Bollettino
Unione Matematica Italiana

Martin Oxenham, Rey Casse

On the resolvability of Hall triple systems

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **1-B** (1998), n.3, p. 639–649.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_1998_8_1B_3_639_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 1998.

Bollettino U. M. I.
(8) **1**-B (1998), 639-649

On the Resolvability of Hall Triple Systems.

MARTIN OXENHAM - REY CASSE

Sunto. – *È ben noto che fra le classi di sistemi ternari di Hall* (HTS), *gli* HTS *Abeliani ammettano una risoluzione siccome sono esattamente gli spazi affini finiti d'ordine* 3; *per questi sistemi una tal risoluzione è fornita dalla relazione di parallelismo. In questa nota viene dimostrato che certe classi di* HTS *non Abeliani costrutti dai gruppi di Burnside* $B(3, r)$, $r \geqslant 3$ *anche ammettono una risoluzione. Allora, questi esempi di* HTS *si possono considerare anche come spazi finiti di Sperner e dunque la nota conclude con un discorso d'una domanda posta di Barlotti in* [1] *riguardo a questi spazi..*

1. – Introduction.

Given a Steiner triple system (STS) S and any point a of S, we define the *symmetry* of S with fixed point a to be the mapping

$$\sigma_a : a \mapsto a \,,$$
$$x \mapsto y \,,$$

whenever $\{a, x, y\}$ is a line of S.

A *Hall Triple System* (HTS) is then a STS S in which each symmetry of S is also a collineation of S. HTS's may also be characterised as those STS's in which any three non-collinear points generate a finite affine plane of order 3. It is then immediate that the finite affine spaces defined over $GF(3)$ give rise to examples of HTS's; these are commonly referred to as the *Abelian* HTS's. The *order* of a HTS S (i.e. the number of points in S) is always of the form 3^s for some s; s is the *size* of S (see [9]). Furthermore, any two minimal generating sets of a HTS have the same number of elements. Writing this number as $n+1$, the *dimension* of the HTS is then defined to be n (see [2]). The smallest non-Abelian HTS has size 4 and dimension 3; all systems with these parameters are isomorphic to one another (see [11]). In [13], Hall constructs this using the Burnside group $B(3, 3)$; this construction can be generalised to produce an infinite family of non-Abelian HTS's of dimension r and size $\binom{r}{3} + r$,

$r \geqslant 3$ (see [14]). For other results concerning the existence and classification of HTS's, see [2], [3].

An *exponent 3 commutative Moufang loop* (3-CM loop) is a loop $(\mathcal{M}, \circ)$ which satisfies the following identities for all $x, y, z \in \mathcal{M}$:

(i) $x \circ y = y \circ x$,

(ii) $x^2 \circ (x \circ y) = y$,

(iii) $(x \circ y) \circ (z \circ x) = (x \circ (y \circ z)) \circ x$.

Note: Setting $y = 1$ in (ii), we have that $x^3 = 1$ for all $x \in \mathcal{M}$ and so the loop is of «exponent 3», (see [3]).

It is reported in [15] that in 1965, M. Hall and R.H. Bruck discovered the close ties which exist between 3-CM loops and HTS's. Given a 3-CM loop $(\mathcal{M}, \circ)$, we can construct a HTS S by taking the elements of $\mathcal{M}$ as the points of S and the triples $\{x, y, (x \circ y)^2\}$ as the lines of S (the incidence relation being that of set inclusion). Conversely, given a HTS S, we can construct a 3-CM loop $\mathcal{M}$ with binary operation $\circ$ as follows:

For any two points a and b of S we define $a \cdot b$ to be the third point on the line containing a and b if a and b are distinct, otherwise it is equal to a. Then the elements of $\mathcal{M}$ are the points of S and for all elements x and y of $\mathcal{M}$, we set

$$x \circ y = (e \cdot x) \cdot (e \cdot y)$$

where e is an arbitrary fixed point of S. Two 3-CM loops constructed in this way via distinct fixed points e and e' are isomorphic. Furthermore the HTS's which arise from these 3-CM loops by the technique described above, are all isomorphic to S. Thus, up to isomorphism, there is a one-to-one correspondence between HTS's and 3-CM loops. (See [3], [12] and [15] for further details.) It may be readily checked that the 3-CM loop corresponding to the finite affine space $AG(n, 3)$, $n \geqslant 2$ is associative. Therefore we conclude the following:

THEOREM 1.1 ([15]). – *The HTS associated with a 3-CM loop $\mathcal{M}$ is a finite affine space if and only if $\mathcal{M}$ is associative.* ■

A HTS is said to be *resolvable* if its line-set can be partitioned into subsets called *resolution classes* in such a way that each point of S lies on a unique line of each *resolution class*. The partition itself is called a *resolution*. Each Abelian HTS S is resolvable as the parallelism relation of the corresponding affine space $AG(n, 3)$ affords a resolution of S. In Section 3, we use the Burnside groups $B(3, r)$, $r \geqslant 3$, to construct an infinite class of resolvable non-Abelian HTS's.

2. – Nilpotence of loops.

The 3-CM loops constitute a subclass of the more general class of loops known as Moufang loops in which only relation (iii) need hold. In addition, the loops are commutative if relation (i) holds. Amongst the best known examples of Moufang loops are the groups. In discussing properties of these loops, especially the 3-CM loops in relation to the HTS's which they coordinatise, one powerful mathematical tool is the concept of central nilpotence of a loop. Before defining central nilpotence, we briefly recall some of the background theory pertaining to loops; for a fuller account see [15].

In the sequel, where it does not lead to ambiguity, we shall denote the loop operation by juxtaposition.

DEFINITION 2.1 ([15], p. 132, 133). – *A* subloop *of the loop* $(G, \circ)$ *is any subset of* $(G, \circ)$ *which is itself a loop with respect to* $\circ$. *A subloop* $(H, \circ)$ *is* normal *in* $(G, \circ)$ *if for all* $x, y, \in G$

(i) $xH = Hx$,

(ii) $(Hx)\, y = H(xy)$,

(iii) $y(xH) = (yx)\, H$.

Given a normal subloop $(H, \circ)$ *of* $(G, \circ)$ *we can define the quotient loop* $G/H = (\{Hg \mid g \in G\}, *)$ *where* $(Hg_1) * (Hg_2) = H(g_1 g_2)$.

DEFINITION 2.2 ([10], [15]). – *Let* $(G, \circ)$ *be a loop. Then*

(i) *The* commutator *of two arbitrary elements* $x, y \in G$ *is the unique element* $[x, y]$ *of* G *which satisfies*

$$(yx)[x, y] = (xy).$$

(ii) *The* associator *of three arbitrary elements* $x, y, z \in G$ *is the unique element* (x, y, z) *of* G *which satisfies*

$$(x(yz))(x, y, z) = (xy)\, z. \qquad \blacksquare$$

NOTE 2.3. – (i) ([15], p. 133). Two elements $x, y \in G$ commute with each other if and only if $[x, y] = 1$ and three elements x, y, z associate with each other (in the given order) if and only if $(x, y, z) = 1$.

(ii) ([10], p. 138). A series of higher order (left) commutators can be defined recursively as follows:

$[x_1, x_2]$ is as already defined,

$$[x_1, x_2, \ldots, x_{n-1}, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n], \qquad n \geqslant 3.$$

A commutator of n elements is said to have *weight* n. ■

DEFINITION 2.4 ([15], p. 134). – *The* centre $Z(G)$ *of a loop* $(G, \circ)$ *is the set*

$$\{z \in G \mid [z, x] = (z, x, y) = (x, z, y) = (x, y, z) = 1 \forall x, y \in G\}. \qquad ■$$

It is immediate by Definition 2.1 that $Z(G)$ is a normal subloop (subgroup) of G.

DEFINITION 2.5 [4]. – (i) *Given a loop* $(G, \circ)$, *the upper central series* (Z_i) *of* G *is defined recursively as given below*:

$Z_0 = \{1\}$,

$Z_i = \{z \in G \mid [z, x], (z, y, x), (x, z, y), (x, y, z) \in Z_{i-1}, \qquad \forall x, y \in G, i \geqslant 1\}$.

(ii) *A loop* $(G, \circ)$ *is said to be* centrally nilpotent *if* $Z_i = G$ *for some* i. *The* central nilpotence class *of such a loop* G *is the smallest integer* i *for which this holds.* ■

We note that if G is commutative, then $(x, y, z) = 1$ implies $(z, y, x) = (x, z, y) = 1$, and so we may rewrite Z_i as

$$Z_i = \{z \in G \mid (x, y, z) \in Z_{i-1} \ \forall x, y \in G\}.$$

In general, a loop need not be nilpotent. However, if the loop is a commutative Moufang loop, then by the celebrated Bruck-Slaby Theorem, it is nilpotent (see [15], p. 136). We shall only require the full statement of this theorem for 3-CM loops.

THEOREM 2.6 ([3]). – *Let* $(G, \circ)$ *be a* 3-*CM loop and write* $|G/D(G)| = 3^n$ *where* $D(G)$ *is the* (*normal*) *subgroup of* G *generated by the associators of* G. *Then* $Z_{n-1}(G) = G$, *i.e.* G *is centrally nilpotent of class at most* $n - 1$. ■

NOTE 2.7 ([3]). – A 3-CM loop is nilpotent of class 1 if and only if $Z_1(G) = G$, if and only if G is Abelian. For results on 3-CM loops of class $\geqslant 2$ see [3] and [15]. ■

When the loop is also a group, the binary operation is necessarily associative and so we may rewrite Z_i as

$$Z_i = \{z \in G \mid [z, x] \in Z_{i-1} \ \forall x \in G\}$$

which coincides with the usual definition of the ith term of the upper series for G (see [10], p. 151.) We may also define nilpotence of a group $(G, \circ)$ in terms of a lower series (Γ_i) which is defined as follows ([10], p. 150):

$$\Gamma_0 = G\,,$$

$$\Gamma_i = \langle [x_1, \ldots, x_{i+1}] \mid x_1, \ldots, x_{i+1} \in G \rangle, \qquad i \geqslant 1\,.$$

G is then *nilpotent* if and only if $\Gamma_i = \{1\}$ for some $i \geqslant 1$. In this case, G is said to be of class i, if i is the smallest integer for which this is true. Thus, as a consequence, we have

COROLLARY 2.8 ([10], p. 153). – *A group G is nilpotent of class i if and only if for some i, each commutator of weight $i+1$ is the identity whereas at least one commutator of weight i is not the identity.* ■

3. – The burnside groups and resolvable HTS's.

Given a group G, a set of subgroups of G is said to form a *partition* $\mathcal{P}$ of G if the subgroups intersect pairwise in only the identity and if the set theoretic union of them is the whole of G. Using such a partition, it is possible to construct a linear space $\mathcal{L}$ as described below:

(i) The points of $\mathcal{L}$ are the elements of G.

(ii) The lines of $\mathcal{L}$ are the left cosets of the subgroups in the partition.

(iii) The incidence relation is set inclusion.

It is immediate that G acts via left multiplication as a collineation group of $\mathcal{L}$. In particular G acts transitively on the point-set of $\mathcal{L}$. This, together with the fact that each element of $G\backslash\{1\}$ lies in a unique subgroup of $\mathcal{P}$, implies that $\mathcal{L}$ is a linear space. Finally, $\mathcal{L}$ is resolvable because we can construct a resolution of $\mathcal{L}$ which has each resolution class consisting of a subgroup in the partition and its proper cosets in G.

In this section, we use this technique to construct resolvable linear spaces from groups G in which $x^p = 1$ for each $x \in G$ and where p is a fixed prime. When $p = 3$ and G is finite, the linear spaces all turn out to be HTS's, a fact which gives rise to our main result: for each $s \geqslant 7$, there exists at least one resolvable non-Abelian HTS of size s.

THE CONSTRUCTION. – Let G be a group in which $x^p = 1$ for each $x \in G$ where p is a given prime. Let $\mathcal{P}$ be the set of subgroups of G of the form $\langle x \rangle$, $x \neq 1$. Each such subgroup has exactly p elements and by invoking Lagrange's theorem and the fact that $\langle x \rangle \cap \langle y \rangle$, $x, y \neq 1$ is again a subgroup of G, it is immediate that any two of these subgroups either coincide or intersect trivially in $\{1\}$. Hence $\mathcal{P}$ is a partition of G and so gives rise to a linear space $\mathcal{L}$ as described previously.

In the case that G is the elementary p-group $\bigoplus_{i=1}^{n} \mathbb{Z}_p$, the resulting linear space $\mathcal{L}$ is simply $AG(n, p)$ and the resolution of $\mathcal{L}$ comprises the usual parallel classes of $AG(n, p)$.

It is worth noting at this point that for all prime powers q there exist infinite classes of finite affine spaces $AG(n, q)$ which admit other resolutions; these are known as skew resolutions and are intimately related to 1-packings of $PG(n, q)$ (see [5], [6], [7]).

Amongst the other groups in which $x^p = 1$ for every element x are the Burnside groups $B(p, r)$, $r \geqslant 1$ and their subgroups. The Burnside group $B(n, r)$, $r, n \geqslant 1$ is constructed from the free group F_r on r generators by factoring out the normal subgroup N of F_r which is generated by all finite products of the form

$$\prod_j (x_j y_j^n x_j^{-1})$$

where x_j, y_j vary over F_r, i.e. $B(n, r) = F_r/N$ (see [8], [10]). It is routine to verify that $x^n = 1$ for every $x \in B(n, r)$. Various results on the cardinality of $B(n, r)$ are known. (To avoid trivialities, we assume that $r > 1$ in the sequel.) Of relevance here is the fact that $B(p, r)$ is finite for only a finite number of primes p (see [8]). When $p = 2$, $B(2, r)$ is elementary Abelian and so, as already mentioned, it gives rise to a finite affine space of order 2. For $5 \leqslant p < 665$, the cardinality of $B(p, r)$ is still an unresolved matter although it is conjectured that the group is infinite. This is known to be the case for all $p > 665$. Finally with $p = 3$, the group is finite for all r and is non-Abelian.

In the remainder of the section, we shall concentrate on determining the nature of the linear spaces $\mathcal{L}$ arising from $B(3, r)$, $r \geqslant 3$ but before doing so it is convenient to note the following results and properties of $B(3, r)$.

(i) $x^3 = 1$ for all $x \in B(3, r)$,

(ii) $|B(3, 1)| = 3$, $|B(3, 2)| = 27$, $|B(3, r)| = 3^{\binom{r}{3} + \binom{r}{2} + \binom{r}{1}}$, $r \geqslant 3$.

For $r \geqslant 3$

(iii) $|Z(B(3, r))| = 3^{\binom{r}{3}}$.

Furthermore, if $B(3, r)$ has generators $x_1, x_2, \ldots, x_r$, then $Z(B(3, r))$ is generated by the $\binom{r}{3}$ commutators $[x_i, x_j, x_k]$, $i < j < k$, and this is a minimal generating set.

(iv) $[a, b, c, d] = 1$ for all $a, b, c, d \in B(3, r)$.

(v) $B(3, r)$ is nilpotent of class 3. (This follows from (iii), (iv) and Corollary 2.8.)

(See [4], p. 292, [10].) ■

From properties (iii) and (iv), we may deduce

LEMMA 3.1. – *For each proper subgroup* N *of* $Z(B(3, r))$, $r \geqslant 3$, $B(3, r)/N$ *is nilpotent of class* 3.

PROOF. – For any commutator of weight k in $B(3, r)/N$ we have that

$$[a_1 N, \dots, a_k N] = [a_1, \dots, a_k] N .$$

Hence by property (iv), $B(3, r)/N$ is nilpotent of class at most 3. However, by (iii), since $N \neq Z(B(3, r))$, there is a commutator $[x_i, x_j, x_k]$, $i < j < k$ of $B(3, r)$ which does not belong to N. Thus

$$[x_i N, x_j N, x_k N] = [x_i, x_j, x_k]\, N \neq N ,$$

and so $B(3, r)/N$ is of class 3. ■

For the sake of brevity, we shall henceforth refer to a group in which $x^3 = 1$ for all $x \in G$ as a *group of exponent* 3.

LEMMA 3.2. – *Let G be a group of exponent* 3. *Then, in the finite linear space* $\mathcal{L}$ *arising from G, we have that*

(i) *the points a, b, c are collinear if and only if* $c = ba^2 b$,

(ii) *the points a, bab,* b^2ab^2 *are collinear for each pair of elements a, b of G.*

PROOF. – (i)

$\{a, b, c\}$ is a line of $\mathcal{L} \Leftrightarrow a\{1, a^2 b, a^2 c\}$ is a coset in $G \Leftrightarrow$

$\{1, a^2 b, a^2 c\}$ is a (cyclic) subgroup of $G \Leftrightarrow a^2 c = (a^2 b)^2 \Leftrightarrow a^2 c = a^2 ba^2 b \Leftrightarrow c = ba^2 b$.

(ii) Let a, $b \in G$ and consider the line through bab and a. If c is the third point of the line, then by (i),

$$c = a(bab)^2 a = (ab)^2 (ba)^2 = (ab)^{-1} (ba)^{-1} =$$

$$(baab)^{-1} = (ba^{-1} b)^{-1} = b^{-1} ab^{-1} = b^2 ab^2 . \quad ■$$

THEOREM 3.3. – *Under the hypotheses of Lemma* 3.2, *the linear space* $\mathcal{L}$ *is a* HTS *if G is finite.*

PROOF. – Since each line of $\mathcal{L}$ has 3 points, it may be equally well regarded as a Steiner triple system.

Let a be an arbitrary point of $\mathcal{L}$ and let σ_a be the symmetry of $\mathcal{L}$ with fixed point a. Let x be a point distinct from a. Then by Lemma 3.2 (i), the line containing a and x is $\{a, x, xa^2x\}$. Hence

$$\sigma_a: a \mapsto a ,$$
$$x \mapsto xa^2x .$$

We now show that σ_a is a collineation of $\mathcal{L}$. Each line of $\mathcal{L}$ is a coset of a non-trivial cyclic subgroup of G. Therefore each line can be written in the form

$$c\{1, b, b^2\} = \{c, cb, cb^2\}$$

for some elements b, c of G. The image of this line under the action of σ_a is the set

$$\sigma_a\{c, cb, cb^2\} = \{ca^2c, cba^2cb, cb^2a^2cb^2\} = c\{(a^2c), b(a^2c)\,b, b^2(a^2c)\,b^2\}.$$

By Lemma 3.2 (ii), $l = \{(a^2c), b(a^2c)\,b, b^2(a^2c)\,b^2\}$ is a line of $\mathcal{L}$. Hence, it follows that $cl = \sigma_a\{c, cb, cb^2\}$ is also a line because G acts via left multiplication as a collineation group of $\mathcal{L}$.

Since each symmetry of $\mathcal{L}$ is also a collineation, it follows by definition that $\mathcal{L}$ is a HTS. ■

If G is a finite group of exponent 3, then the HTS $\mathcal{L}$ arising from G can be coordinatised by a 3-CM loop $\mathcal{M}$. By the discussion presented in the introduction, the 3-CM loop corresponding to the point 1 of $\mathcal{L}$ is isomorphic to $\mathcal{M}$. Therefore we identify these two loops. Letting $\circ$ denote the binary operation of $\mathcal{M}$ and applying the result of Lemma 3.2 (i), we have that

$$a \circ b = (1 \cdot a) \cdot (1 \cdot b) = (a^2) \cdot (b^2) = b^2ab^2$$

for each pair of elements $a, b \in G$.

This 3-CM loop is a special case of a class of commutative Moufang loops studied by Bruck in [4] (see p. 307). He showed that given any Moufang loop $(\mathcal{M}', \bullet)$ in which $x \mapsto x^3$ is an endomorphism of $\mathcal{M}'$ into its centre, then the elements of $\mathcal{M}'$ under the binary operation $*$ defined by

$$x * y = x^{-1} \bullet y \bullet x^2$$

constitute a commutative Moufang loop. (Note: In a Moufang loop $(\mathcal{M}, \bullet)$ any two elements generate a subloop which is also a subgroup. Hence for each $x \in \mathcal{M}$, the inverse x^{-1} is well-defined and $x^{-1} \bullet (y \bullet x^2) = (x^{-1} \bullet y) \bullet x^2$ for all x, $y \in \mathcal{M}$. Thus the expression $x^{-1} \bullet y \bullet x^2$ is unambiguous, (see [15], p. 134).) Furthermore if $(\mathcal{M}', \bullet)$ is a group, then $(\mathcal{M}', *)$ is centrally nilpotent of class $\leqslant 2$ with equality holding if and only if $(\mathcal{M}', \bullet)$ is nilpotent of class 3. Thus, if G is a group of exponent 3, then $x \mapsto x^3 (= 1)$ is an endomorphism of G into its centre and so, considered as a Moufang loop, G gives rise to a 3-CM loop under the operation $*$ where

$$a * b = b * a = b^{-1}ab^2 = b^2ab^2 \qquad (\text{since } b^3 = 1)$$

which is identical to the operation $\circ$ obtained from the HTS $\mathcal{L}$ arising from G (when G is finite). Summarising these results, we now have

THEOREM 3.4. – *Let G be a finite group of exponent* 3. *Then the HTS $\mathcal{L}$ arising from G is non-Abelian* (*i.e. not an affine space of order* 3) *if G is of class* 3. ■

We can now prove our main result:

THEOREM 3.5. – *For each $s \geq 7$, there exists at least one non-Abelian resolvable* HTS *of size s.*

PROOF. – Let G be the group which is the direct product of $B(3, 3)$ and $(s-7)$ copies of $(\mathbb{Z}_3, +)$.

G is a group of exponent 3 and is nilpotent of class 3 because $B(3, 3)$ is of class 3 and each $\mathbb{Z}_3$ is of class 1. Hence by Theorem 3.3, since G is finite, it gives rise to a HTS $\mathcal{L}$, and by Theorem 3.4 $\mathcal{L}$, is non-Abelian. Furthermore, the order of $\mathcal{L}$ is

$$|B(3, 3)| \times |\mathbb{Z}_3|^{s-7} = 3^7 \cdot 3^{s-7} = 3^s ,$$

so $\mathcal{L}$ has size s.

Finally, by the initial construction, $\mathcal{L}$ is resolvable. ■

Note: We can also construct non-Abelian resolvable HTS's of various sizes by using combinations of the Burnside groups $B(3, r)$, $r \geq 3$ and the factor groups $B(3, r)/N$, $r \geq 3$ (where N is a proper subgroup of $Z(B(3, r))$; see Lemma 3.1), with any finite number of isomorphic copies of $(\mathbb{Z}_3, +)$.

4. – Conclusion.

A linear space $\mathcal{L}$ is said to be *uniform* and of *order m* if each line of $\mathcal{L}$ is incident with a fixed (possibly infinite) number m of points of $\mathcal{L}$. If, in addition, $\mathcal{L}$ is resolvable then it is a *Sperner space* or alternatively a *weak affine space*. As a consequence of a Sperner space's admitting a resolution, it can be deduced that through each point P of the space there pass exactly k lines where k is independent of P. There are three possibilities for the form of k:

(i) k is finite and of the form $(m^n - 1)/(m-1)$,

(ii) k is finite but not of the form $(m^n - 1)/(m-1)$,

(iii) k is infinite.

If k is of form (i), then $\mathcal{L}$ is said to be of *dimension* n. If k is of form (iii), then it is said to be *infinite dimensional*. Finally if k is of form (ii), then its *dimension* is *undefined*.

The classical examples of Sperner spaces are the affine spaces which is the motivation behind the term «weak affine space». Thus it is of interest to determine the extent to which a Sperner space may behave like an affine space without actually being an affine space. It is shown in [1] that a Sperner space of dimension 2 is necessarily a finite affine plane. Hence, for a finite Sperner space, the degree of «affineness» of the space may be gauged by the number of subspaces of dimension 2 which it possesses. As an example, for a finite Sperner space, consider the set $\{n_P\}$ where n_P is the number of planes passing through the point P. Then we have

THEOREM 4.1 ([1], p. I-14). – *A finite* 3*-dimensional Sperner space of order* m *with* $\min\{n_P\} > m^2 + m - 1$ *is the finite affine space* $AG(3, m)$. ■

It was mentioned in [1] that it was an unresolved problem as to whether this result might be improved. If we interpret this as generalising the result to finite Sperner spaces of higher dimension, then the non-Abelian resolvable HTS's which we constructed in Section 3 show that for Sperner spaces of dimension $n \geqslant 7$ and order 3, no similar result holds because for these HTS's, n_P for each point P is the same as the number of planes through a point of $AG(n, 3)$. The cases in which the spaces have order 3 and dimension 4, 5 or 6 are still open.

REFERENCES

[1] A. BARLOTTI, *Some topics in finite geometrical structures*, Institute of Statistics Mimeo Series, **439**, University of North Carolina (1965).

[2] L. BENETEAU, *Topics about Moufang loops and Hall triple systems*, Simon Stevin, **54** no. 2 (1980), 107-124.

[3] L. BENETEAU, *Hall triple systems and related topics*, Ann. Discrete Maths., **18** (1983), 55-60.

[4] R. H. BRUCK, *Contribution to the theory of loops*, Trans. Amer. Math. Soc., **60** (1946), 245-354.

[5] R. FUJI-HARA - S. A. VANSTONE, *Orthogonal resolutions of lines in* $AG(n, q)$, Discrete Math., **41** (1982), 17-28.

[6] R. FUJI-HARA - S. A. VANSTONE, *Affine geometries obtained from projective planes and skew resolutions of* $AG(3, q)$, Ann. Discrete Math., **18** (1983), 355-376.

[7] R. FUJI-HARA - S. A. VANSTONE, *Hyperplane skew resolutions and their applications*, J. Combin. Theory A, **47** (1988), 134-144.

[8] N. GUPTA, *On groups in which every element has finite order*, Amer. Math. Monthly, **96**, no. 4 (1989), 297-308.
[9] J. I. HALL, *On the order of Hall triple systems*, J. Combin. Theory A, **29** (1980), 261-262.
[10] M. HALL JR., *Theory of groups*, Chelsea Publishing Company (1959).
[11] M. HALL JR., *Automorphisms of Steiner triple systems*, I.B.M. J. Res. Develop., **4** (1960), 460-471.
[12] M. HALL JR., *Group theory and block designs*, in *Proceedings, International Conference on the Theory of Groups, Canberra*, Gordon and Breach, New York (1965).
[13] M. HALL JR., *Incidence axioms for affine geometries*, J. Algebra, **21** (1972), 535-547.
[14] M. G. OXENHAM, *On n-covers of $PG(3, q)$ and related structures*, Doctoral Thesis, University of Adelaide (1991).
[15] D. K. RAY-CHAUDHURI - R. ROTH, *Hall triple systems and commutative Moufang exponent 3 loops: the case of nilpotence class 2*, J. Combin. Theory A, **36** (1984), 129-162.

Dept. of Pure Mathematics, University of Adelaide - Adelaide, South Australia, 5001

Pervenuta in Redazione
l'11 giugno 1996 e, in forma rivista, l'11 marzo 1997