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Sonia Brivio

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On Rank 2 Semistable Vector Bundles Over an Irreducible Nodal Curve of Genus 2.

SONIA BRIVIO (*)

Sunto. – *Sia C una curva irriducibile nodale di genere aritmetico $p_a = 2$. In queste note vogliamo mostrare come il sistema lineare delle quadriche, contenenti un opportuno modello proiettivo della curva, permette di descrivere i fibrati vettoriali semistabili, di rango 2, su C.*

Introduction.

Let C be a smooth, complex, projective curve of genus 2 and let $\mathcal{SU}(2, \omega_C)$ be the moduli space of semistable, rank 2 holomorphic, vector bundles over C, with determinant isomorphic to the canonical line bundle ω_C. As it is well known, this moduli space was studied by Narasimhan and Ramanan, in [N-R1]. They proved that $\mathcal{SU}(2, \omega_C)$ is naturally isomorphic to the 3-dimensional linear system $|2\Theta|$, where Θ is a symmetric theta divisor on the Jacobian variety $J(C)$ of C.

In general, if C is an irreducible projective curve, it is well known that the existence of singular points implies that torsion free sheaves are not necessary locally free. Even in the case of line bundles, there is no hope to obtain complete moduli spaces unless we include all torsion free sheaves, (see [D]). Actually, all the theory of vector bundles can be properly modified and applied to torsion free sheaves, to prove the existence of a coarse moduli space $M(r, d)$ for semistable torsion free sheaves of rank r and degree d over C, (see [S2] or [N]); in particular for $r = 1$, $M(1, d)$ is called the Generalized Jacobian and denoted by $\mathrm{Jac}^d(C)$.

The aim of these notes is to give a geometrical description of semistable rank-2 vector bundles, with determinant ω_C, over an irreducible projective curve C, (not necessary smooth), with arithmetic genus $p_a = 2$. At this end, we restrict our attention to curves C whose only singularities are ordinary double points, (i.e. nodes). In fact, for such a curve the canonical sheaf ω_C is inverti-

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ble, moreover the fibre, at any point, of a torsion free sheaf F is completely known (see [S2]). More precisely: for $t \in \mathrm{Pic}^2(C)$ let $O_C(H) = \omega_C(2t)$, then the linear system $|H|$ gives an embedding $C \hookrightarrow \boldsymbol{P}^4$, with $|I_C(2)| \simeq \boldsymbol{P}^3$, where I_C is the ideal sheaf of C; let $X \subset M(2, 6)$ be the closure of the subset corresponding to vector bundles over C with determinant $O_C(H)$, we produce an isomorphism (see Th. 4.1)

$$\phi \colon X \to |I_C(2)| ,$$

which completely describes X in term of the quadrics of $\boldsymbol{P}^4$ containing the curve C.

If C is smooth, then actually $X \simeq \mathcal{SU}(2, \omega_C)$, so Th. 4.1 furnishes an alternative proof of the result of [N-R1].

Let C be singular: we show that vector bundles of X corresponds to quadrics containing C, whose singular locus does not contain any node of C, (see Prop. 3.4 and 3.7). Actually, for any node $p \in C$, there is a sheaf $N_p \subset |I_C(2)|$, of quadrics whose wertex contains p, each quadric of N_p corresponds to a torsion free sheaf in X which is not stable, hence can be represented by $A_1 \oplus A_2$, with both A_i not invertible at p, (see Prop. 4.7). Finally, these turns out to be the only sheaves which are limits of vector bundles with determinant ω_C, over C.

Moreover, let $X_{ss} \subset X$ be the closed subset of not stable points: if C is smooth, then as it is well known $X_{ss} \simeq \mathcal{SU}(2, \omega_C)_{ss}$, which is isomorphic to the Kummer surface K associated to the Jacobian variety $J(C)$, see [N-R1]. We generalize this result as follows: we produce a regular involution $\bar{i}$: $\mathrm{Jac}^1(C) \to \mathrm{Jac}^1(C)$, which for any $L \in \mathrm{Pic}^1(C)$ is the natural involution $L \to \omega_C \otimes L^{-1}$, (see Prop. 4.3), and we prove, (see 4.9) that

$$X_{ss} \simeq \frac{\mathrm{Jac}^1(C)}{\bar{i}} .$$

We will call this quotient the generalized Kummer surface of C. Finally, the discriminant surface in $|I_C(2)|$ is reducible into two components: one of these is a quartic surface S_4 which turns out to be the image (via ϕ) of the generalized Kummer surface.

We point out that our method applies to curves of any genus p_a, though involves more delicate technical problems, (see [B-V] for the smooth case).

Finally, I wish to thank Alessandro Verra for his huseful suggestions on the matter.

1. – Preliminaries.

Let C be a complex, projective, irreducible curve, with at most nodes as singularities, with arithmetic genus $p_a \geq 2$. We recall that such a curve C is stable «in the sense of moduli», i.e. it has only finitely many automorphisms. We recall the following results:

PROPOSITION 1.1. – *Let C be a stable irreducible curve, ω_C its dualizing sheaf, $\pi: \tilde{C} \to C$ be the normalization of C, and $p_1 \dots p_d$ be the nodes of C. Then:*

a) *ω_C is invertible;*

b) *$\pi^* \omega_C = \omega_{\tilde{C}} \otimes O_{\tilde{C}}(D)$, where $D = \sum_{i=1}^{d} P_i^+ + \sum_{i=1}^{d} P_i^-$, such that $\pi(P_i^+) = \pi(P_i^-) = P_i$, for each $i = 1, \dots, d$;*

c) *$h^0(C, \omega_C) = p_a(C)$;*

d) $\deg(\omega_C) = 2p_a(C) - 2$;

e) *If $p_a(C) \geqslant 2$, then ω_C is ample, and ω_C^n is very ample for $n \geqslant 3$.*

For the proof see [B]. Note that if $p_a(C) = 2$, then C has at most two nodes.

THEOREM 1.2. – *Let C be a stable irreducible curve and let H be a Cartier divisor on C. If* $\deg(H) \geqslant 2p_a(C) + 1$, *then H is very ample on C; if* $\deg(H) \geqslant 2p_a(C) - 1$ *then $h^1(C, O_C(H)) = 0$, moreover if* $\deg H = 2p_a(C) - 2$ *then $h^1(O_C(H)) \geqslant 1$ if and only if $H \equiv K_C$.*

For the proof of see [C-F-R].

Let C be a stable irreducible curve with $p_a(C) = 2$ and H a Cartier divisor on it of degree 6. By (1.2) H is very ample, so we can assume

$$C \hookrightarrow \boldsymbol{P}^4 = \boldsymbol{P}(H^0(C, O_C(H))^*),$$

and let I_C denote its ideal sheaf.

PROPOSITION 1.3. – *In the above hypothesis:*

i) *$|I_C(2)| \simeq \boldsymbol{P}^3$, and the general element is a rank 5 quadric;*

ii) *C is projectively normal in $\boldsymbol{P}^4$;*

iii) *$C = \Sigma \cdot Q$, where Σ is a rational normal cubic scroll of $\boldsymbol{P}^4$, $Q \in |O_{\boldsymbol{P}^4}(2)|$.*

PROOF. – First of all note that $C \subset \boldsymbol{P}^4$ is linearly normal. Let's consider now the exact sequence of sheaves:

(1.3.1) $$0 \to I_C(2) \to O_{\boldsymbol{P}^4}(2) \to O_C(2H) \to 0$$

which induces the global sections sequence

$$(1.3.2)\quad 0 \to H^0(I_C(2)) \to H^0(O_{P^4}(2)) \xrightarrow{r_2} H^0(O_C(2H)) \to H^1(I_C(2)) \to 0\,,$$

one can easily see that $h^0(I_C(2)) \geq 4$.

Since C is an extremal curve in $\boldsymbol{P}^4$, then by Castelnuovo theory, (see [A-C-G-H]), C is projectively normal; in particular r_2 is surjective, so that $h^0(I_C(2)) = 4$. Moreover for such a curve the ideal I_C is generated by quadrics.

Let's consider the natural multiplication map

$$(1.3.3)\qquad H^0(\omega_C) \otimes H^0(O_C(H-K_C)) \xrightarrow{\mu} H^0(O_C(H))\,,$$

it is non degenerate, and thus gives rise to a rational normal cubic scroll $\Sigma \subset \boldsymbol{P}^4$ containing the curve, and all the lines spanned by the linear system $|\omega_C|$, see [E-H], Th. 2. Moreover, $|I_\Sigma(2)| \simeq \boldsymbol{P}^2$, and Σ is smooth if and only if μ is surjective. Let $H \not\equiv 3K_C$, since ω_C is base points free, (see [C-F], Cor 2.5), we have:

$$(1.3.4)\qquad 0 \to \omega_C^{-1} \to H^0(\omega_C) \otimes O_C \to \omega_C \to 0\,,$$

by tensoring with $O_C(H-K_C)$

$$(1.3.5)\quad 0 \to (O_C(H-2K_C)) \to H^0(\omega_C) \otimes O_C(H-K_C)) \to O_C(H) \to 0\,,$$

since $h^1(O_C(H-2K_C)) = 0$, passing to cohomology we have

$$(1.3.6)\ 0 \to H^0(O_C(H-2K_C)) \to H^0(\omega_C) \otimes H^0(O_C(H-K_C)) \xrightarrow{\mu} H^0(O_C(H)) \to 0\,,$$

that is μ is surjective. So for $H \not\equiv 3K_C$, we can conclude that Σ is the embedding of the rational surface F_1 by the line bundle $O_{F_1}(\sigma + 2f)$, where σ is the fundamental section and f is the fibre. One can easily verify that $C \equiv 2\sigma + 4f$, since Σ is projectively normal and generated by quadrics, this implies $C = Q \cdot \Sigma$, with $Q \in |O_{P^4}(2)|$.

REMARK 1.4. – Let's consider the following subvariety of $|I_C(2)|$:

$$(1.4.1)\qquad \Delta := \{Q \in |I_C(2)| :\ \mathrm{rk}\, Q \leq 4\}\,,$$

since $|I_\Sigma(2)| \subset \Delta$, it is a reducible surface: $\Delta = S_4 \cup |I_\Sigma(2)|$, and S_4 is a quartic surface. Assume that C is singular, let $p \in \mathrm{Sing}(C)$, we can consider the set

$$(1.4.2)\qquad N_p := \{Q \in \Delta :\ p \in \mathrm{Sing}(Q)\}\,,$$

then it is easy to verify that N_p is is a double line in Δ, which intersects $|I_\Sigma(2)|$ in a unique point, let's denote it by Q_p.

1.5. Here we show a natural way to associate a quadric to any rank 2 vector bundle over a projective curve C, see [B-V]. We consider pairs (E, V) with the following properties:

a) E is a rank 2 vector bundle on C,

b) V is a 4-dimensional vector space in $H^0(E)$,

c) $\det O_C(H)$ is a very ample line bundle on C.

We associate to every pair (E, V):

i) the evaluation map $e_V \colon V \otimes O_C \to E$,

ii) the determinant map $d_V \colon \wedge^2 V \to H^0(OC(H))$,

iii) the Grassmannian $G_V^* \subset \boldsymbol{P}(\wedge^2 V^*)$ of 2-dimensional subspaces of V^*.

For $x \in C$, we set $V_x^* = \operatorname{Im} e_{V_x}^*$, if e_V is generically surjective we have a rational map

$$g_V \colon C \to G_V^* \tag{1.5.1}$$

by associating to x the point $\wedge^2 V_x^* \in G_V^*$. We call g_V *the Gauss map of the pair* (E, V).

Let p be an equation for G_V^*, we consider the dual map

$$d_V^* \colon H^0(O_C(H))^* \to \wedge^2 V^* , \tag{1.5.2}$$

we define $q(E, V) \in \operatorname{Sym}^2(H^0(O_C(H)))$ the pull back of p.

Since H is very ample, we can assume the curve

$$C \hookrightarrow \boldsymbol{P}^n = \boldsymbol{P}(H^0(O_C(H))^*) , \tag{1.5.3}$$

if $q(E, V)$ is not identically zero, its zero locus is a quadric in $\boldsymbol{P}^n$: $Q = Q(E, V)$, with rank $r \leqslant 6$ containing C. Q can be considered as a cone of vertex $\boldsymbol{P}(\operatorname{Ker}(d_V^*))$ over the quadric $\boldsymbol{P}(\operatorname{Im}(d_V^*)) \cap G_V^*$, in particular $\boldsymbol{P}(\operatorname{Ker}(d_V^*)) = \operatorname{Sing} Q$ if and only if $\boldsymbol{P}(\operatorname{Im}(d_V^*))$ is transversal to G_V^*, see [B-V].

LEMMA 1.6. – *In the above hypothesis. Let r be the rank of $q(E, V)$. Then:*

i) $r \leqslant 4$ *if and only if* $\exists\ L \subset E$ *such that* $\dim V_L \geqslant 2$;

ii) $r = 0$ *if and only if* $\exists\ L \subset E$ *such that* $\dim V_L \geqslant 3$;

iii) e_V *is not generically surjective if and only if* $\exists\ L \subset E$ *such that* $\dim V_L = 4$.

For the proof see [B-V], Prop. (1.11).

LEMMA 1.7. – *Let* (E_1, V_1) *and* (E_2, V_2) *pairs with the following properties:*

i) $Q(E_1, V_1) = (E_2, V_2) = Q$, *with* $\operatorname{Sing} Q \cap C = \emptyset$, $\operatorname{rk} Q = 5$;

ii) V_i *generates* E_i;

then (E_1, V_1) *and* (E_2, V_2) *are isomorphic.*

For the proof see [B-V], Lemma 1.18.

Finally, in the sequel we recall some definitions and results about coherent sheaves over an irreducible stable curve C, (cf. [S2], [N]). Let F be a torsion free sheaf on C of rank r. The degree of F is defined as follows:

$$\deg F := \chi(F) - r\chi(O_C),$$

where χ is the Eulero Characteristic of F. For $x \in C$, let $\mathcal{M}_x$ be the sheaf of ideals defining x, we set

$$F_x := \frac{F}{\mathcal{M}_x F},$$

then F_x is a torsion sheaf with support x, and it is called the «fibre» of F at x. The fibre of a torsion free sheaf F need not have costant dimension as a vector space: in fact if x is not singular, then $F_x \simeq rO_x$, while if x is a node, then $F_x \simeq aO_x \oplus (r-a)\, \mathcal{M}_x$ for some integer $0 \leq a \leq r$, (see [S2], chap. 8).

LEMMA. – 1.8. – *Let* F *be a torsion free sheaf of rank* r *over* C, *let* $\pi' \colon C' \to C$ *be a partial normalization of* C *at the points* $x_i \in C$, $i = 1, \ldots, n$, $n \leq d$, *where* F *is not locally free. Then there exists* F' *locally free on* C' *such that* $\pi'_* F' = F$ *if and only if* $F_{x_i} \simeq r\mathcal{M}_{x_i}$. *Moreover, up to isomorphism* F' *is unique and* $\deg F' = \deg F - rn$.

For the proof see [S2].

Note that if $r = 1$, then every F which is not locally free can be identified with a line bundle F' on a unique partial normalization of C.

We say that a subsheaf $G \subset F$ is a subbundle of F is the quotient F/G is torsion free too. We define the *slope* of F as:

$$\mu(F) := \frac{\deg F}{\operatorname{rk} F},$$

we say that a torsion free sheaf F is *semistable* (*resp. stable*) if for every non zero proper subbundle G of F we have:

$$\mu(G) \leq \mu(F) \qquad (\text{resp.} <).$$

If F is semistable, then we can define a Jordan Holder filtration $\{F_i\}$ and

$\mathrm{Gr}(F) = \oplus F_i / F_{i+1}$ is uniquely defined, so that we can introduce the relation of S-equivalence as for vector bundles: F and G are said S-equivalent if and only if $\mathrm{Gr}(F) \simeq \mathrm{Gr}(G)$, see [S1].

Finally, we will consider flat families of torsion free sheaves, i.e. sheaves $\mathcal{F}$ over $S \times C$, flat over S, whose restriction $\mathcal{F}_s$ to $s \times C$ is torsion free for all $s \in S$. We have the fundamental result:

THEOREM 1.9. – *There exists a coarse moduli space $M_s(r, d)$ for stable torsion free sheaves on C of rank r and degree d, which admits a natural compactification to a projective variety $M(r, d)$, whose points correspond to classes of semistable torsion free sheaves over C under the relation of S equivalence.*

This result is also true for an irreducible projective singular curve, (with more general singularities), see [N], and can be opportunely extended to reducible curves too, see [S2]. Actually, if the curve C is stable we have the following further information: for $r \leqslant 2$, $M(r, d)$ is reduced and irreducible, (see [D] for $r = 1$ and [S2] for $r = 2$), moreover it has an open dense subset corresponding to locally free sheaves over C. For this reason, in the case $r = 1$, this moduli space is also called generalized Jacobian, and it is denoted by $\mathrm{Jac}^d(C)$, (see f.e. [A]), moreover it is Cohen Macaulay variety. If C is a smooth curve, every torsion free sheaf is locally free, so that $M(r, d) = U(r, d)$ is the well known moduli space of rank r semistable vector bundles on C with degree d.

2. – The moduli space $M(2, 6)$.

2.1. Here we fix the notations we will follow all over the paper:

C will be a stable irreducible curve with $p_a(C) = 2$ and $M(2, 6)$ the moduli space of semistable torsion free sheaves over C of rank 2 and degree 6; $M_s(2, 6) \subset M(2, 6)$ will be the open subset of stable torsion free sheaves, $M_{ss}(2, 6) = M(2, 6) - M_s(2, 6)$ the closed subset of not stable sheaves. Moreover, if $\mathrm{Sing}\, C \neq \emptyset$, for any node $p \in C$, we will introduce the following subsets of $M(2, 6)$, see [S2], chap. 8:

$$(2.1.1) \qquad \begin{cases} U_p^2 = \{E \in M(2, 6)\colon\ E_p \simeq 2O_p\}\,, \\ U_p^1 = \{E \in M(2, 6)\colon\ E_p \simeq O_p \oplus \mathcal{M}_p\}\,, \\ U_p^0 = \{E \in M(2, 6)\colon\ E_p \simeq 2\mathcal{M}_p\}\,, \end{cases}$$

with the properties: $\overline{U}_p^2 = U_p^2 \cup U_p^1 \cup U_p^0$ and $\overline{U}_p^1 = U_p^1 \cup U_p^0$. Let's call $V \subset M(2, 6)$ the open dense subset of $M(2, 6)$ corresponding to rank 2 vector bundles over C, note that if $\mathrm{Sing}(C) = \{p\}$, then $V \simeq U_p^2$, while if $\mathrm{Sing}(C) = \{p, q\}$, then $V \simeq U_p^2 \cap U_q^2$.

LEMMA. – 2.2. – *Let* $E \in M(2, 6)$, *then*

(1) $h^0(E) = 4$;

(2) if $E \in V$ *is not globally generated at some point* x *then* $h^0(E \otimes \omega_C^{-1}) \geq 1$;

(3) *for any subbundle* $L \subset E$ *we have* $h^0(L) \leq 2$, *moreover* $h^0(L) = 2$ *if and only if either* E *is not stable or* $L = \omega_C$.

PROOF. – (1) By applying Riemann Roch theorem we have $h^0(E) \geq 4$, if $h^1 > (E)0$ then there exists a morphism $\phi \in \mathrm{Hom}(E, \omega_C)$,

$$\phi\colon E \to \omega_C; \tag{2.2.1}$$

let's consider the subbundle $\ker\phi \subset E$, we have

$$\deg(\ker\phi) \geq \deg(\mathrm{E}) - \deg(\omega_C) = 4\,, \tag{2.2.2}$$

which contradics the semistability of E.

(2) We prove that $h^0(E \otimes \omega_C^{-1}) = 0$, implies E is a rank 2 vector bundle globally generated. Look at the exact sequence:

$$0 \to \mathcal{M}_x E \to E \to E_x \to 0\,, \tag{2.2.3}$$

where $\mathcal{M}_x \subset O_x$ is the sheaf of ideals of x, passing to cohomology we have

$$0 \to H^0(\mathcal{M}_x E) \to H^0(E) \to H^0(E_x) \to H^1(\mathcal{M}_x E) \to 0\,, \tag{2.2.4}$$

then E is generated at x if and only if $h^0(\mathcal{M}_x E) = 2$. Note that $H^0(E \otimes \omega_C^{-1})$ is the kernel of the multilication map

$$\nu\colon H^0(E) \otimes H^0(\omega_C) \to H^0(E \otimes \omega_C)\,, \tag{2.2.5}$$

hence if $h^0(E \otimes \omega_C^{-1}) = 0$ then ν is injective. Let $S_x = \nu^{-1}(H^0(E \otimes \omega_C \otimes \mathcal{M}_x))$, then

$$S_x = H^0(\omega_C \otimes \mathcal{M}_x) \otimes H^0(E) + H^0(\omega_C) \otimes H^0(E \otimes \mathcal{M}_x)\,, \tag{2.2.6}$$

since $E \otimes \omega_C$ is globally generated, then $\dim S_x = 6$, which implies $h^0(\mathcal{M}_x E) = 2$, for every $x \in C$.

(3) Since E is semistable, for any rank 1 torsion free sheaf $L \subset E$ we have $\deg L \leq 3$. If $\deg L = 3$ then $h^0(L) = 2$ and E is not stable. Let $\deg L \leq 2$: if L is invertible then then $h^0(L) \leq 2$ and actually $h^0(L) = 2$ if and only if $L = \omega_C$, see (1.2); if L is not locally free at $p \in \mathrm{Sing}(C)$, then $L = \pi_* \widetilde{L}$ with $\deg(\widetilde{L}) \leq 1$, see (1.8), of course this implies $h^0(L) \leq 1$.

LEMMA 2.3. – *There exists a natural surjective morphism*

$$\det\colon V \to \mathrm{Pic}^{(6)}(C),$$

by associating to each vector bundle E its determinant $\det E \simeq \wedge^2 E$.

PROOF. – Let $\mathcal{E}$ be a vector bundle over $S \times C$, where S is an irreducible variety with the property: $\mathcal{E}_{|s \times C}$ is a semistable rank 2 vector bundle on C with degree 6. Then $\wedge^2 \mathcal{E}$ is a line bundle over $S \times C$ and of course $(\wedge^2 \mathcal{E})_{|s \times C} \simeq \det(\mathcal{E}_{|s \times C})$, is an element of $\mathrm{Pic}^{(6)}(C)$. So we have a map

$$\sigma\colon S \to \mathrm{Pic}^{(6)}(C), \tag{2.3.1}$$

by sending $s \to (\wedge^2 \mathcal{E})_{|s \times C}$.

Let $L \in \mathrm{Pic}^{(6)}(C)$: we can consider vector bundles E given by the extensions

$$0 \to O_C \to E \to L \to 0, \tag{2.3.2}$$

of course $\det E \simeq L$, moreover these bundles are parametrized by $\boldsymbol{P}(\mathrm{Ext}^1(L, O_C))$. Actually, for a general element $e \in \boldsymbol{P}(\mathrm{Ext}^1(L, O_C))$ the corresponding bundle is semistable, see [BE], hence it defines a point of V.

If the curve C is smooth, then actually $V = U(2, 6)$, det is a surjective morphism with fibre the moduli space $SU(2, L)$, of semistable bundles with determinant isomorphic to $L \in \mathrm{Pic}^{(6)}(C)$. Note that if $E \in M(2, 6)$ is not locally free at p, then $\wedge^2 E$ is not a line bundle: actually the fibre at the point p is not even torsion free, since it contains $\wedge^2 \mathcal{M}_p$. Anyway the bidual sheaf $(\wedge^2 E)^{**}$ turns out to be torsion free.

LEMMA 2.4. – *Let $E \in M(2, 6)$, then E is an extension of some torsion free sheaves A and B:*

$$0 \to A \to E \to B \to 0;$$

moreover $(\wedge^2 E)^{**} \simeq (A \otimes B)^{**}$.

PROOF. – Let L be an invertible sheaf such that $F = E \otimes L$ is globally generated, i.e. the restriction map $H^0(F) \to F_x$ is surjective for any $x \in C$; it is enough to prove the assertion for F.

Let's consider the following subset of $H^0(F)$:

$$Y = \{ s \in H^0(F),\ \exists x \in C,\ s(x) = 0 \}, \tag{2.4.1}$$

it is easy to verify that $\dim Y \leq h^0(F) - 1$, which implies the existence of a section s with $s(x) \neq 0$ for any $x \in C$. This induces an injective map of sheaves $O_C \to F$. If F/O_C is actually torsion free, then we have an exact sequence

$$0 \to O_C \to F \to B \to 0\,, \tag{2.4.2}$$

and the assertion follows.

On the contrary, if $O_C \subset F$ is not a subbundle, then there exists a torsion free sheaf with $O_C \subset A \subset F$ and F/A is torsion free too, see [N], so the assertion follows.

Let E be extension of two torsion free sheaves A and B, then the sheaf $\wedge^2 E$ has a finite filtration as follows, see [H]:

$$\begin{cases} 0 \to E_1 \to \wedge^2 E \to \wedge^2 B \to 0\,, \\ 0 \to \wedge^2 A \to E_1 \to A \otimes B \to 0\,. \end{cases} \tag{2.4.3}$$

By applying the funtor Hom to these exact sequences we have $(\wedge^2 E)^{**} \simeq (A \otimes B)^{**}$.

We have the following:

Proposition 2.5. – *There is a rational map d: $M(2, 6) \to \mathrm{Jac}^6(\mathrm{C})$ by associating to $E \to (\wedge^2 E)^{**}$, such that for any vector bundle $d(E) = \det E$.*

Proof. – It follows from Lemmas 2.3 and 2.4 and the following facts:

1) $\mathrm{Hom}\,(\mathcal{M}_p, O_p) \simeq \mathrm{End}\,(\mathcal{M}_p) \simeq \mathrm{Hom}\,(\mathcal{M}_p \otimes \mathcal{M}_p, O_p) \simeq \mathcal{M}_p$, see [S2], chap. 8.

2) For any rank 1 torsion free sheaf A over C, $A^{**} \simeq A$.

Let $A = \pi^* \tilde{A}$, we have $\mathrm{Hom}\,(\pi^* \tilde{A}, O_C) \simeq \pi^*(\mathrm{Hom}\,(\tilde{A}, O_{\tilde{C}})) \otimes \omega_{\tilde{C}}^{-1}$, see [C], which implies the claim.

Actually, let E be extension of two torsion free sheaves A and B: if both are locally free, then obviously $(\wedge^2 E)^{**} \simeq \det E$. Assume that A is invertible and B is not. Then

$$(\wedge^2 E)^{**} \simeq A \otimes B^{**} \simeq A \otimes B\,, \tag{2.5.1}$$

hence it is an element of $\mathrm{Jac}^6(C)$, but not of $\mathrm{Pic}^6(C)$. Finally, if both A and B are not locally free at the same point p: then as we have already seen $A = \pi_* \tilde{A}$ and $B = \pi_* \tilde{B}$, one can verify that

$$(A \otimes B)^{**} \simeq \pi_*(\tilde{A} \otimes \tilde{B})\,, \tag{2.5.2}$$

so that $\deg\,(A \otimes B)^{**} \leq 5$, hence d is not defined at E.

Remark. – 2.6. – Let $H_1, H_2 \in \mathrm{Pic}^{(6)}(C)$, we prove that the open subsets of V: $d^{-1}(O_C(H_1))$ and $d^{-1}(O_C(H_2))$ are isomorphic. In fact, let $t \in \mathrm{Pic}^{(0)}(C)$ such that $O_C(H_1) \otimes 2t \simeq O_C(H_2)$; we have a natural map

$$\nu_t \colon d^{-1}(O_C(H_1)) \to d^{-1}(O_C(H_2)),$$

by sending E to $E \otimes t$, which actually turns out to be an isomorphism.

3. – The fundamental map.

3.1. For any line bundle $O_C(H) \in \mathrm{Pic}^{(6)}(C)$, we will consider the open subset

$$d^{-1}(O_C(H)) \subset V, \tag{3.1.1}$$

and we will study its closure in $M(2, 6)$, let's denote it by X. We will introduce the following subsets of X:

$$V_X := X \cap V, \tag{3.1.2}$$

the subset of X corresponding to vector bundles,

$$\begin{cases} X_s := X \cap M_s(2, 6), \\ X_{ss} := X \cap M_{ss}(2, 6), \\ \Omega = \{E \in X \colon h^0(E \otimes \omega_C^{-1}) \geq 1\}, \end{cases} \tag{3.1.3}$$

finally

$$W := V_X \cap X_s \cap (X - \Omega).$$

Moreover, by lemma 2.5 we can conclude that $X - V_X \subset \bigcup U_p^0$, for $p \in \mathrm{Sing}(C)$.

3.2. Assume $C \hookrightarrow \boldsymbol{P}^4$ by the linear system $|H|$, and I_C denote its ideal sheaf. Let $E \in X$ be a rank 2 vector bundle, by Lemma 2.2, $h^0(E) = 4$, and E does not contain any subbundle L of rank 1 with $h^0(L) \geq 3$: so we can consider the quadratic form $q(E, H^0(E))$ defined in (1.5), which actually is not zero. Hence its zero locus defines uniquely a quadric $Q(E, H^0(E))$ in the linear system $|I_C(2)|$. This allow us to define the following map

$$\phi \colon X \to |I_C(2)|, \tag{3.2.1}$$

by sending E to $Q(E, H^0(E))$. ϕ will be said the *fundamental map*. Let $Q = \phi(E)$: note that $\mathrm{rk}\, Q \leq 5$, moreover by (2.2), $\mathrm{rk}\, Q = 4$ if and only if either E is not stable or $\omega_C \subset E$ as a subbundle. Then we have the following result:

PROPOSITION 3.3. – *$\phi \colon V_X \to |I_C(2)|$ is a morphism.*

PROOF. – Let $\mathcal{V}$ be a family of rank 2 vector bundles on S, where S is an irreducible variety, s.t. $\forall s \in S$, $\mathcal{V}_s = \mathcal{V}_{|s \times C}$ is a semistable vector bundle of rank 2, with $\det E = O_C(H)$. As usual, we have a morphism

$$\sigma\colon S \to X\,, \tag{3.3.1}$$

just sending s to the equivalence class of $\mathcal{V}_s$.

We consider the sheaf on S:

$$p_{1_*}\mathcal{V}\,, \tag{3.3.2}$$

where $p_1\colon S \times C \to S$ is the first projection. Since $h^0(\mathcal{V}_s) = 4$, for any $s \in S$, then $p_{1_*}\mathcal{V}$ is a rank 4 vector bundles on S. Then we can consider the determinant map

$$d\colon \wedge^2(p_{1_*}\mathcal{V}) \to p_{1_*}(\wedge^2\mathcal{V})\,, \tag{3.3.3}$$

which at each point $s \in S$ is so defined:

$$d_s\colon \wedge^2 H^0(\mathcal{V}_s) \to H^0(O_C(H))\,. \tag{3.3.4}$$

Note that there exists a natural symmetric bilinear pairing

$$p\colon \wedge^2(p_{1_*}\mathcal{V}) \times \wedge^2(p_{1_*}\mathcal{V}) \to \wedge^4(p_{1_*}\mathcal{V})\,, \tag{3.3.5}$$

by sending $(s_1 \wedge s_2, s_3 \wedge s_4)$ to $(s_1 \wedge s_2 \wedge s_3 \wedge s_4)$; p turns out to be a global section of the space $H^0(S, \mathrm{Sym}^2(\wedge^2(p_{1_*}\mathcal{V}))$. Finally we consider the following map of sheaves:

$$d^2\colon \mathrm{Sym}^2(\wedge^2(p_{1_*}\mathcal{V})) \to \mathrm{Sym}^2(p_{1_*}(\wedge^2\mathcal{V}))\,, \tag{3.3.6}$$

which induces the global sections map

$$d_S^2\colon H^0(\mathrm{Sym}^2(\wedge^2(p_{1_*}\mathcal{V}))) \to H^0(\mathrm{Sym}^2(p_{1_*}(\wedge^2\mathcal{V})))\,, \tag{3.3.7}$$

we set $q := d_S^2(p)$, i.e. the image of the section p. Then by construction it turns out that $q(s) = q(\mathcal{V}_s, H^0(\mathcal{V}_s)) \in \mathrm{Sym}^2(H^0(O_C(H)))$, by (2.1), it is not identically zero. So we have a morphism

$$\phi_S\colon S \to |I_C(2)| \tag{3.3.8}$$

by sending s to $\operatorname{div} q(s)$; so we can conclude that ϕ is a morphism too.

Let's define the following open subset of $|I_C(2)|$:

$$V_5 := \{Q \in |I_C(2)|\, \mathrm{rk}\, Q = 5\}\,,$$

of course $V_5 = |I_C(2)| - \varDelta$, then we have the following result:

PROPOSITION 3.4. – *$f_{|W}\colon W \to V_5$ is a biregular morphism.*

PROOF. – Since V_5 is smooth, it is enought to prove that $\phi_{|W}$ is bijective.

Let $Q \in V_5$: it can be considered as a smooth hyperplane section of the Grassmannian variety $G = G(2, V) \subset \boldsymbol{P}^5$, where V is a 4-dimensional vector space, let $\mathcal{U}$ and $\mathcal{Q}$ be respectively the universal and the quotient bundle over G, then as it is well known $\det(\mathcal{U}^*) \simeq O_G(1)$, $H^0(\mathcal{U}^*) \simeq V$ and

$$0 \to \mathcal{U} \to V \otimes O_G \to \mathcal{Q} \to 0. \tag{3.4.1}$$

Moreover, as it is well known the restrictions of $\mathcal{U}^*$ and $\mathcal{Q}$ to any smooth hyperplane section of G are isomorphic. Let $i: C \to Q$ the natural inclusion, we define

$$E := i^*(\mathcal{U}^*_{|Q}), \qquad V_E := i^* V. \tag{3.4.2}$$

Then E is a rank 2 vector bundle on C, with $\det(E) \simeq i^*(O_Q(1)) \simeq O_C(H)$, and $V_E \subset H^0(E)$ is a 4-dimensional subspace which generates E. Moreover it's easy to verify that actually $Q(E, V_E) = Q$. It remains to prove that E is stable. Let $L \subset E$ be a destabilizing subbundle, with $\deg L \geqslant 3$, and let M be the quotient, since $h^1(L) = 0$, we have an exact sequence

$$0 \to H^0(L) \to H^0(E) \to H^0(M) \to 0\,, \tag{3.4.3}$$

from which we can see $\dim(V_E \cap H^0(L)) \geqslant 2$, which implies $\operatorname{rk} Q \leqslant 4$, see (1.6), and contradicts the hypothesis. Hence we can conclude that $E \in W$ and $\phi(E) = Q$.

Finally, let $E_1, E_2 \in W$ such that $\phi(E_1) = \phi(E_2) = Q \in V_5$. Since E_1 and E_2 are stable and globally generated, see (2.2), then by (1.7) they are isomorphic. This concludes the proof.

3.5. Let $E \in \Omega \cap V_X$: then E fits into a non splitting exact sequence

$$0 \to \omega_C \to E \to O_C(H - K_C) \to 0\,, \tag{3.5.1}$$

it is well known that all such extensions are parametrized by $\boldsymbol{P}(\mathrm{Ext}^1(O_C(H - K_C), \omega_C)) \simeq \boldsymbol{P}(H^0(O_C(H - K_C))^*) = \boldsymbol{P}^2$, and give rise to vector bundles with determinant $O_C(H)$. This get a well defined rational extension map, (see [BE]):

$$\varepsilon\colon \boldsymbol{P}(\mathrm{Ext}^1(O_C(H - K_C), \omega_C)) \simeq \boldsymbol{P}^2 \to \Omega \cap V_X \tag{3.5.2}$$

by associating to each $x \in \boldsymbol{P}^2$, the bundle $\varepsilon(x)$ obtained from the corresponding extension. We have the following result:

LEMMA 3.6. – *Assume $C \subset \boldsymbol{P}^2$, by the linear system $|H - K_C|$. Then the bundle $\varepsilon(x)$ is semistable for any $x \in \boldsymbol{P}^2$, $\varepsilon(x)$ is stable if and only if $x \in \boldsymbol{P}^2 - C$.*

PROOF. – For $x \in \boldsymbol{P}^2$, let $E = \varepsilon(x)$ the bundle corresponding to the extension x,

$$0 \to \omega_C \to E \to O_C(H - K_C) \to 0 . \tag{3.6.1}$$

We prove that E is semistable. Let $E \to L \to 0$ a destabilizing torsion free sheaf, with $\deg L \leqslant 2$, then $\alpha\colon\ \omega_C \to L$ is defined: if α is zero, then $\beta\colon\ O_C(H - K_C) \to L$ is defined but this is impossible since $\deg L \leqslant 2$. So α is not zero, actually is an isomorphism, $L = \omega_C$ and the extension splits, which is impossible.

Let $x \in C - \mathrm{Sing}(C)$: then it is easy to see that E admits the quotient $E \to \omega_C(x) \to 0$, hence E is not stable, and can be identified by S-equivalence to to $O_C(H - K_C - x) \oplus \omega_C(x)$.

Finally let $x = p \in \mathrm{Sing}(C)$: in this case E admits the quotient $E \to \omega_C \otimes \pi_* O_{\tilde{C}}$, where $\pi\colon \tilde{C} \to C$ is a partial normalization of C at p, so that E is not stable, and we can identify, by S-equivalence the bundle with $(\omega_C \otimes \pi_* O_{\tilde{C}}) \oplus (O_C(H - K_C) \otimes \mathcal{M}_p)$.

Note that the bundle $\varepsilon(X)$ fails to be globally generated if and only if $x \in C$.

PROPOSITION 3.7. – *The following restrictions are bijective:*

$$\phi_{|\Omega \cap V_X}\colon\ \Omega \cap V_X \to |I_\Sigma(2)| - \{Q_p\}_{p \in \mathrm{Sing}(C)}, \qquad \phi_{|X_{ss} \cap V_X}\colon\ X_{ss} \cap V_X \to S_4 - \cup N_p .$$

PROOF. – Let $E \in \Omega \cap V_X$: then the inclusion $\omega_C \subset E$, implies $H^0(\omega_C) \subset H^0(E)$ and consequently the quadric $\phi(E) \in |I_\Sigma(2)|$.

Let $Q \in |I_\Sigma(2)|$, with $Q \neq Q_p$. We claim that there exists a unique vector bundle $E \in \Omega$, with $\phi(E) = Q$. The two rulings of Q defines on C two pencils:

$$|A_1| = |K_C| , \qquad |A_2| \subset |H - K_C| ; \tag{3.7.1}$$

note that A_2 corresponds to the pencil of lines in $\boldsymbol{P}^2 = \boldsymbol{P}(H^0(O_C(H - K_C))^*)$ of center $x \in \boldsymbol{P}^2$: let's consider the bundle

$$E := \varepsilon(x) \in \Omega \cap V_X , \tag{3.7.2}$$

given by the extension associated to x. By Lemma 3.6, E is semistable, moreover

$$H^0(E) = H^0(\omega_C) \oplus V_2, \quad \boldsymbol{P}(V_2) = |A_2| ; \tag{3.7.3}$$

so it's easy to see that $Q(E, H^0(E)) = Q$, which implies $\phi(E) = Q$. Note that E is not stable if and only if $x \in C$.

Finally, by (3.7.1) and (3.7.3) follows immediately that E is the unique bundle of $\Omega \cap V_X$ with $\phi(E) = Q$.

Let $Q \in S_4$, $Q \notin \cup N_p$. As above let's consider the two rulings of Q: they cut two pencils

(3.7.4) $$|A_i|,\ \deg A_1 = \deg A_2, \qquad A_1 \otimes A_2 = O_C(H).$$

Let's define:

(3.7.5) $$E := A_1 \oplus A_2,$$

it's easy to verify that $E \in X_{ss} \cap V_X$ and $\phi(E) = Q$. Moreover, each bundle F with $\phi(F) = Q$ is obtained as an extension of A_i, so it defines the same point in X_{ss}.

Note that if $Q \notin |I_\Sigma(2)|$, then by (2.2) the pencils $|A_i|$ are base points free; conversely if $Q \in |I_\Sigma(2)| \cap S_4$, it's easy to see that one of the pencil is just $|\omega_C(x)|$, so that E is not globally generated at the point x, and $E \in \Omega \cap X_{ss} \cap V_X$.

We can conclude that ϕ induces a bijective morphism between the open subsets V_X and $|I_C(2)| - \bigcup N_p$.

If the curve is smooth, then $X = V_X = SU(2, O_C(H))$, so we have the result:

COROLLARY 3.8. – f: $\mathcal{SU}(2, O_C(H)) \to |I_C(2)|$ *is an isomorphism.*

PROOF. – In this case $\phi\colon \mathcal{SU}(2, O_C(H)) \to |I_C(2)|$ is a bijective morphism: since $\mathcal{SU}(2, O_C(H))$ is normal and $|I_C(2)| \simeq \boldsymbol{P}^3$ is smooth, this implies the claim.

4. – The main result.

We completely devote this section to prove our main result:

THEOREM 4.1. – *Let C be a stable irreducible curve with $p_a = 2$, then the fundamental map*

$$\phi\colon X \to |I_C(2)|$$

is an isomorphism.

4.2. Let C be a stable, irreducible curve with $p_a = 2$. As it is well known, there is a natural involution over $\mathrm{Pic}^1(C)$:

(4.2.1) $$i\colon L \to \omega_C \otimes L^{-1};$$

we recall that if C is smooth then the quotient $K := \mathrm{Pic}^1(C)/i$ is called the Kummer surface associated to $J(C)$, see [G-H].

PROPOSITION 4.3. – *There exists a regular involution* $\bar{i}$: $\mathrm{Jac}^1(C) \to \mathrm{Jac}^1(C)$ *which is an extension of i.*

PROOF. – Let $L \in \mathrm{Jac}^1(C)$, a torsion free sheaf which is not locally free over C. Then, as usual, let $\pi: \tilde{C} \to C$ be a partial normalization of C at the points where L is not invertible: there exists $\tilde{L} \in \mathrm{Pic}^{\tilde{g}-1}(\tilde{C})$, (either $\tilde{g} = 1$ or $\tilde{g} = 0$), s.t. $\pi_* \tilde{L} = L$. Let $\tilde{i}(\tilde{L}) = \omega_{\tilde{C}} \otimes \tilde{L}^{-1}$, the corresponding involution defined on $\mathrm{Pic}^{\tilde{g}-1}(\tilde{C})$, we set

$$\bar{i}(L) := \pi_* (\tilde{i}(\tilde{L})). \tag{4.3.1}$$

It 's immediate to verify that: $\bar{i}(L) \in \mathrm{Jac}^1(C)$ and $\bar{i}(\bar{i}(L)) = L$. It remains to prove that actually $\bar{i}$ is a morphism, which is an extension of i. At this end, we prove that

$$\bar{i}(L) = \mathrm{Hom}(L, \omega_C). \tag{4.3.2}$$

This is obvious if L is invertible. Let $L \notin \mathrm{Pic}^1(C)$ then we have, see [C]:

$$\mathrm{Hom}(\pi_*(\tilde{L}), \omega_C) \simeq \pi_* \mathrm{Hom}(\tilde{L}, \omega_{\tilde{C}}) \simeq \pi_*(\tilde{L}^{-1} \otimes \omega_C), \tag{4.3.3}$$

from which follows the claim.

4.4. Assume $O_C(H) = \omega_C \otimes O_C(2t)$, with $O_C(t) \in \mathrm{Pic}^2(C)$. We have a natural map

$$\nu: \mathrm{Jac}^1(C) \to M(2, 6)_{ss} \tag{4.4.1}$$

which is defined for each $L \in \mathrm{Jac}^1(C)$ by associating the rank 2 torsion free sheaf

$$\nu(L) := L(t) \oplus \bar{i}(L)(t), \tag{4.4.2}$$

which actually turns out to be semistable (not stable). Note that ν factors through the involution $\bar{i}$, so it is a finite map of degree 2.

Moreover, if $L \in \mathrm{Pic}^1(C)$, then

$$\nu(L) = L(t) \oplus (\omega_C \otimes L^{-1})(t) \tag{4.4.3}$$

is a vector bundle with determinant $O_C(H)$, hence an element of X_{ss}, so that $\nu(\mathrm{Pic}^1(C)) \subset V_X \cap X_{ss}$. Since X_{ss} is a closed subset of $M(2, 6)_{ss}$, this implies that $\nu(\mathrm{Jac}^1(C))$ is an irreducible component of X_{ss}.

From the above considerations we can conclude with the following:

PROPOSITION 4.5. – *X_{ss} has an irreducible component X'_{ss} which can be identified with the quotient $\overline{K} = \mathrm{Jac}^1(C)/\bar{i}$.*

We will call $\overline{K}$ the generalized Kummer surface of $\mathrm{Jac}^1(C)$.

4.6. Let's consider now the composition of maps $\psi := \phi \cdot \nu$,

$$\psi\colon \operatorname{Jac}^1(C) \to X'_{ss} \to S_4\,, \tag{4.5.1}$$

ψ is a rational map, which factors trough the involution $\bar{\imath}$. Actually we have the following result:

PROPOSITION 4.7. – *j induces a bijection between the quotient* $\operatorname{Jac}^1(C)/\bar{\imath}$ *and the quartic surface S_4.*

PROOF. – We will show an injective map

$$\varrho\colon S_4 \to \frac{\operatorname{Jac}^1(C)}{\bar{\imath}} \tag{4.7.1}$$

which is set theoretically the inverse map of ψ.

Let $Q \in S_4$: assume $Q \notin N_p$, as we saw in Prop. (3.7) the two rulings of Q cut two linear systems over C

$$|A_1|\,,\ |A_2|\,, \tag{4.7.2}$$

with $A_1 \otimes A_2 = O_C(H)$. Set $A_i(-t) = L_i$ then: $L_i \in \operatorname{Pic}^1(C)$ and $\bar{\imath}(L_1) = L_2$. So we have a well defined map by setting

$$\varrho(Q) := [L_1]\,, \tag{4.7.3}$$

where $[L_1]$ stands for the equivalence class of L_1 in the quotient $\operatorname{Jac}^1(C)/\bar{\imath}$. It is easy to verify that ϱ is injective, and $\psi(\varrho(Q)) = Q$.

Let now $Q \in N_p$. Assume first that $\operatorname{Sing}(C) = \{p\}$. Here the rulings cannot define any divisor on C. Let $\pi\colon \widetilde{C} \to C$ be the normalization at p: the rulings induce univoquely the existence of $\widetilde{A}_1$, $\widetilde{A}_2$ locally free of degree 2 over $\widetilde{C}$ with

$$\widetilde{A}_1 \otimes \widetilde{A}_2 = \pi^* O_C(H) \otimes O_{\widetilde{C}}(-x_1 - x_2)\,, \tag{4.7.4}$$

with $\{x_1, x_2\} = \pi^{-1}(p)$. Then $\pi_* \widetilde{A}_i$ is torsion free of degree 3, for $i = 1, 2$; set

$$L_i := \pi_* \widetilde{A}_i(-t)\,, \tag{4.7.5}$$

one can easily verify that $\bar{\imath}(L_1) = L_2$. So we can extend the definition $\varrho(Q) =: [L_1]$. This allow us to identify N_p with $\operatorname{Pic}^0(\widetilde{C})/\tilde{\imath}$.

Finally, assume that $\operatorname{Sing}(C) = \{p, q\}$. Let $Q_{p,q} \in N_p \cap N_q$: similar arguments as before, allow us to conclude that for each node we have

$$N_p - \{Q_{p,q}\} \simeq \frac{\operatorname{Pic}^0(\widetilde{C})}{\tilde{\imath}}\,. \tag{4.7.6}$$

On the other end, ϱ can be defined at $Q_{p,q}$ as follows: let $\pi: \widetilde{C} \to C$ is the total normalization of C, since $\operatorname{rk} Q_{p,q} = 3$, the two rulings coincide so we have a unique $\widetilde{A} \in \operatorname{Pic}^1(\widetilde{C})$. By setting L as in (4.7.5), we define $\varrho(Q) := [L]$, the unique point of $\operatorname{Pic}^{-1}(\widetilde{C})/\tilde{i}$. This concludes the proof.

As a consequence of the preceding results we have:

PROPOSITION 4.8. – *The restriction* $\phi: X'_{ss} \to S_4$ *is bijective.*

Finally we can prove our fundamental result.

PROOF OF TH. 4.1. – We proved that $\phi: X \to |I_C(2)|$ is a surjective birational map. Since $|I_C(2)|$ is smooth, by the main theorem of Zariski we can conclude that either ϕ is an isomorphism or ϕ is the blow up of $|I_C(2)|$ at some points. Actually we prove that ϕ is an isomorphism, $X'_{ss} = X_{ss}$, so the only torsion free sheaves of X which are limits of vector bundles are exactly the sheaves of $\phi^{-1}(N_p)$, for any $p \in \operatorname{Sing}(C)$.

At this end, assume first that $\operatorname{Sing}(C) = \{p\}$. Note that as a consequence of lemma 2.5 we have

$$X - V_X \subset U_p^0\,; \tag{4.1.1}$$

actually, let E such a sheaf we claim that E is not stable. In fact, as we have already seen, $E = \pi_* \widetilde{E}$, where $\widetilde{E}$ is a rank 2 vector bundle over $\widetilde{C}$ of degree 4, so $\widetilde{E}$ is not stable, see [AT]. So we can conclude that $E \in X'_{ss}$, and $\phi(E) \in N_p$. This immediately implies that ϕ is an isomorphism.

Assume now $\operatorname{Sing}(C) = \{p, q\}$. Again we have

$$X - V_X \subset U_p^0 \cup U_q^0\,. \tag{4.1.2}$$

Let E be such a sheaf: if $E \in X \cap U_p^0 \cap U^2 q$, then an argument similar to the preceding can show that E is not stable, hence $E \in X'_{ss}$, and $\phi(E) \in N_p$. The same for $X \cap U_q^0 \cap U_p^2$. Actually we claim that the closed sets

$$\mathcal{E}_1 = X \cap (U_p^0 \cap \overline{U_q^1})\ \mathcal{E}_2 = X \cap (U_q^0 \cap \overline{U_p^1}), \tag{4.1.3}$$

consist of a single point, the bundle $E \in U_p^0 \cap U_q^0$, with $\phi(E) = Q_{p,q} = N_p \cap N_q$. In fact, suppose on the contrary that these sets are not reduced to a point, since ϕ is an isomorphism outside the closed set $\mathcal{E}_1 \cup \mathcal{E}_2$, this implies that each $\mathcal{E}_i$ turns out to be an exceptional divisor of ϕ, but this is impossible, since $\mathcal{E}_1 \cap \mathcal{E}_2 \neq \emptyset$. This concludes the proof.

As an immediate consequence we have the following:

COROLLARY 4.9. – $X_{ss} \simeq \operatorname{Jac}^1(C)/\bar{i} \simeq S_4$.

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Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10 - 10123 Torino

e-mail: Brivio@dm.unito.it

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