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Lyapunov Exponents, KS-Entropy and Correlation Decay in Skew Product Extensions of Bernoulli Endomorphisms.

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Sunto. – *Viene considerata una classe di sistemi dinamici del toro bidimensionale* $\mathbb{T}^2$. *Tali sistemi presentano la forma di un prodotto skew fra l'endomorfismo Bernoulli* $B_p(x) = px \bmod 1$, $p \in \mathbb{Z} \backslash \{-1, 0, 1\}$, *definito sul toro undidimensionale* $\mathbb{T}^1 \equiv [0, 1)$ *ed una traslazione del toro stesso. Si dimostra che gli esponenti di Liapunov e l'entropia di Kolmogorov-Sinai della misura di Haar invariante possono essere calcolati esplicitamente. Viene infine discusso il decadimento delle correlazioni per i caratteri.*

1. – Introduction and statement of the results.

We study here mappings of the 2-torus $\mathbb{T}^2$ defined by

$$M_\phi(x, y) \equiv (px, y + \phi(x)) \bmod 1 , \tag{1.1}$$

where p is any integer number different from $-1, 0, 1$ and ϕ stands for an appropriate real valued function of the 1-torus $\mathbb{T}^1$. By $\mathbb{T}^1$ we mean the Abelian group $(\mathbb{R}/\mathbb{Z}, +)$ endowed with the distance $d_1 \colon \mathbb{T}^1 \times \mathbb{T}^1 \to \mathbb{R}_+$

$$d_1(x_1, x_2) \equiv \min \{ |x_1 - x_2|, 1 - |x_1 - x_2| \}, \qquad x_1, x_2 \in \mathbb{T}^1 , \tag{1.2}$$

which makes $(\mathbb{R}/\mathbb{Z}, +)$ compact. In what follows the torus $\mathbb{T}^1$ is parametrized by the interval $[0, 1)$. We denote with $\mathfrak{B}_1$ the Borel σ-algebra of $(\mathbb{T}^1, d_1)$ and by μ_1 the normalized Haar measure on $(\mathbb{T}^1, \mathfrak{B}_1)$. The real function ϕ in (1.1) will be μ_1-measurable. Analogously we set $\mathbb{T}^2 \equiv \mathbb{T}^1 \times \mathbb{T}^1 = (\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}, +)$, with the metric

$$d_2((\xi_1, \xi_2), (\eta_1, \eta_2)) \equiv \max_{i=1,2} \min \{ |\xi_i - \eta_i|, 1 - |\xi_i - \eta_i| \}, \tag{1.3}$$

$$(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{T}^2$$

and introduce on the Borel σ-algebra $\mathfrak{B}_2$ of the compact $(\mathbb{T}^2, d_2)$ the corresponding normalized Haar measure. Moreover, the unit square $[0, 1)^2$ will provide the usual parametrization of $\mathbb{T}^2$. With respect to the invariant product measure $\mu_2 = \mu_1 \times \mu_1$ on $\mathfrak{B}_2 = \mathfrak{B}_1 \times \mathfrak{B}_1$ (1.1) can be viewed as the skew product of the toral maps

$$B_p(x) \equiv px \bmod 1\,, \qquad x \in \mathbb{T}^1 \tag{1.4}$$

and

$$T_x(y) \equiv y + \phi(x) \bmod 1\,, \qquad y \in \mathbb{T}^1\,. \tag{1.5}$$

The triple $(\mathbb{T}^1, \mu_1, B_p)$ constitutes the so-called dynamical system acting on the base of the skew product and is well-known to be a Bernoulli endomorphism, whereas $(\mathbb{T}^1, \mu_1, T_x)$ provides a family of toral translations measurably dependent on x and acting on the fibers. Of course, both base and fibers coincide with $\mathbb{T}^1$ for this class of mappings.

With the particular choice $p = 2$ and $\phi(x) = \omega + \varepsilon x$, $\varepsilon, \omega \in \mathbb{R}$, maps of the form (1.1) were originally considered in relation to models of modulated diffusion for Hamiltonian systems subjected to deterministic noise [2], [12]. A full characterization of ergodicity, weak and strong mixing was given in [14] for any value of the parameters ε and ω, whereas the exactness of the skew product for irrational ε was proved in [6] by Parry, who also applied Perron-Frobenius techniques to show that the correlation decay of characters is exponential. More recently, spectral methods have been successfully applied to deal with the case of arbitrary $p \in \mathbb{Z}\backslash\{-1, 0, 1\}$ and to estimate the rate of correlation decay for analytic and sufficiently smooth (depending on the choice of p, ε, ω) observables [13].

In the present work we discuss some questions about existence of characteristic exponents, computation of Kolmogorov-Sinai entropy of the invariant measure μ_2 and mixing. In Section 2 we prove the result below.

PROPOSITION 1. – *Let ϕ be a real differentiable function of $\mathbb{T}^1$, with bounded derivative. Then all the points of $\mathbb{T}^2$ are regular* [4] *for the map M_ϕ. The Lyapunov exponents are constant everywhere and take the values* $\log|p|$ *and* 0 *respectively.*

The statement does not follows from Oseledec's multiplicative ergodic theorem [5], [9], [11], since the regularity condition occurs on the whole torus and not only μ_2-almost everywhere. Moreover the constancy of Lyapunov exponents can be established without any hypothesis about the ergodicity of the map M_ϕ (which could actually be non-ergodic) and the values are explicitly computed. Proposition 1 allows us to get the subsequent Proposition 2, whose proof is presented in Section 3.

PROPOSITION 2. – *Let M_ϕ be ergodic. Then the Kolmogorov-Sinai entropy of the invariant measure μ_1 is* $\log|p|$.

The final issue of the paper concerns the correlation decay of characters and generalizes Parry's results in [6].

PROPOSITION 3. – *Let ϕ be Lipschitz continuous in $\mathbb{T}^1$. Consider any non-constant character of $\mathbb{T}^2$*

$$e_{a,b}(x, y) \equiv e^{i2\pi(ax+by)}(a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}, \qquad (x, y) \in \mathbb{T}^2. \tag{1.6}$$

Then the autocorrelations of $e_{a,b}$

$$C_n(a, b) \equiv \int_{\mathbb{T}^2} \overline{e_{a,b}(x, y)}\, e_{a,b} \circ M_\phi^n(x, y)\, d\mu_2(x, y), \qquad n \geq 0 \tag{1.7}$$

decay exponentially to zero if there exists no constant $\alpha \in \mathbb{R}$ such that the equation

$$R(B_p(x)) = e^{i2\pi[a(p-1)x + b\phi(x)]} R(x)\, e^{i\alpha} \tag{1.8}$$

admits a solution $R\colon \mathbb{T}^1 \to \mathbb{C}$, $R \in L^2(\mathbb{T}^1, \mathfrak{B}_1, \mu_1)$, which is not null μ_1-almost everywhere.

Notice that if the previous condition occurs $\forall (a, b) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ then the map M_ϕ is strong mixing. The proposition is shown and commented in Section 4.

2. – Lyapunov exponents.

We prove Proposition 1. To this end we preliminary observe that the Jacobian matrix of M_ϕ at $(x, y) \in \mathbb{T}^2$ takes the form

$$DM_\phi(x, y) \equiv \begin{pmatrix} p & 0 \\ \phi'(x) & 1 \end{pmatrix} \tag{2.1}$$

where $\phi'(x)$ is the derivative of ϕ at x. The regularity of $(x, y) \in \mathbb{T}^2$ for M_ϕ is equivalent to the convergence of the matrix sequence

$$[(M^n_{(x,y)})^\dagger M^n_{(x,y)}]^{1/2n} \tag{2.2}$$

where we have posed

$$M^n_{(x,y)} \equiv \tag{2.3}$$

$$DM_\phi(M_\phi^{n-1}(x, y))\, DM_\phi(M_\phi^{n-2}(x, y)) \dots DM_\phi(M_\phi(x, y))\, DM_\phi(x, y),$$

denoted with $(M^n_{(x,y)})^\dagger$ the adjoint of $M^n_{(x,y)}$ and taken the $2n$-th root of the real symmetric non-negative matrix $(M^n_{(x,y)})^\dagger M^n_{(x,y)}$. Throughout the paper the orbit of an arbitrary $(x_0, y_0) \in \mathbb{T}^2$ will be written as $(x_n, y_n) \equiv M_\phi^n(x_0, y_0)$, $n \geq 0$.

It is then simple to show by induction that $\forall n \geqslant 1$ and $\forall (x_0, y_0) \in \mathbb{T}^2$ there holds

$$(2.4) \qquad M^n_{(x_0,\, y_0)} = \begin{pmatrix} p^n & 0 \\ \sum_{j=0}^{n-1} p^j \phi'(x_j) & 1 \end{pmatrix}$$

and consequently

$$(2.5) \qquad (M^n_{(x_0,\, y_0)})^\dagger M^n_{(x_0,\, y_0)} = \begin{pmatrix} p^{2n} + \left(\sum_{j=0}^{n-1} p^j \phi'(x_j) \right)^2 & \sum_{j=0}^{n-1} p^j \phi'(x_j) \\ \sum_{j=0}^{n-1} p^j \phi'(x_j) & 1 \end{pmatrix}.$$

This leads to the expression

$$(2.6) \qquad [(M^n_{(x_0,\, y_0)})^\dagger M^n_{(x_0,\, y_0)}]^{1/2n} = \frac{1}{\lambda_n^+ - \lambda_n^-} \cdot$$

$$\begin{pmatrix} (\lambda_n^+ - 1)(\lambda_n^+)^{1/2n} - (\lambda_n^- - 1)(\lambda_n^-)^{1/2n} & b_n[(\lambda_n^+)^{1/2n} - (\lambda_n^-)^{1/2n}] \\ b_n[(\lambda_n^+)^{1/2n} - (\lambda_n^-)^{1/2n}] & (1 - \lambda_n^-)(\lambda_n^+)^{1/2n} + (\lambda_n^+ - 1)(\lambda_n^-)^{1/2n} \end{pmatrix}$$

where $b_n \equiv \sum_{j=0}^{n-1} p^j \phi'(x_j)$ and

$$(2.7) \qquad \lambda_n^\pm \equiv \frac{1}{2}(1 + p^{2n} + b_n^2) \pm \frac{1}{2}\sqrt{(1 + p^{2n} + b_n^2)^2 - 4p^{2n}}\,.$$

According to our hypothesis there exists $K \equiv \sup_{x \in \mathbb{T}^1} |\phi'(x)| \in \mathbb{R}_+$ and it is then straightforward to achieve the basic inequality

$$(2.8) \qquad p^{2n} \leqslant \lambda_n^+ \leqslant [2 + K^2(|p| - 1)^{-2}]\, p^{2n}, \qquad \forall n \in \mathbb{N}$$

which together with the trivial identity $\lambda_n^+ \lambda_n^- = p^{2n}$ gives

$$(2.9) \qquad \lim_{n \to +\infty} \frac{(\lambda_n^+)^{1/2n} - (\lambda_n^-)^{1/2n}}{\lambda_n^+ - \lambda_n^-} = 0\,.$$

Hence we deduce, for every $(x_0, y_0) \in \mathbb{T}^2$,

$$(2.10) \qquad \lim_{n \to +\infty} (M^n_{(x_0,\, y_0)})^\dagger M^n_{(x_0,\, y_0)} = \begin{pmatrix} |p| & 0 \\ 0 & 1 \end{pmatrix}$$

so that the Lyapunov exponents are $\log|p|$ and $\log 1 = 0$, everywhere constant. The proof is complete. ■

3. – KS-entropy. Proof of proposition 2.

For the entropy of the product measure $\mu_2 = \mu_1 \times \mu_1$ Abramov and Rokhlin [1], [8] gave the formula

$$h(\mu_1 \times \mu_1) = h(\mu_1) + \widehat{h}\,, \tag{3.1}$$

where $h(\mu_1) = \log|p|$ is the entropy of the factor B_p and $\widehat{h}$ is a non-negative constant obtained by taking the supremum, over all the finite partitions β of the fiber space $\mathbb{T}^1$, of the non-negative quantity

$$h_B(\beta, T) = \lim_{n \to +\infty} \frac{1}{n} \int_{\mathbb{T}^1} H(\beta_0^n(x))\, d\mu_1(x)\,. \tag{3.2}$$

The symbol $H(\beta_0^n(x))$ in the r.h.s. stands for the entropy of the partition $\beta_0^n(x) = \beta \vee \beta_1^n(x)$ defined by

$$\beta_1^n(x) = T_x^{-1}\beta \vee T_x^{-1} T_{B_p(x)}^{-1}\beta \vee \ldots \vee T_x^{-1} T_{B_p(x)}^{-1} \ldots T_{B_p^{n-1}(x)}^{-1}\beta \tag{3.3}$$

on having denoted with T_x^{-1} the inverse of the mapping T_x. The direct computation of $\widehat{h}$ is quite cumbersome, but we can avoid it. Owing to the ergodicity and differentiability of M_ϕ on the finite dimensional manifold $\mathbb{T}^2$ — the support of the invariant measure μ_2 coincides with the whole torus, which is compact —, Ruelle's formula [10] holds and allows to bound the entropy of $\mu_1 \times \mu_1$ by the sum of the positive Lyapunov exponents of M_ϕ

$$h(\mu_1 \times \mu_1) \leqslant \log|p|\,. \tag{3.4}$$

As a consequence, since (3.1) implies $h(\mu_1 \times \mu_1) \geqslant \log|p|$, we conclude that

$$\widehat{h} = 0 \qquad \text{and:}\ h(\mu_1 \times \mu_1) = \log|p| \tag{3.5}$$

which completes the proof. ■

4. – Decay of correlations for characters. Proof of Proposition 3.

We preliminary observe that $\forall n \geqslant 1$ the definition (1.1) implies

$$(x_n, y_n) = \left(p^n x_0,\ y_0 + \sum_{j=0}^{n-1} \phi(x_j)\right) \bmod 1 \tag{4.1}$$

so that the autocorrelations (1.7) of $e_{a,b}$ become

$$C_n(a, b) = \int_{\mathbb{T}^1} e^{i2\pi\left[a(p^n - 1)x_0 + b\sum_{j=0}^{n-1}\phi(x_j)\right]}\, d\mu_1(x_0) \tag{4.2}$$

and since $a(p^n - 1)x_0 \bmod 1 = a(p-1)\sum_{j=0}^{n-1} x_j \bmod 1$, take the equivalent form

$$C_n(a, b) = \int_{\mathbb{T}^1} e^{i2\pi \sum_{j=0}^{n-1}[a(p-1)x_j + b\phi(x_j)]}\, d\mu_1(x_0)\,. \tag{4.3}$$

Let us pose now $f(x) \equiv a(p-1)\,x + b\phi(x)$ and consider, for simplicity's sake, the case of positive p. It is understood that the final results are still valid when $p \leqslant -2$, even if the calculations are slightly different. Following [6] we introduce the symbolic dynamics of the Bernoulli map B_p. In the space $\Sigma_p^+ \equiv \prod_{i=0}^{\infty} \{0, 1, \ldots, p-1\}$ of the one-sided sequences of p symbols 0, 1, 2, …, $p-1$, let $\mathfrak{M}_p$ be the σ-algebra generated by the cylindrical sets

$$C(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{j+l}) \equiv \equiv \{\omega \equiv (\omega_i)_{i=0}^{\infty} \in \Sigma_p^+ : \omega_k = \alpha_k,\ \forall k = j+1, \ldots j+l\}. \tag{4.4}$$

A probability measure m_p is then uniquely determined on $\mathfrak{M}_p$ by posing, for each cylinder

$$m_p(C(\alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_{j+l})) \equiv p^{-l}. \tag{4.5}$$

The shift map σ on the probability space $(\Sigma_p^+, \mathfrak{M}_p, m_p)$ is given by

$$\sigma(\omega) = \omega', \qquad \omega'_i = \omega_{i+1}, \qquad \forall i \geqslant 0, \qquad \forall \omega \in \Sigma_p^+ \tag{4.6}$$

and admits m_p as an invariant measure. The Bernoulli shift obtained can be easily conjugated (mod 0) with $(B_p, \mathbb{T}^1, \mathfrak{B}_1, \mu_1)$ by means of the mapping

$$\chi(\omega) \equiv \sum_{i=0}^{\infty} \omega_i p^{-(i+1)} \tag{4.7}$$

defined in Σ_p^+ onto $\mathbb{T}^1$. χ is also one-to-one in $\mathbb{T}^1 \backslash D_p$, on having denoted with D_p the set of p-adyc numbers in the unit interval $[0, 1)$, such that $\mu_1(D_p) = 0$. Finally it maps the Bernoulli measure m_p on Σ_p^+ to the Haar-Lebesgue measure μ_1, so that the autocorrelations

$$C_n(a, b) = \int_{\mathbb{T}^1} e^{i2\pi \sum_{j=0}^{n-1} f(x_j)}\, d\mu_1(x_0) \tag{4.8}$$

can be rewritten into the form

$$C_n(a, b) = \int_{\Sigma_p^+} e^{i2\pi \sum_{j=0}^{n-1} f[B_p^j(\chi(\omega))]}\, dm_p(\omega) = \int_{\Sigma_p^+} (\mathcal{L}^n 1)(\omega)\, dm_p(\omega), \tag{4.9}$$

where $1 : \Sigma_p^+ \to \mathbb{R}$ is the constant function of value 1 and $\mathcal{L}$ the complex Ruelle-Perron-Frobenius (RPF) operator

$$(\mathcal{L}h)(\omega) \equiv \sum_{\overline{\omega} :\, \sigma(\overline{\omega}) = \omega} e^{-\log p + i2\pi f[\chi(\overline{\omega})]} h(\overline{\omega}),\ h : \Sigma_p^+ \to \mathbb{C}, \qquad h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p). \tag{4.10}$$

Now notice that:

i) with respect to the distance $d_{1/p} : \Sigma_p^+ \times \Sigma_p^+ : \to \mathbb{R}_+$ defined by $d_{1/p}(\omega, \overline{\omega}) \equiv p^{-N}$, being N the largest integer such that $\omega_i = \overline{\omega}_i$, $0 \leqslant i < N$, the conjugation χ is Lipschitz continuous

$$|\chi(\omega) - \chi(\varpi)| \leq d_{1/p}(\omega, \varpi), \qquad \forall \omega, \ \varpi \in \Sigma_p^+ \tag{4.11}$$

and so is $f \circ \chi$

$$\begin{aligned} |f \circ \chi(\omega) - f \circ \chi(\varpi)| &\leq \\ &|a(p-1)|\,|\chi(\omega) - \chi(\varpi)| + |b|\,|\phi \circ \chi(\omega) - \phi \circ \chi(\varpi)| \leq \\ &(|a(p-1)| + |b|\,\kappa)|\chi(\omega) - \chi(\varpi)| \leq (|a(p-1)| + |b|\,\kappa)\, d_{1/p}(\omega, \varpi), \end{aligned} \tag{4.12}$$

where κ stands for the Lipschitz constant of ϕ;

ii) the shift (Σ_p^+, σ) is obviously aperiodic [4];

iii) the associated real RPF operator

$$(\mathcal{L}_r h)(\omega) = \sum_{\varpi:\, \sigma(\varpi) = \omega} e^{-\log p} h(\varpi), \quad \forall h: \Sigma_p^+ \to \mathbb{C}, \quad h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p) \tag{4.13}$$

is normalized, i.e. $\mathcal{L}_r 1 = 1$. Therefore m_p is the only equilibrium probability measure for $\mathcal{L}_r$, according to Ruelle-Perron-Frobenius theorem [3], [7].

Then it is known [7] that the RPF operator (4.10), considered on the (suitably normed) space $C(\Sigma_p^+)$ of continuous functions in $(\Sigma_p^+, d_{1/p})$, has a spectral radius strictly less than 1 if the isometric operator

$$(Vh)(\omega) \equiv e^{-i2\pi f[\chi(\omega)]} h(\sigma(\omega)), \qquad h: \Sigma_p^+ \to \mathbb{C}, \qquad h \in L^2(\Sigma_p^+, \mathfrak{M}_p, m_p) \tag{4.14}$$

admits no eigenfunction (with eigenvalue necessarily of the form $e^{i\alpha}$, a phase factor). It is obvious that the latter condition, reprojected to $L^2(\mathbb{T}^1, \mathfrak{B}_1, \mu_1)$ via χ, coincides with (1.8). Since $1 \in C(\Sigma_p^+)$ we conclude that

$$|C_n(a, b)| = \left| \int_{\Sigma_p^+} (\mathcal{L}^n 1)(\omega)\, dm_p(\omega) \right| \leq K r^n, \qquad \forall n \in \mathbb{N} \tag{4.15}$$

for some constants $K > 0$ and $r \in (0, 1)$. ■

It is noticeable that when the solvability of (1.8) can be excluded for any non-integer $e^{i\alpha}$, Parry's technique [6] leads to the curious result that weak mixing of M_ϕ implies unsolvability of (1.8) for $e^{i\alpha} \in \mathbb{Z}$ too, and therefore mixing. Of course the ergodicity of M_ϕ is fully characterized by Anzai's criterion [8].

Another remark concerns correlation decay. Even if Proposition 3 were satisfied for any $(a, b) \in \mathbb{Z}^2 \setminus \{0, 0\}$, the RPF operator would not provide an explicit relation between (a, b) and the decay rate of $C_n(a, b)$. Therefore it would not be possible to bound the correlation decay for observables other than characters or finite linear combinations of them. With regard to this subject the only available results are those obtained in [13] for $\phi(x) = \omega + \varepsilon x$, by means of a detailed Fourier analysis.

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