X, is pretopologically maximal at x_0 in the class of paratopologies on X (resp. pseudotopologies on X) if and only if for each countably based filter $\mathcal{H}$ (resp. each filter), with $x_0 \in \mathrm{adh}_\tau(\mathcal{H} \backslash x_0)$, there exists $G \subset X$ such that $x_0 \in \mathrm{cl}_\tau(G \backslash x_0)$ and*

$$\forall_{H \in \mathcal{H}}\ x_0 \notin \mathrm{cl}_\tau(G \backslash H \backslash x_0)\,. \tag{7}$$

COROLLARY 4. – *A pseudotopology τ, on a set X, is paratopologically maximal at x_0 in the class of pseudotopologies on X if and only if for each filter $\mathcal{H}$ with $x_0 \in \mathrm{adh}_\tau(\mathcal{H} \backslash x_0)$, there exists a countably based filter $\mathcal{G}$ such that $x_0 \in \mathrm{adh}_\tau(\mathcal{G} \backslash x_0)$ and*

$$\forall_{H \in \mathcal{H}}\ x_0 \notin \mathrm{adh}_\tau(\mathcal{G} \backslash H \backslash x_0)\,. \tag{8}$$

EXAMPLE 5 (*A pseudotopology ξ such that $\xi > P_\omega \xi > P\xi > T\xi$*). – Recall that the *Stone topology* on the set βX of the ultrafilters of X admits the following base: $\beta W = \{\mathcal{U} \in \beta X \colon W \in \mathcal{U}\}$, $W \subset X$.

A pseudotopology on X is determined by the collection

$$\{\mathfrak{B}(x) \colon x \in X\} \tag{9}$$

of families of the ultrafilters convergent to each $x \in X$; conversely, every class (9) with the property that the principal ultrafilter of x belongs to $\mathfrak{B}(x)$ for each $x \in X$, determines a pseudotopology.

It has been shown in [2] that a pseudotopology on X is a paratopology if and only if all the sets in (9) are closed for the topology $G_\delta(\beta)$, i.e., the topology generated by the base for open sets composed of the countable intersections of the Stone open sets.

On the other hand, a pseudotopology on X is a pretopology if and only if the class (9) consists of Stone closed sets.

Consider a subset $\mathfrak{B}$ of free ultrafilters on an infinite countable set X which is $G_\delta(\beta)$-closed but not Stone closed. Such sets exist, because for each strictly decreasing sequence $(W_n)_n$ of infinite subsets of X, the set $\bigcap_n \beta W_n$ is not open. On the other hand, as the Stone interior of $\bigcap_n \beta W_n$ is not empty hence the discrete sets for the Stone topology and for $G_\delta(\beta)$ coincide such sets are not discrete. Let $\mathcal{U} \in \mathrm{cl}_{G_\delta(\beta)} \mathfrak{B} \setminus \{\mathcal{U}\}$. Consider the set $\{0\} \cup \{1/n: 0 \neq n \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} X_n$, where X_n are copies of X. Define the following pseudotopology ξ the points of $\bigcup_{n \in \mathbb{N}} X_n$ are isolated, the ultrafilters convergent to $1/n$ are: the principal ultrafilter and all the elements of $\mathfrak{B}_n \setminus \{\mathcal{U}_n\}$ (the n-th copies of $\mathfrak{B} \setminus \{U\}$); each free ultrafilter on $\{1/n: 0 \neq n \in \mathbb{N}\}$ converges to 0. $\mathcal{U}_n$ is the only ultrafilter that converges to $1/n$ for $P_\omega \xi$ but not for ξ. All the ultrafilters of $\mathrm{cl}_\beta \mathfrak{B}_n$ converge to $1/n$ for P_ξ. Finally the trace on $\bigcup_{n \in \mathbb{N}} X_n$ of the topological neighborhood filter of 0 is $\bigcup_n \bigcup_{k \geqslant n} \bigcup_{W \in \mathrm{cl}_\beta \mathfrak{B}_k} W$.

Therefore, ξ, $P_\omega \xi$, $P\xi$, $T\xi$ are examples of convergences that are not maximal in any of the discussed cases.

REFERENCES

[1] G. Choquet, *Convergences*, Ann. Univ. Grenoble, **23** (1947-48), 55-112.

[2] S. Dolecki, *Convergence-theoretic methods in quotient quest.*, Topology Appl., **73** (1996), 1-21.

[3] S. Dolecki - G. H. Greco, *Topologically maximal pretopologies*, Studia Math., **77** (1984), 265-281.

[4] S. Dolecki - M. Pillot, *Topologically maximal convergences and types of accessibility*, submitted (1996).

[5] V. Kannan, *Ordinal invariants in topology*, Memoirs Amer. Math. Soc., **32**, No. 245 (1981), 1-164.

[6] D. C. Kent, *Convergence quotient maps*, Fund. Math., **65** (1969), 197-205.

[7] F. Siwiec, *Sequence-covering and countably bi-quotient mappings*, Gen. Topology Appl., **1** (1971), 143-154.

[8] G. T. Whyburn, *Mappings on inverse sets*, Duke Math. J., **23** (1956), 237-240.

[9] G. T. Whyburn, *Accessibility spaces*, Proc. Amer. Math. Soc., **24** (1970), 181-185.

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