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Finite Groups Whose Poset of Conjugacy Classes of Subgroups is Isomorphic to the One of an Abelian Group.

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Sunto. – *Si determinano i gruppi finiti il cui insieme parzialmente ordinato delle classi di coniugio dei sottogruppi è isomorfo a quello di un gruppo abeliano.*

Let G be a finite group and $\mathcal{C}(G)$ be the set of all conjugacy classes of subgroups of G endowed with the following relation:

$$\{H^g \mid g \in G\} \leqslant \{K^g \mid g \in G\}$$

if and only if there exists $x \in G$ such that $H^x \leqslant K$. Since G is finite, $\leqslant$ is a partial ordering on $\mathcal{C}(G)$. We shall call this poset the *frame* of G. A natural question when dealing with weaker structures related to a group G, such as the subgroup lattice or, in this case, the frame, is to determine how strongly these influence the structure of G itself (see Preface of [5]). In particular we are interested in the following question: given a class χ of groups determine the class $\bar{\chi}$ of all finite groups whose frame is isomorphic to the one of a group of the class χ. We shall call $\bar{\chi}$ the *closure* of χ and we shall say that χ is *closed* if $\chi = \bar{\chi}$. In [1] Brandl showed that the classes of finite p-groups, finite abelian p-groups, metacyclic p-groups are closed. The same result holds for the class of finite modular p-groups and for the class of p-groups of maximal class ([3]). The last result depends on the fact that in a finite p-group the (conjugacy classes containing) normal subgroups can be characterized within the frame. This is clearly no more possible for groups of composite order for e.g. the groups S_3 and C_6 have isomorphic frames but nonisomorphic normal subgroup lattices. More generally Brandl, Cutolo and Rinauro proved that the closure of the class of finite cyclic groups is the class of finite groups with cyclic Sylow subgroups ([2] Theorem A). In this paper we continue their investigation and determine the closure of the class of finite abelian groups. The result we shall prove is the following.

THEOREM 1. – *Let G be a finite group. Then there exists an abelian group A and a poset isomorphism $\varphi\colon \mathcal{C}(G) \to \mathcal{C}(A)$ if and only if G is a supersolv-*

able T-group with abelian Sylow subgroups. In this case for every noncyclic Sylow p-subgroup P of G, the (unique) element, say R, of $[P]^\varphi$ *is a Sylow r-subgroup of A whose lattice is isomorphic to the one of P.*

Note that, by [5] Theorem 2.2.6 and Theorem 2.6.8., this implies that if P is not cyclic, then $R \cong P$ and if P is cyclic, then R is cyclic and $\log_p(|P|) = \log_r(|R|)$.

Most notations are standard (see e.g. [4]). In addition to that for every subset $\mathcal{S}$ of a poset $(\mathcal{L}, \leqslant)$ we denote with $\mathcal{S}^<$ the set $\{x \mid x \in \mathcal{L},\ x \geqslant s,\ \forall s \in \mathcal{S}\}$ and with $\mathcal{S}^>$ the set $\{x \mid x \in \mathcal{L},\ x \leqslant s,\ \forall s \in \mathcal{S}\}$. We shall denote with $\mathcal{S}^+$ and $\mathcal{S}^-$ respectively the set of minimal elements of $\mathcal{S}^<$ and the set of maximal elements of $\mathcal{S}^>$. If $a, b \in \mathcal{L}$ with $a \geqslant b$, we denote with $[a\colon b]$ the set $\{c \mid c \in \mathcal{L},\ a \geqslant c \geqslant b\}$. If G is a group we denote with $\mathcal{L}(G)$ its subgroup lattice. If H and P are subgroups of G with $H \leqslant P$ we denote with $[H]_P$ the set of all conjugates of H under P; in particular if $P = G$ and there is no ambiguity, we write simply $[H]$ instead of $[H]_G$.

LEMMA 2. – *Let G be a finite group.*

1) *Let* $\pi_G\colon \mathcal{L}(G) \to \mathcal{C}(G)$ *be the map that sends every subgroup of G in its conjugacy class. Then* π_G *is a poset epimorphism and it is an isomorphism if and only if G is abelian or hamiltonian.*

2) *Let* $P \leqslant G$ *and* $H \leqslant P$ *such that* $G = N_G(H)\,P$. *Then* $[H]_P = [H]_G$.

3) *If* $P \trianglelefteq G$ *then* $\mathcal{C}(G/P)$ *is isomorphic to the interval* $[[G]_G\colon [P]_G]$ *of* $\mathcal{C}(G)$.

PROOF. – The first two assertions follow immediately from the definition of the order relation in $\mathcal{C}(G)$. For the last one note that if $[H]_G \in [[G]_G\colon [P]_G]$ then $P \leqslant H$ and it is easy to see that the position $[H]_G \mapsto [H/P]_{G/P}$ for every $[H]_G \in [[G]_G\colon [P]_G]$ defines a poset isomorphism between $[[G]_G\colon [P]_G]$ and $\mathcal{C}(G/P)$. ■

LEMMA 3. – *Let* H_1 *and* H_2 *be subgroups of a finite group G. For every element* $c \in \{[H_1],[H_2]\}^+$ *(resp.* $\{[H_1],[H_2]\}^-$*) there exist* $g \in G$ *such that* $c = [\langle H_1, H_2^g \rangle]$ *(resp.* $c = [H_1 \cap H_2^g]$*).*

PROOF. – We prove the lemma only for $c \in \{[H_1],[H_2]\}^+$, a dual argument proves the other case. Let $L \in c$, then L is a subgroup of G that contain a conjugate $H_1^{g_1}$ of H_1 and a conjugate $H_2^{g_2}$ of H_2 for suitable elements g_1 and g_2 in G. Set $g = g_2 g_1^{-1}$, then $c = [L] = [L^{g_1^{-1}}] \geqslant [\langle H_1, H_2^g \rangle]$ on the other hand $[\langle H_1, H_2^g \rangle] \geqslant [H_i]$ for $i \in \{1, 2\}$ hence $[\langle H_1, H_2^g \rangle] \in \{[H_1],[H_2]\}^<$ and minimality of c forces $c = [\langle H_1, H_2^g \rangle]$. ■

LEMMA 4. – *Let P be a Sylow p-subgroup of a group G. Suppose that every subgroup of P is normal in G. Then there is a poset isomorphism*

$$\gamma\colon \mathcal{C}(G) \to \mathcal{C}(P) \times \mathcal{C}(G/P)$$

such that

1) $([H]_G)^\gamma \in \mathcal{C}(P) \times \{[\{1_{G/P}\}]_{G/P}\}$ *if H is a p-subgroup of G and*

2) $([H]_G)^\gamma \in \{[\{1_P\}]_P\} \times \mathcal{C}(G/P)$ *if H is a p'-subgroup of G.*

PROOF. – For any conjugacy class $[H]_G$ consider the pair $([H \cap P]_P, [HP/P]_{G/P})$. Assume $[H]_G = [K]_G$. Then $K = H^g$ for a $g \in G$ and, since every subgroup of P is normal in G, we have

$$[K \cap P]_P = [H^g \cap P]_P = [(H \cap P)^g]_P = [H \cap P]_P$$

and

$$[KP/P]_{G/P} = [H^g P/P]_{G/P} = [(HP/P)^{gP}]_{G/P} = [HP/P]_{G/P}\,.$$

Thus we may define a map $\gamma\colon \mathcal{C}(G) \to \mathcal{C}(P) \times \mathcal{C}(G/P)$ sending $[H]_G$ to the pair $([H \cap P]_P, [HP/P]_{G/P})$. It is immediate to see that γ satisfies 1) and 2).

We prove that

1) *γ is injective.*

Suppose $([H]_G)^\gamma = ([L]_G)^\gamma$, that is

$$([H \cap P]_P, [HP/P]_{G/P}) = ([L \cap P]_P, [LP/P]_{G/P})\,.$$

Then we have that $[H \cap P]_P = [L \cap P]_P$ and, since every subgroup of P is normal, this means that

$$H \cap P = L \cap P\,.$$

By the Theorem of Schur-Zassenhaus there is a p'-complement Q of P in HP that is contained in H and $H = (H \cap P)Q$. Since $[HP/P]_{G/P} = [LP/P]_{G/P}$, there exists $g \in G$ such that $(HP)^g = LP$ hence $Q^g \leqslant LP$ and Q^g is a p'-complement of P in LP. Again by the theorem of Schur-Zassenhaus there is a conjugate Q^{gh} of Q^g that is contained in L and

$$L = (L \cap P)\, Q^{gh} = (H \cap P)\, Q^{gh} = (H \cap P)^{gh}\, Q^{gh} = H^{gh}\,,$$

that is $[H]_G = [L]_G$.

2) *γ is surjective.*

Consider a pair $([M]_P, [N/P]_{G/P})$ in $\mathcal{C}(P) \times \mathcal{C}(G/P)$. Then $M \leqslant P \leqslant N$. Let R

be a p'-complement of P in N and consider the subgroup MR. Then

$$M = MR \cap P \quad \text{and} \quad N/P = RP/P = MRP/P ,$$

that is $([M]_P, [N/P]_{G/P}) = ([MR]_G)^\gamma$.

3) *γ is a poset homomorphism.*

Let $[H]_G \leqslant [L]_G$, then there exists $g \in G$ such that $H \leqslant L^g$. Thus

$$HP/P \leqslant L^g P/P = (LP/P)^{gP} ,$$

which implies that

$$[HP/P]_{G/P} \leqslant [LP/P]_{G/P} .$$

Moreover

$$H \cap P \leqslant L^g \cap P = (L \cap P)^g = (L \cap P) ,$$

which implies that $[H \cap P]_P \leqslant [L \cap P]_P$ and it follows that $([H]_G)^\gamma \leqslant ([L]_G)^\gamma$.

4) *γ is a poset isomorphism.*

Suppose $([H]_G)^\gamma \leqslant ([L]_G)^\gamma$, then

$$[H \cap P]_P \leqslant [L \cap P]_P \text{ and } [HP/P]_{G/P} \leqslant [LP/P]_{G/P} .$$

Since the subgroups of P are normal in G the first inequality implies that $H \cap P \leqslant L \cap P$, and the second implies that there exists $g \in G$ such that $HP/P \leqslant L^g P/P$. By Schur-Zassenhaus there are p'-complements Q_1 and Q_2 of P in HP and $L^g P$ respectively with $Q_1 \leqslant Q_2$ and

$$H = (H \cap P)\, Q_1 \leqslant (L \cap P) Q_2 = L^g ,$$

that is $[H]_G \leqslant [L]_G$. ■

We prove now Theorem 1.
Let

$$\varphi \colon \mathcal{C}(G) \to \mathcal{C}(A)$$

be a poset isomorphism where A is an abelian group and G is a finite group. We show that G is a supersolvable T-group with abelian Sylow subgroups. Suppose not and let G be a counterexample of least possible order. Throughout this proof, if $H \trianglelefteq G$, we shall denote with $\overline{H}$ the unique element of $[H]^\varphi$. Let N be a nontrivial normal subgroup of G, then we have

$$\mathcal{C}(G/N) \cong [[G] \colon [N]] \cong [[G]^\varphi \colon [N]^\varphi] \cong [[A] \colon [\overline{N}]] \cong \mathcal{C}(A/\overline{N}) .$$

Since $A/\overline{N}$ is abelian and N is nontrivial, minimality of G implies that

5) *For every nontrivial normal subgroup N of G, G/N is a supersolvable T-group with abelian Sylow subgroups.*

Since $\mathcal{C}(G) \cong \mathcal{C}(A) \cong \mathcal{L}(A)$ and $\mathcal{C}(G)$ is finite, we have that $\mathcal{L}(A)$ (hence $\mathcal{C}(G)$) is a finite modular lattice. By [1], Proposition 1, it follows that

6) *G is supersolvable and $\mathcal{C}(G)$ is a finite modular lattice.*

This means that every pair $[H]$, $[K]$ of elements of $\mathcal{C}(G)$ there is a unique element of $\{[H],[K]\}^+$ and a unique element of $\{[H],[K]\}^-$ which we denote, as usual, respectively with $[H] \vee [K]$ and with $[H] \wedge [K]$. Let p the greatest of the primes dividing $|G|$ and $P \in Syl_p(G)$. Then $P \trianglelefteq G$ by a theorem of Zappa (see [4], 5.4.8 pag. 145). We prove that

7) *if H is a normal subgroup of P, then H is normal in G.*

Suppose the assertion were false. Let q be the greatest prime such that there are a q-element g of G and a normal subgroup H of P which is not normalized by g. Clearly $p > q$, since $H \trianglelefteq P$. Let ϱ be the set of primes which are greater than q and set $N = O_\varrho(G)$. By [4] (5.4.8) N is a ϱ-Hall subgroup of G. Let K be a ϱ'-complement of N containing g. By the choice of q

8) *every element of N normalizes H.*

and since $K \cong G/N$, which is a proper factor of G, 5) implies that,

9) *every element of K normalizes $\langle g \rangle$.*

Let $t \in G$, since $G = NK$, there exist $n \in N$ and $k \in K$ such that $t = nk$. By 8) and 9) it follows that

$$\langle g, H^t \rangle = \langle g, H^{nk} \rangle = \langle g, H^k \rangle = \langle g^k, H^k \rangle = \langle g, H \rangle^k .$$

By Lemma 3 this implies that

10) $[H] \vee [\langle g \rangle] = [\langle H, g \rangle]$.

Now we have that

$$H < \langle H, H^g \rangle \leqslant P .$$

Hence, as $|\langle g \rangle|$ and $|P|$ are coprime and $P \trianglelefteq G$, it follows that

$$[H] < [\langle H, H^g \rangle] \leqslant [P] \wedge [\langle H, g \rangle] = [P \cap \langle H, g \rangle] < [\langle H, g \rangle] = [H] \vee [\langle g \rangle] .$$

On the other hand

$$[\langle g \rangle] \wedge [\langle H, H^g \rangle] \leqslant [\langle g \rangle] \wedge [P] = [\{1_G\}]$$

which implies that $\mathcal{C}(G)$ is not modular since

$$([H] \vee [\langle g \rangle]) \wedge [\langle H, H^g \rangle] = [\langle H, H^g \rangle] \neq [H] = [H] \vee ([\langle g \rangle] \wedge [\langle H, H^g \rangle]),$$

a contradiction that proves 7). We prove now that

11) *every subgroup of P is normal in G.*

By 7) (or by the proof of the theorem of Zappa [4], 5.4.8 pag. 145) there is a subgroup Z of order p and normal in G. By 5) G/Z is a supersolvable T-group with abelian Sylow subgroups, in particular, by the Correspondance Theorem,

12) *every subgroup of P containing Z is normal in G.*

If Z is not contained in the Frattini subgroup of P then, since $Z \leqslant Z(P)$, P is isomorphic to the direct product $Z \times P/Z$. Since P/Z is abelian, P is abelian and 11) follows from 7). Assume now that

13) *Z is contained in the Frattini subgroup of P.*

We may assume that P/Z is not cyclic, otherwise, by 13), P is cyclic and we have nothing more to prove. By 12)

$$\mathcal{L}(P/Z) \cong [[P]:[Z]] \cong [[\overline{P}]:[\overline{Z}]] \cong \mathcal{L}(\overline{P}/\overline{Z}).$$

Therefore $\overline{P}/\overline{Z}$ is an abelian group which is lattice isomorphic to the noncyclic abelian p-group P/Z. By a theorem of Suzuki (see [5] Theorem 2.2.6, pag. 52), $|\overline{P}/\overline{Z}| = |P/Z|$ hence, by a theorem of Baer (see [5], Theorem 2.6.8, pag. 104) we have that

14) *$\overline{P}/\overline{Z} \cong P/Z$, in particular $\overline{P}/\overline{Z}$ is a p-group.*

Since Z is contained in every maximal subgroup of P, $\overline{Z}$ is contained in the Frattini subgroup of $\overline{P}$ and it follows, by 14), that

15) *$\overline{P}$ is an abelian p-group.*

Let X be a cyclic subgroup of P. Then $[[X]:[\{1\}]]$ is a chain hence $\mathcal{L}(\overline{X})$ is a chain and $\overline{X}$ is cyclic. We prove that

16) $X \trianglelefteq G$.

If $Z \leqslant X$ then $X \trianglelefteq P$ by 12). Thus assume $Z \nleqslant X$ and $X \neq \{1\}$. Then, since $Z \trianglelefteq G$, by Lemma 3, we have $[Z] \vee [X] = [ZX]$ and $[Z] \wedge [X] = \{1\}$, hence

17) *ZX is a normal abelian subgroup of G of type (p, p^{α}), where $p^{\alpha} = |X|$.*

On the other hand we have

$$\mathcal{L}(\overline{X}) \cong [[\overline{X}]:[\{1\}]] \cong [[X]:[\{1\}]],$$

$$\mathcal{L}(\overline{Z}) \cong [[\overline{Z}]:[\{1\}]] \cong [[Z]:[\{1\}]]$$

and

$$[\overline{X} \cap \overline{Z}] = [\overline{X}] \wedge [\overline{Z}] = ([X] \wedge [Z])^{\varphi} = [X \cap Z]^{\varphi} = [\{1\}]^{\varphi} = [\{1\}].$$

Hence

18) *$\overline{X}$ and $\overline{Z}$ are cyclic and $\overline{Z}\overline{X} \cong ZX$.*

Let Y be a G-conjugate of X, then $[X] = [Y]$ and, since ZX is normal in G, Y is a maximal subgroup of ZX. By 18), $[[ZX]:[\{1\}]] \cong \mathcal{L}(ZX)$. Hence, by 17), there are exactly $p+1$ maximal elements $[X_1] = [X], \dots, [X_{p+1}]$ of $[[ZX]:[\{1\}]]$. Since each one of these contain a maximal subgroup of ZX and ZX has exactly $p+1$ maximal subgroups, for every $i \in \{1, \dots, p+1\}$ there is only one maximal subgroup of ZX which is contained in $[X_i]$. In particular it follows that $Y = X$ and X is normal in G. We have proven that every cyclic subgroup of P is normal in G and this clearly implies 11).

By 5) and [4], 13.4.5. G is a supersolvable T-group and its Sylow subgroups are abelian since they are either isomorphic with P, which is abelian by 11) or isomorphic with a Sylow subgroup of G/P and these are abelian by 5), the final contradiction.

Conversely, suppose G is a finite supersolvable T-group with abelian Sylow subgroups. Let t be the lenght of a Sylow tower of G and $P_1, \dots, P_t$ be a set of representatives of the conjugacy classes of the Sylow subgroups of G. For every $i \in \{1, \dots, t\}$ let Q_i be a group isomorphic to P_i and let A be the direct product of the Q_i's. Then A is abelian since every P_i is abelian. By Lemma 4 and induction on t, we have that there is a poset isomorphism

$$\gamma\colon \mathcal{C}(G) \to \mathcal{C}(P_1) \times \mathcal{C}(P_2) \times \dots \times \mathcal{C}(P_t).$$

such that

$$[P_i]^{\gamma} \in \{[\{1_{P_1}\}]_{P_1}\} \times \{[\{1_{P_2}\}]_{P_2}\} \times \dots \times \mathcal{C}(P_i) \times \dots \times \{[\{1_{P_t}\}]_{P_t}\}.$$

By [6], Theorem 4, pag. 5 (or by [5], Lemma 1.6.4) and Lemma 2, it is easy to see that we may construct a poset isomorphism

$$\delta\colon \mathcal{C}(G) \to \mathcal{C}(A).$$

Finally, let B be another abelian group and

$$\mu\colon \mathcal{C}(G) \to \mathcal{C}(B)$$

be a poset isomorphism. Let π_A and π_B be defined as in Lemma 2 and set $R_i = [P_i]^{\mu\pi_B^{-1}}$ then R_i is a direct factor of $\mathcal{L}(B)$, hence, by [6], Theorem 4, pag. 5,

19) *R_i is a Hall subgroup of B.*

Since $\pi_B\mu^{-1}\delta\pi_A^{-1}$ is a poset isomorphism, we have that

$$\mathcal{L}(R_i) = [R_i\colon \{1_B\}] \cong [R_i\colon \{1_B\}]^{\pi_B\mu^{-1}\delta\pi_A^{-1}} = [Q_i\colon \{1_A\}]\ \mathcal{L}(Q_i) \cong \mathcal{L}(P_i)\,.$$

Thus R_i and P_i are lattice-isomorphic abelian groups.

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