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On Real Algebraic Links in S^3.

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Sunto. – *Viene presentata una costruzione che, dato un arbitrario nodo* $L \subseteq S^3$, *produce allo stesso tempo*: 1) *un'applicazione polinomiale* $f: (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ *con singolarità (debolmente) isolata in* 0 *e* L *come tipo di nodo della singolarità*; 2) *una risoluzione delle singolarità di* f *nel senso di Hironaka. Specializzando la costruzione ai nodi fibrati otteniamo una versione debole (a meno di scoppiementi e nella categoria analitica reale) di un reciproco per il teorema di fibrazione di Milnor.*

1. – Introduction.

Milnor's fibration theorem has been established in wide generality (arbitrary dimension, complex and real case) in the celebrated book [1].

Though several concepts we are going to develop should make sense in wide generality, in some points we shall use peculiar facts of the low dimensional case, so we will just settle such a case.

First fix few notations. For each $\varepsilon > 0$, for each $x \in \mathbb{R}^k$, set $D^k(x, \varepsilon)$ the closed ball centred at x with ray ε, $D^k = D^k(0, 1)$; $S^{k-1}(x, \varepsilon)$ (resp. S^{k-1}) denotes the boundary of $D^k(x, \varepsilon)$ (resp. S^{k-1}).

From now on U denotes an open neighbourhood of 0 in $\mathbb{R}^4$

$$f = (f_1, f_2) : U \to \mathbb{R}^2$$

is a *continuous* map such that $f(0) = 0$.

DEFINITION 1.1. – We say that f has an *isolated singularity at* 0 (resp. a *weak isolated singularity at* 0) if f is smooth on $U \setminus \{0\}$ and there exists $\delta > 0$ such that for each $y \in (U \setminus \{0\}) \cap D^4(0, \delta)$ (resp. $y \in (f^{-1}(0) \setminus \{0\}) \cap D^4(0, \delta)$), rank $\mathrm{d}f_y = 2$.

DEFINITION 1.2. – Let f have a weak isolated singularity at 0. We say that f has *tame zero set* at 0 if there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$, $Y = f^{-1}(0)$ is transverse to the sphere $S^3(0, \varepsilon)$. In particular each

$$L(f, \varepsilon) = \varepsilon^{-1}(Y \cap S^3(0, \varepsilon) \subset S^3$$

is a (tame) link in S^3. Let f have an isolated singularity at 0. We say that f is *tame* at 0 if it has tame zero set and for each $0 < \varepsilon \leq \varepsilon_0$ there exists $\delta_0 = \delta_0(\varepsilon)$

such that for each $0 < \delta \leqslant \delta_0$,

$$B(f,\, \varepsilon,\, \delta) = f^{-1}(D^2(0,\, \delta)) \cap D^4(0,\, \varepsilon)$$

is a 4-ball with as boundary a smooth 3-sphere with corners, $S(f,\, \varepsilon,\, \delta)$ say, the corners being along $f^{-1}(S^1(0,\, \delta)) \cap S^3(0,\, \varepsilon)$ which is the boundary of a tubular neighbourhood of $\varepsilon L(f,\, \varepsilon)$. In both cases such a $D^4(0,\, \varepsilon_0)$ is called a *Milnor ball* for f.

DEFINITION 1.3. – A link

$$L = \bigcup_{i=1\ldots k} L_i \subset S^3$$

is said a *fibred link* if there exist a smooth map

$$\pi\colon S^3 \backslash L \to S^1$$

pairwise disjoint tubular neighbourhoods

$$T_i, \quad i = 1 \ldots k \ \text{ of } L_i$$

trivializations

$$\tau_i\colon D^2 \times S^1 \to T_i, \qquad \tau_i(\{0\} \times S^1) = L_i$$

such that:

a) π is a locally trivial fibration;

b) for each $i = 1 \ldots k$ and for each $(x,\, y) \in (D^2 \backslash \{0\}) \times S^1$

$$\pi\tau_i(x,\, y) = x\|x\|^{-1}.$$

It follows that for each $t \in S^1$

$$F_t = \pi^{-1}(t) \cup L$$

is an orientable compact surface embedded in S^3 bounded by L (i.e. F_t is a *Seifert surface* of L).

We can state now the Milnor Fibration Theorem as follows

THEOREM 1.4. – *a*) *Let f have tame zero set at* 0. *Then the links $L(f,\, \varepsilon) \subset S^3$ are ambient isotopic each other, so that the* link type of f at 0 $(S^3,\, L = L(f))$ (*simply L*) *is well defined.*

a') *If f is an analytic map with weak isolated singularity at* 0, *then f has tame zero set at* 0.

b) *If f is tame at* 0, *then $(S^3,\, L)$ is a fibred link.*

b′) If f is an analytic map with isolated singularity at 0*, then f is tame at* 0.

Few remarks about the fibration theorem.

REMARK 1.5. – 1) Simple dimensional considerations show that a «generic» analytic f singular at 0 has critical set of dimension 1, hence the maps with isolated singularity at 0 are in principle quite *rare*.

2) A basic way to construct examples of such maps is to consider complex analytic function $g\colon \mathbb{C}^2 \to \mathbb{C}$ with isolated singularity at zero (that are «generic» in the complex setting) and to forget the complex structure; in such a way one produces widely studied special link types (*special iterated torus links*, also said (*complex*) *algebraic links*); a natural question, already posed by Milnor in [1], is wether there are other essentially different examples, more generally which fibred links realize the link type of analytic (polynomial) maps with isolated singularity; in particular Milnor asked it for the simplest potential counterexample, the «figure 8» knot.

3) Looijenga [2] detected a class of fibred links, called *odd links* (the link and the trivializations are invariant with respect to the antipodal maps) that are link types of analytic (in fact odd polynomial) maps with isolated singularity, moreover he proved that a suitable «double» of any fibred knot gives an odd knot; in such a way one can realize a lot of non complex algebraic examples.

One can find constructions of real-not-complex algebraic knots also in [3].

For the «figure 8» knot, which is not odd, explicit equations have been firstly established by Perron in [4], answering affirmatively to such a specific Milnor's question.

Lee Rudolph, using a mixing of complex and real coordinates, found a quite efficient way to construct other explicit examples, including the «figure 8» and also the Borromean rings as particular cases (see for instance [5]).

All the known examples have rather special topological properties and the general question of which classical fibred links arise from isolated singularities seems to be quite open.

4) Akbulut and King [6] proved that every link in S^3 (not necessarily fibred) arises as the link type of polynomial maps with weak isolated singularity at 0, briefly every link in S^3 is *weakly real algebraic*; fibred links seem to be not distinguished by that «algebrization» procedure.

5) The fibration theorem (points a') and b')) actually holds under weaker assumptions: for instance if f is a continuous (semialgebraic) subanalytic map. Moreover the link type is uniquely well defined if we use instead of the small spheres around $\{0\}$, the level surfaces (actually diffeomorphic to the

sphere), for sufficently small values, of any non negative continuous (semialgebraic) subanalytic function $h\colon U \to \mathbb{R}$, such that $\{h=0\} = \{0\}$.

6) It is quite easy to see, by means of the «cone construction» (cf. [2], [5]) that we will discuss later, that every fibred link is the link type of C^r semialgebraic tame maps (r being arbitrarily large) as well of tame smooth maps (but flat at 0).

The present paper would outline a program to settle out the full reciprocal of the fibration theorem for classical links; that is we state the following conjecture.

CONJECTURE 1.6. – *Every fibred link in S^3 is the link type of a polynomial* map *f with isolated singularity at* 0, *briefly every fibred link is* real algebraic.

Unfortunately, we are not able, for the moment, to achieve the whole program, but we already got the following results that we state now in a slightly informal way; we will precise them in the sequel of the paper. They give the conjecture few evidence.

We are firstly able to improve the result of [6] as follows

THEOREM 1.7. – *It is settled an «algebrization procedure» that for any given link $L \subset S^3$, produces at the same time a polynomial map f with weak isolated singularity at* 0 *having L as link type, and also an explicit* embedded *algebraic resolution of the singularity of f via a finite tower of blowing-up of non singular centres over $0 \in \mathbb{R}^4$ (in the sense of Hironaka).*

Specializing the general procedure to fibred knot, we get a weak reciprocal *up to blowing-up* of the fibration theorem

THEOREM 1.8. – *For any non trivial fibred knot L (adopting the notations of the previous theorem), there exists a tame map g, sharing with the polynomial map f a Milnor ball B and the zero set $Y \subset B$, such that, denoting $\pi\colon B^* \to B$ the blowing-up composition resolving f, then $h = g\pi\colon B^* \to \mathbb{R}^2$ is an analytic map and for each $x \in B^* \setminus (\pi^{-1}(0))$,* rank $\mathrm{d}h_x = 2$.

Remark that the zero set of h is the union (in general position in B^*) of $\pi^{-1}(0)$, that is the total *exceptional divisor* of π, and the *strict transform* Y^* of Y which is a non singular surface in B^*, transverse to $\partial B^* = S^3$, so that $(S^3, Y^* \cap S^3)$ is the link type of f.

We have stated the result for fibred knot, but the method does work for any fibred link such that the fibre of the fibration carries a *divisorial spine* (see later for the definition); actually it could be presumably generalized to any fibred link, allowing divisorial spines with *immersed* and not only *embedded* components, but we will not insist on this technical point.

The main step we lack to complete the program is clearly a suitable «blowing-down» lemma (better, a refinement of the construction so that such a lemma does hold) so that one could take the map g to be analytic; in such a case well established «finite-sufficency» results should allow to finally convert g into a polynomial map.

Two words about the few ideas developed in the paper. The first idea is a refinement of an already extensively exploited one, mostly in Akbulut-King works on the topology of real algebraic sets (see [6], [7]): *to find a good topological counterpart of the algebraic resolution of singularities and then try to make it «algebraic» via suitable approximations*. The refinement consists in dealing with the *embedded* resolutions of singularities, rather than the abstract ones: the topological resolution shall be embedded in a tower of genuine algebraic blow-up along non singular centres. This is the main ingredient to prove our improvement of [6].

The proof of the weak reciprocal of the fibration theorem is already quite subtle: it consists, roughly speaking, in a careful approximation result of *topologically monomial like* maps by analytic map, without changing the critical set (a non trivial task - recall the above remark 1). We note that, at present, we are able to develop this last method strictly in the analytic cathegory; for instance, if we would try it in the *Nash* case, the method should produce *non-affine* Nash manifolds, and this not satisfactory.

The proofs shall be somewhat concise, as we prefer to point out the few basic new ideas, instead to write down all the full details that in some case should be a little heavy.

Finally we remark that our work could be seen also as a generalization of [2]: *odd links* are resolved after the single blowing-up of $0 \in \mathbb{R}^4$.

The first author presented the content of this paper at the *Real Algebraic and Analytic Geometry Meeting* in Geneve, September 16 to 20 1996; this is, in a sense, an expanded version of the text of that talk.

2. – Divisorial spines.

In this section we will work in arbitrary dimension. From now on we denote by W a compact connected smooth n-manifold with non empty boundary $\partial W = Z$. Let

$$E = E_1 \cup \ldots \cup E_h \subset (W \setminus Z)$$

where

a) each E_i is a compact connected closed smooth hypersurface embedded in $W\backslash Z$;

b) the family $\mathcal{E}$ of such hypersurfaces is in general position in $W\backslash Z$.

DEFINITION 2.1. – $x \in E$ is said of *depth* j if there are $E_{i_1}, \ldots, E_{i_j}$ distinct hypersurfaces of $\mathcal{E}$, such that

$$x \in \bigcap_{s=1, \ldots, j} E_{i_s}$$

but there are not $j+1$ distinct hypersurfaces such that x belongs to their intersection. Note that if x is of depth j then $\{E_{i_s}\}_{s=1\ldots j}$ is uniquely determined. Set

$$E^j = \{x \in E;\ \text{depth}\,(x) = j\}$$

E^j is a submanifold of $W\backslash Z$ (not necessarily closed) of codimension j; each E^j has a finite number of connected components $\{S_r^j\}_{r=1\ldots h_j}$ and the union of $\{S_r^j\}_{j=1\ldots n,\, r=1\ldots h_j}$ makes a *natural stratification of E*.

For each $i = 1, \ldots n$ define a map

$$\pi_i\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$$

$$\pi_1(x_1, \ldots, x_n) = (x_2, \ldots, x_n)$$

for $i = 2, \ldots, n$

$$\pi_i(x_1, \ldots, x_n) = (x_1^2 - x_2^2, \ldots, x_1^2 - x_i^2, x_{i+1}, \ldots, x_n)$$

DEFINITION 2.2. – Let W and E be as above. A retraction $r\colon W \to E$ is said a normal retraction if for each $x_0 \in E$ of depth j there exists a neighbourhood V of x_0 in E and a smooth diffeomorphism

$$\phi\colon r^{-1}(V) \to V_j' = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n;\ |x_1 \ldots x_j| \leq 1\}$$

such that:

$$\phi(x_0) = 0 \qquad \text{and} \qquad \phi(V) = \{x_1 \ldots x_j = 0\},$$

$\forall x \in V$

$\phi(r^{-1}(x)) =$

$$\{\text{the connected component of } \pi_j^{-1}(\pi_j(\phi(x)) \cap V_j') \text{ containing } \phi(x)\}.$$

If W and each E_i are analytic manifolds and each ϕ is an analytic diffeomorphism we say that r is a *normal analytic retraction*. If $r\colon W \to E$

is a normal retraction, then we say that E is a *divisorial spine*, briefly a $D-$*spine*, of W.

It is not hard to prove that the property to be a normal retraction is a *coherent* property, that is

LEMMA 2.3. – *If $x_0 \in E$ has a neighbourhood V as stated by the above definition then each $x \in V \setminus x_0$ has such a kind of neighbourhood.*

If $E \subset W$ is a $D-$spine, then clearly W is a *regular neighbourhood* of E (that is $W \searrow E$ and $(W \setminus E) \simeq Z \times (0, 1])$. In fact the notion of normal retraction is a natural extention to E of the notion of *tubular neighbourhood* of a smooth submanifold, and it is not so hard to prove a generalization of the *uniqeness of tubular neighbourhood up to ambient isotopy* for normal retractions. Note also that if E is not necessarily a D-spine of W it is a D-spine for its closed regular neighbourhoods in W.

It is useful to recognize a D-spine E of W with its normal retraction r in terms of suitable *boundary data*. First of all we remark how to lift to the boundary Z the natural stratification of E. Let S_i^j be a stratum of E. $r^{-1}(S_i^j) \cap Z$ is a (not necessarily closed) submanifold of Z of codimension $j-1$. Taking its connected components and varying the stratum of E we get a stratification $\mathcal{F}$ of Z. If we denote by ϱ the restriction of r to Z, ϱ is actually a *stratified map* from the stratified boundary Z onto the stratified spine E, more it is a local diffeomorphism on every stratum of $\mathcal{F}$. Denote by $Z^j = Z^j_{\mathcal{F}}$ the union of stata of Z of codimension j and by

$$\mathcal{Z}^j = \bigcup_{j \leq i \leq n} Z^i$$

the codimensional j-skelton of $\mathcal{F}$. For each x in some $S_i^j \subset E$

$$\varrho^{-1}(x) \subset Z^{j-1}$$

and

$$\# \varrho^{-1}(x) = 2^j.$$

Of course W is naturally identified with the mapping cylinder of $\varrho: Z \to E$. As Z^0 is open an dense in Z, ϱ (whence W and r) is determined by its restriction ϱ' to Z^0. Moreover the equivalence relation induced by ϱ' on Z^0 is in fact the relation of an involution $\tau: Z^0 \to Z^0$. The triple

$$(Z, \mathcal{F}, \tau)$$

is the triple of *boundary data* associated to the D-spine

$$(W, E, r)$$

and it is enough to determine it.

3. – Embedded resolution of singularities.

The construction of blowing-up a non singular algebraic variety (an analytic manifold) along a non singular centre (an analytic submanifold) is classical and well known (see for instance [9] for a definition in the real algebraic case). This construction can be straightforwardly extented to the smooth case (see for instance [8] where one uses such constructions in a low dimensional setting similar to the present one). Let us recall briefly how. Let M be a smooth manifold and Int (M) its interior. Let N be a closed submanifold of Int (M) of codimension s. Denote by $\pi\colon \nu_M(N) \to N$ the normal bundle of N in M; an associated D^s-bundle

$$\pi\colon D(\nu_M(N)) \to N$$

make a neighbourhood of the zero section $N \times \{0\} \simeq N$ and can be naturally identified with a tubular neighbourhood $T = T_M(N)$ of N in M. Let

$$p\colon \mathbb{P}(\nu_M(N)) \to N$$

be the projectivization of $\nu_M(N)$; it is a $\mathbb{P}^{s-1}$-bundle over N. There is a *tautological* line bundle

$$q\colon \gamma_M(N) \to \mathbb{P}(\nu_M(N))$$

with associated D^1-bundle $D(\gamma_M(N))$. Then there is a natural diffeomorphism

$$\mu\colon D(\gamma_M(N)) \backslash \mathbb{P}(\nu_M(N)) \to D(\nu_M(N)) \backslash N$$

μ can be extended to a map

$$\delta\colon D(\gamma_M(N)) \to D(\nu_M(N))$$

by setting $\delta|_{\mathbb{P}(\nu_M(N))} = p$. Now the *blowing-up* of M along the *smooth centre* N

$$\sigma = \sigma_N\colon M^* = B(M, N) \to M$$

is obtained by glueing $D(\gamma_M(N))$ to $T \backslash N$ via μ, the smooth map σ is defined by $\sigma|_{(M \backslash N)} = id$ and $\sigma|_{\mathbb{P}(\nu_M(N))} = \delta$. By construction

$$\sigma|\colon B(M, N) \backslash \sigma^{-1}(N) \to M \backslash N$$

is a diffeomorfism and we got $B(M, N)$ «replacing» N by $F = F(M, N) = \mathbb{P}(\nu_M(N))$ which is called the *exceptional divisor* of σ.

Let Y be a closed subset of M such that $Y \backslash N$ is dense in Y. The closure $Y^* = Y^*(M, N)$ of $\sigma^{-1}(Y \backslash N)$ in $B(M, N)$ is called the *strict transform* of Y. If Y is a closed submanifold of M transverse to N,

$$\sigma| : Y^* \to Y$$

is naturally identified with

$$\sigma_{Y \cap N} \colon B(Y, Y \cap N) \to Y .$$

Moreover Y^* is transverse to $F(M, N)$ along $F(Y, Y \cap N)$. If N is a submanifold of Y, Y^* is identified with $B(Y, N)$ and it is transverse to $F(M, N)$ along $F(Y, N)$. Remark also that if M is a tubular neighbourhood of N then $F(M, N)$ is a D – spine of $B(M, N)$. In particular if $M = D^n$ and $N = \{0\}$, then $B(M, N) \simeq \mathbb{P}^n \backslash D^n$, $F(M, N) \simeq \mathbb{P}^{n-1}$ and the boundary data of such a spine are $(S^{n-1}, \{S^{n-1}\}$, $\tau =$ *the antipodal map*). As we will use such an example , we give it a name:

$$(P_n, K_n, \alpha_n)$$

If M is a non singular algebraic variety (an analytic manifold) and N is a non singular centre (a smooth analytic centre), it turns out that $B(M, N)$ is also a non singular algebraic variety (an analytic manifold) and σ is a regular (analytic) map, and so on. In any case we shall speak of *blowing-up along a smooth centre*. If we have a finite *tower* of blowing-up along smooth centres and we also denote $\sigma \colon M^{**} \to M$ the composition, then the notion of strict transform of $Y \subset M$ via σ and the *total* exceptional divisor of σ are naturally defined.

Let $r \colon W \to E$ be now a divisorial spine with $(Z, \mathcal{F}, \tau)$ its boundary data. We want to discuss now its behaviour up to blowing-up of smooth centres suitably placed in E.

DEFINITION 3.1. – A compact closed connected submanifold X of W is said *well placed in E* if

a) $X \subset E$

b) X is completely contained in some $\bigcap_{s=1 \dots r} E_{i_s}$ and the family $\{X, E_t\}_{X \not\subset E_t}$ is in general position in $W \backslash Z$.

Assume that X is well placed in $E = \bigcup_{i=1\dots h} E_i$. Consider the blowing-up

$$\sigma\colon W^* = B(W, X) \to W .$$

Set:

$$E^* = \bigcup_{i=1\dots h} E_i^* ,$$

$$E' = F(W, X) \cup E^* .$$

It is easy to see that E' is a D-spine of W^*. For if $T = T_W(X)$ is a small closed tubular neighbourhood of X in W, clearly $W \searrow (K = E \cup T)$ and r induced a retraction $s\colon W \to K$ factorized by the inclusion $\partial T \subset T$; remark that we may assume that ∂T is in general position with $\mathcal{E}$ so that T induces a tubular neighbourhood $T_{i_1\dots i_s}$ of each $X_{i_1\dots i_s} = X \cap E_{i_1\dots i_s}$ in $E_{i_1\dots i_s}$, $E_{i_1\dots i_s} = \bigcap_{j=1\dots s} E_{i_j}$. To get $r'\colon W^* \to E'$, we have only to modify r on $B(T, X)$; we prefer to describe it in terms of the boundary data $(Z, \mathcal{F}', \tau')$ (note that, clearly, $\partial W = \partial W^* = Z$).

The simplest case. Assume that the centre X is entirely contained in a codimension 1 stratum $S = S_i^1$ of E, $s = \operatorname{codim}_W X$. Consider a closed tubular neighbourhood T of X in W inducing a tubular neighbourhood T' of X in S. On T it is defined a natural involution η operating on each fibre of the associated D^s-bundle like the *antipodal map* and such that T' is η-invariant. Set

$$\mathcal{U} = r^{-1}(T') \cap Z , \qquad U = r^{-1}(\operatorname{int} T') \cap Z$$

where $\operatorname{int} Y$ denotes the interior of Y. $U = \operatorname{int} \mathcal{U}$. U is an open set in $r^{-1}(S) \cap Z$, locally diffeomorphic onto $\operatorname{int} T'$. Using r, in fact s, we can construct a diffeomorphism of U onto $\partial T \backslash S$ and lift the involution η to η' defined on U. We can describe now $(Z, \mathcal{F}', \tau')$. $R = r^{-1}(S) \cap Z$ is union of strata of $Z^0_{\mathcal{F}}$ containing $\mathcal{U}$; split R into the connected components of $R \backslash \mathcal{U}$ and of U. The other codimension 0 strata of $\mathcal{F}$ are untouched. In this way we get $Z^0_{\mathcal{F}'}$. To get the whole $\mathcal{F}'$ add as strata the connected components of $\partial \mathcal{U}$. $\tau' = \tau$ on $Z^0_{\mathcal{F}'}$, $\tau' = \eta'$ on U.

The general case. Consider again the tube T along X in W inducing a regular neighbourhood T' of X in E stratified by the above $T_{i_1\dots i_s}$. $\mathcal{U} = r^{-1}(T') \cap Z$ is a regular neighbourhood of $r^{-1}(X) \cap Z$ such that $\partial \mathcal{U}$ is in general position with the codimension 1 skelton $\mathcal{Z}^1_{\mathcal{F}}$. Set $U = \operatorname{int} \mathcal{U}$. Hence

$$(\partial \mathcal{U} \cup \mathcal{Z}^1_{\mathcal{F}}) \backslash U$$

is naturally stratified and makes the new codimension 1 skelton $\mathcal{Z}^1_{\mathcal{F}'}$. The new $Z^0_{\mathcal{F}'}$ is obtained by taking $\{Z^0_{\mathcal{F}} \backslash \mathcal{U}, U\}$ stratified by the connected components. Using the retraction s we send isomorphically U onto $\partial T \backslash E$ and we lift to U, as before, the fiber-wise antipodal involution on ∂T, obtaining an involution η'

on U. Set

$$\tau' = \tau \quad \text{on} \quad Z^0_{\mathcal{F}} \backslash \mathcal{U}$$

$$\tau' = \eta' \quad \text{on} \quad U .$$

In this way we have completely described $(Z, \mathcal{F}', \tau')$.

DEFINITION 3.2. – Let (W, E, r) and (W', E', r') be the data of two divisorial spines. We say that (W', E', r') is a *modification* of (W, E, r) if there exists a finite tower $(W_1, E_1, r_1), \ldots, (W_s, E_s, r_s)$ of divisorial spines such that

a) $(W_1, E_1, r_1) = (W, E, r)$ and $(W_s, E_s, r_s) = (W', E', r')$

b) for each i there is a centre X_i well placed in E_i so that $(W_{i+1}, E_{i+1}, r_{i+1})$ is obtained from (W_i, E_i, r_i) by blowing-up along X_i.

DEFINITION 3.3. – Let (W, E, r), (X, H, s) be the data of two divisorial spines. Set $Y = \partial X$ and assume that $\dim X < \dim W$. Call $\mathcal{E}$ the family of hypersurfaces in E and by $\mathcal{H}$ the family of H. We say that (X, H, s) is *properly embedded* in (W, E, r) if

a) (X, Y) is a proper submanifold of (W, Z) i.e. $X \cap Z = Y \cap Z$ and Y is transverse to Z;

b) $\{X, \mathcal{E}\}$ is in general position in W;

c) $\mathcal{H} = X \cap \mathcal{E}$;

d) s is the restriction of r.

Proper embedding can be naturally recognized in terms of boundary data. Let $(Z, \mathcal{F}, \tau)$ and $(Y, \mathcal{G}, \eta)$ be the corresponding data. Then Y and $\mathcal{F}$ are in general position in Z, so that we get by intersection a stratification of Y that is actually equal to $\mathcal{G}$, η is obtained by restriction of τ.

Note that if (X, H, s) is properly embedded into (W, K, r), and Y is a well placed centre in the first one, then it is well placed also in the second, and by blow-up both the divisorial spines along Y we get two other divisorial spines, the first properly embedded into the second one.

DEFINITION 3.4. – Let L be a closed submanifold of $S^{n-1} = \partial D^n$. Consider the pair $(D^n, c(L))$ where $c(L)$ is the geometric cone with base L and centre $0 \in D^n$. An *embedded resolution* of $(D^n, c(L))$ is given by

a) a modification (W', E', r') of $(P_n, K_n, \alpha_n))$;

b) a divisorial spine (X, H, s) such that $L = \partial X$;

c) a proper embedding $(X, H, s) \subset (W', E', r')$.

If V is a germ at $\{0\}$ of an analytic set with isolated singularity, then the *locally cone structure* (see [9]) says that (B, V), B being a «small» ball around $\{0\}$, is isomorphic to $(D^n, c(L))$, $L = \partial B \cap V$; of course Hironaka's embedded resolution theorem implies an embedded resolution in our sense. Later we shall see directly the existence of such embedded resolutions (with further refinements) when L is a link in S^3.

We end this section posing a general question which is probably not easy.

QUESTION 3.5. – *Let L be as in the above definition. Assume that L is (abstractly) a boundary. Does there exist an embedded resolution of $(D^n, c(L))$?*

A positive (topological) answer should allow, for example, to avoid the use of Hironaka's algebraic resolution of singularities (that is somewhat «blind» in a contest that is, otherwise, rather constructive) in Mihalkin's solution of the Nash «topological rationality» question (see [8], [10]). Probably one could easierly deal when L bounds a submanifold of S^{n-1}, like links in S^3, but this case is not relevant to Mihalkin's application.

4. – Embedded resolutions for classical links in S^3.

The *input* of the contruction consists of the following data

1) A link L in S^3.

2) A connected Seifert surface F for L.

3) A divisorial spine $\varrho: F \to E$.

For any link L there exists such a pair (F, E).

The *output* will be an explicit embedded resolution of $(D^4, c(L))$.

$V = F \times [-1, 1]$ is embedded in S^3 with boundary S isomorphic to the «double» of F (after corners smoothing). E is regarded as a subset of $F \times \{0\}$ is a spine (not a divisorial one) of V. We can consider another copy of V, V' say, with boundary S', isotopic to V, such that

1) V' contains $F \times \{0\}$ (hence E is also a spine for V').

2) S and S' intersect transversely along L.

Infact we will produce simultaneously embedded resolutions of $(D^4, c(S))$ and $(D^4, c(S'))$ in such a way that a resolution of $(D^4, c(L))$ is properly embedded into both, actually being their transverse intersection.

Before the general construction let us show the elementary basic idea. Blow-up D^4 at the centre and get the starting divisorial spine that we denote simply (W_0, K_0, r_0). Assume that T is a closed submanifold of S^3, M is closed tubular neighbourhood of T in S^3 and $N = T \times [-1, 1]$ (T is identified with the *zero*-section) is properly embedded into M. Thus T is a, not necessarily divisorial, spine of M and a divisorial one for N. Assume that M is contained in the upper half-sphere of S^3, so that the restriction of r_0 to M is an embedding in K_0. For any subset A of M we denote A' its image in K_0. Then T is a well placed centre for (N, T) and T' is so for (W_0, K_0). Blow-up M and N along T and W_0 along T'. We get: M_1 with a divisorial spine H_1; (N_1, T_1), that in this case is isomorphic to (N, T), that properly embeds into (M_1, H_1); (W_1, K_1) so that M_1 naturally embeds into K_1. In fact (N_1, T_1) *almost* properly embeds into (W_1, H_1) in the sense that one can embed it with compatible normal retractions, but not in general position (in terms of boundary data, this means that the boundary of N_1 embeds into the stratification of S^3 associated to the boundary data of K_1, with compatible involutions, but it is not in general position with respect to this stratification). To get a proper embedding blow-up M_1 and N_1 along T_1 and W_1 along the image T_1' of T_1, that is the intersection of the two hypersurfaces making the spine K_1. We get M_2 with a divisorial spine H_2; (N_2, T_2) that in this case is again isomorphic to the initial one and that embeds into (M_2, H_2); (W_2, K_2) so that M_2 naturally embeds into K_2. Now (N_2, T_2) properly embeds into (W_2, K_2). (M_2, H_2) almost properly embeds into (W_2, K_2) in the above sense. A further blowing-up of M_2, along the proper transform in H_2 of H_1 and of W_2 along its image, creates finally $(M_3, H_3), (N_3, T_3), (W_3, K_3)$ so that all the spines are divisorial and there is a natural chain of proper embeddings.

The idea is to extend the above construction to our actual situation, where (F, E) should play the role of (N, T) and both V and V' the role of M; of course a complication comes from the fact that E is not made by a single component, so one should work step by step with respect to the strata of the natural stratification of E. But the essential behaviour already manifests in the above elementary case.

As in the example we start with (W_0, K_0) and we assume that $V \cup V'$ is contained in the upper half-shere, so that it embeds into K_0. At each step we have:

1) A modification (F_i, E_i) of $(F = F_0, E = E_0)$, that is contained in the intersection $Z_i = V_i \cap V_i'$ of a modification $B_i = V_i \cup V_i'$ of $B_0 = V \cup V'$.

2) H_i a common, not necessarily divisorial, spine of both V_i and V_i' that gives E_i by intersection.

3) A modification (W_i, K_i) of (W_0, K_0) so that B_i is naturally embedded into K_i.

4) a centre X_i well placed either in (F_i, E_i) or in H_i (when it becomes a divisorial spine; in such a case the intersection will be well placed in E_i), so that its image X_i' in K_i is also well placed in (W_i, K_i).

Then we get (F_{i+1}, E_{i+1}), $B_{i+1}Z_{i+1}$ and (W_{i+1}, K_{i+1}) by blowing-up of X_i and X_i'.

Step by step we shall get, essentially in the order: a modification of (F_0, H_0) that properly embeds into a modification of (W_0, K_0) and this property shall be preserved by the eventual further modifications; divisorial spines for modifications of V and V' and, finally, chains of proper embeddings of divisorial spines.

We are ready to describe the actual tower of blowing-up.

X_0 consists of the strata of depth 2 of E_0, that is of the finite set of double points of E_0.

X_1 consists of the strict-transform of the components of E_0 that are components of E_1.

X_2 consists of the set of double points of E_2.

X_3 is the strict transform in E_3 of the exceptional divisor of the blow-up that produced F_1.

X_4 is the strict transform in E_4 of the components of the initial E_0.

One verifies that we have already realized a modification (F_5, E_5) that properly embeds into the modification (W_5, K_5). Note that till now the dimension of the blow-up centres is ≤ 1.

Futher blow-up of the same type allow to complete the program. We left the reader to do it, suggesting to follow the construction also in terms of boundary data, and remarking that the centres are always of dimension ≤ 2, centres X_i' of dimension 2 are exceptional divisors of blow-up of components of previous K_j along centres of dimension ≤ 1.

5. – Weakly real algebraic links.

We are able now to outline the proof of the result on weakly real algebraic links stated in the introduction.

For technical reasons it is better to consider D^4 as an half-disk of S^4, so that S^3 divides S^4 in two parts. Perform symmetrically all the constructions we have done or we are going to do in D^4 also in the other copy. The advantage, referring to the content of the previous section, is that the constructed proper embeddings of divisorial spines in modifications of D^4, «double» to embedded *closed* submanifolds in modifications of S^4, and it is easier to apply suitable *algebraic approximation* results . For simplicity we keep for these doubled modifications the same name, so that, for example W_0 denotes now the blow-up of S^4 at two points (the two centres of the two copies of D^4) and so on.

Recall that a non singular projective (hence affine) real algebraic variety is said *totally algebraic* if the whole $\mathbb{Z}_2$-homology is generated by the fundamental class of algebraic subsets. The first remark is that we can assume that each centre X_i' of the blow-up producing W_{i+1} is a totally algebraic projective non singular real algebraic variety of dimension ≤ 2. The claim on the dimension comes from the topological construction; again from the construction each centre is a submanifold of submanifolds of dimension 3 (some components of the spine K_i), and the eventual 2-dimensional centres are exceptional divisors of previous blowing-up of such 3-submanifolds along 0 or 1-dimensional centres. Thus, arguing like e.g. in [8], we can assume, up to little isotopy and inductively, that each centre of dimesion ≤ 1, whence every centre X_i' is non singular algebraic and totally algebraic (connected algebraic ≤ 1-dimensional centres are trivially totally algebraic by dimensional reasons).

It follows we can assume that:

1) The blow-up towers producing the modifications W_j are towers of genuine real algebraic blow-up along non singular centres.

2) Every W_j is a totally algebraic non singular real algebraic variety.

3) The spine K_j is the union of non singular totally algebraic real hypersurfaces.

Let W^* be the last modification of the tower, with spine K^* and containing properly embedded modifications V^* (V'^*) of V (V'). In the present situation, algebraic approximation arguments (a variation in the series of results inagurated by the so called *Nash-Tognoli* theorem, see [7], [9]) shows that there is a regular rational map $h = (g, g')$: $W^* \to \mathbb{R}^2$ such that:

1) $g^{-1}(0) = K^* \cup A$, $g'^{-1}(0) = K^* \cup A'$.

2) A is close to V^*, A' is close to V'^*, $A \cap A'$ is close to the corresponding modification F^* of F, so that $A \cap A' \cap S^3$ is a link close (hence isotopic to) the initial link L.

3) The rank of $\mathrm{d}h$ is equal to 2 on $(A \cap A') \setminus K^*$.

As h is regular rational and the tower is made by algebraic blow-up, one can blow-down h to a regular algebraic map f: $S^4 \to \mathbb{R}^2$ that (considering its restriction to the first copy of D^4) realizes the link L as a weakly real algebraic link, and, as promised, we already dispose of an algebraic resolution of the singularities of f.

Note that in this section we have used some special facts holding for low dimensions.

6. – R-cones and momomial like maps.

For a while we will treat arbitrary dimensions. Let $E \subset W$ be a D-spine and r: $W \to E$ be a normal retraction. $Z = \partial W$, $n = \dim W$.

DEFINITION 6.1. – Let $f\colon Z \to \mathbb{R}^m$, $m \leq n$, be a smooth map. Let $F\colon Z \times (0, 1] \to \mathbb{R}^m$ defined by

$$F(x, t) = tf(x).$$

A smooth extension

$$g\colon W \to \mathbb{R}^m$$

of f is called an *r-cone* of f if there exists a diffeomorphism

$$\psi\colon Z \times (0, 1] \to W \backslash E,$$

such that

$$g\psi = F$$

and $\forall x \in Z$

$$\psi(x, 1) = x,$$

and $\psi(\{x\} \times (0, 1]$ is a connected component of $r^{-1}(r(x)) \backslash r(x)$.

Denoting

$$\pi\colon Z \times (0, 1] \to (0, 1]$$

the natural projection, we note that the function

$$d\colon W \to \mathbb{R}, \qquad d = \pi\psi^{-1} \quad \text{on} \quad W \backslash E, \ d \equiv 0 \ \text{on } E$$

is a non negative continuous function such that $\{d = 0\} = E$; set

$$W_t = h^{-1}((0, t]), \qquad Z_t = h^{-1}(t)$$

then $(W_t, \partial W_t = Z_t)$ is a submanifold of W, isomorphic to (W, Z) and having E as D-spine. Each Z_t is transverse to the foliation by arcs induced by the normal retraction r.

An r-cone g is said *regular* if the map F is regular that is dF is of maximal rank everywhere; of course a regular r-cone is regular, as a smooth map, on $W \backslash E$. Assume that W and each hypersurface $E_i \subset E$ are analytic manifolds. If an r-cone g of f is an analytic map and the associated function d is analytic on $W \backslash E$ and globally subanalytic, then it is called an *analytic* r-cone.

EXAMPLE 6.2. – This is our basic example. Assume that L is a fibred link in S^3; let $\beta\colon [0, 1] \to [0, 1]$ be a smooth function such that $\beta = 0$ on $[0, 1/3]$, $\beta = 1$ on $[2/3, 1]$, β is strictly monotone on $(1/3, 2/3)$. Let $p\colon D^2 \times S^1 \to D^2$ be the natural projection. Adopting the notations of 1.3, set $T = \bigcup T_i$, $\tau = \bigcup \tau_i$,

$\varrho = \tau^{-1}$. Define the smooth map

$$\phi: S^3 \to \mathbb{R}^2,$$

$$\phi = \pi \quad \text{on} \quad S^3 \setminus T,$$

$$\phi(x) = (1 - \beta(\|p\varrho(x)\|)\, p\varrho(x) + \beta(\|p\varrho(x)\|)\, \pi(x), \qquad \forall x \in T \setminus L,$$

$$\phi(x) = 0, \qquad \forall x \in L.$$

It is clear that rank $d\phi = 2$ on a neighbourhood of L and rank $d\phi_x \geq 1$, $\forall x \in S^3$. Moreover $\phi^{-1}(0) = L$. Let (W, E, r) be any modification of (P_4, K_4, α_4). Any r-cone of ϕ is regular and $g^{-1}(0) = E \cup \psi(L \times (0, 1])$. Applying a similar «cone» construction (see [2]) to the natural radial retraction of D^4 onto $\{0\}$, we get L as the link type of (continuous, but also, with minor changes, C^r semi-lagebraic or smooth, but flat at $\{0\}$) tame maps, in the sense of the introduction.

We stipulate that the topology of smooth maps is the usual Whitney strong topology.

LEMMA 6.3. – *If g is a regular r-cone of f and f' is close enough to f then any r-cone of f' is regular.*

PROOF OF 6.3. – Set $X_1 = f^{-1}(0)$ and $X_2 = \{x \in Z;\ f \text{ is not regular at } x\}$. Set similarly X_1', X_2' for f'. rank $df_x =$ rank $dF_{(x,t)}$ for each $(x, t) \in X_1 \times (0, 1]$, so that $X_1 \cap X_2 = \emptyset$. If f' is close enough to f, there exists an open set $U \subset Z$, $X_1 \subset U$ such that

$$X_2 \cap U = \emptyset, \qquad X_1' \subset U, \qquad X_2' \cap U = \emptyset.$$

Clearly F' is regular on $U \times (0, 1]$; since F is regular on $Z \setminus U$, then $\forall x \in Z \setminus U$, Im df_x and $f(x)$ span $\mathbb{R}^m$. As $Z \setminus U$ is a compact set, if f' is close to f, F' keeps this property and it is regular on the whole $Z \times (0, 1]$. ∎

Let $g = (g_1, \ldots, g_m): W \to \mathbb{R}^m$ be an r-cone. Set

$$A_i = g_i^{-1}(0).$$

DEFINITION 6.4. – An r-cone g is said *tame* if it is regular and

a) for every i $A_i = E \cup Y_i$ and Y_i is a smooth hypersurface;

b) the family of hypersurfaces $\{E_j, Y_i\}$ is in general position in (W, Z). We note that each Z_t is transverse to each Y_i.

Given a normal retraction $r: W \to E$, $E = \bigcup_{i=1,\ldots,h} E_i$ we can construct in a natural way systems of (open) tubes U_i along each hypersurface E_i of E and involutions $\iota_i\colon U_i \to U_i$ such that:

$$\forall i, \qquad E_i = \{\iota_i(x) = x\},$$

$$\iota_i \iota_j = \iota_j \iota_i \qquad \text{on} \qquad U_i \cap U_j,$$

$$\forall i, \qquad r\iota_i = \iota_i r \qquad \text{on} \qquad U_i.$$

We admit that the tubes could be arbitrarily small. Assume such a system to be fixed.

DEFINITION 6.5. – A tame r-cone g is said *monomial like* if there is a map

$$\varrho = (\varrho_1, \ldots, \varrho_m)\colon \{1, \ldots, h\} \to \{0, 1\}^m$$

such that

$$\forall i \;\; g_s = (-1)^{\varrho_s(i)} g\iota_i \qquad \text{on} \qquad U_i.$$

7. – Fibred links.

In this section we assume that L is a non trivial fibred knot or more generally a fibred link such that the fibre F of the fibration carries a divisorial spine E.

Our aim is to outline the prove of the weak reciprocal to the Milnor fibration theorem stated in the introduction.

The first ingredient is to specialise the previous contructions to the present situation; adopting the notations of the other sections, now (F, E) is a fibre of the fibration endowed with the fixed spine; V and V' are made by fibres of the fibration, with boundary S and S' made by two couple of fibres glued along L. Then we can perform the blow-up procedure following also the behavior of the r-cones (that we have defined before in the basic example) till we get finally, at the last step of the tower, a *monomial like* r-cone.

The second one is summarized in the following proposition. With the notations of the section on r-cones.

THEOREM 7.1. – *Let $g\colon W \to \mathbb{R}^2$ be a monomial like smooth r-cone of a map f, relatively to a normal retraction $r\colon W \to E$. Assume that*

a) $\dim W = 4$;

b) W, the hypersurfaces E_j, Y_1, Y_2 are analytic manifolds.

Then one can construct an analytic tame r-cone g' of an analytic map f' close to f such that $\{g' = 0\} = \{g = 0\}$.

We have focused the hypothesis on the dimension because we will use few peculiar facts holding for low dimensions (in particular $3 = \dim \partial W$).

The rest of the section is devoted to outline the proof of this theorem. For simplicity we will neglige Y_1 and Y_2 in the discussion, because the part of the statement that regards them can be easily obtained invoking well-known *relative* versions of approximation results of smooth maps by analytic ones.

We state a lemma useful in the sequel.

LEMMA 7.2. – *Assume that* $W = \{x \in \mathbb{R}^4;\ |x_1 \dots x_4| \leqslant 1\}$, $E_i = \{x_i = 0\}$, *and* $r\colon W \to E$ *is defined by* π_4 (*with the notation of the section on the divisorial spines*). *Let* V_1 *and* V_2 *be connected open sets in* ∂W, *and let* $b\colon V_1 \to V_2$ *be an analytic diffeomorphism such that* $b(V_1 \cap r^{-1}(E_i)) = V_2 \cap r^{-1}(E_i)$ *for* $i = 1, 4$. *Then there exists an analytic diffeomorphism* $B\colon r^{-1}(r(V_1)) \to r^{-1}(r(V_2))$ *extending* b *and such that*

1) $B(r^{-1}(r(x))) = r^{-1}(rb(x))$ *for* $x \in V_1$;

2) $B(r(V_1)) = r(V_2)$.

The proof is a little long but based on direct calculations. We left it to the reader.

Let us come now to the main steps of the proof.

STEP 1. – In the hypothesis of the theorem we may assume that r is an analytic normal retraction and that the involutions ι_i of the monomial-like structure are analytic involutions.

More precisely: fix for any component E_i of the spine E a positive even integer $m(i)$ and a non negative analytic function h_i such that $E_i = h_i^{-1}(0)$ and at each point x_0 of E_i the germ of h_i is the $m(i)$-*th*-power of an analytic regular function germ. Set h the product of the h_i. Then we may assume that (adopting the notations of the previous sections):

1) The above h coincides with the function d of r-cones definition, relatively to an r-cone g of f (and an analytic diffeomorphism ψ).

2) Using the *analytic local coordinates* of the analytic normal retraction r, at each point $x_0 \in E$ we can assume, for some $i = 1, \dots, 4$,

$$W = \{x \in \mathbb{R}^4;\ |x_1 \dots x_i| \leqslant 1,\ |x_i^2 - x_2^2| \leqslant 1,\ \dots,\ |x_1^2 - x_i^2| \leqslant 1,$$
$$|x_{i+1}| \leqslant 1,\ \dots,\ |x_4| \leqslant 1\}.$$

3) $E = \{x_1 \dots x_i = 0\}$, $x_0 = 0$.

4) $\{r^{-1}(x);\ x \in E\} = \{$connected components of the levels of $\pi_i\}$.

5) $h(x) = x_1^{m(1)} \dots x_i^{m(i)} a(x)$ where a is a positive analytic fuction on W.

6) The tubes U_i of the definition of the *monomial like* notion make a

covering of W, they are the inverse image by the analytic retraction r of open neighbourhoods of E_i in E, and, as already stated, the involutions ι_i are analytic.

This step actually should hold without dimensional limitation.

Let us sketch the proof.

Let h_i and h be as before. For each j-tuple of distinct indices $J = (i_1, \ldots, i_j)$, set

$$E_J = \bigcap_{i = i_1, \ldots, i_j} E_i \setminus \bigcup_{i \neq i_1 \ldots i_j} E_i.$$

Let $p_J \colon U_J \to E_J$ be a small analytic tubular neighbourhood in W, such that:

1) $U_J \cap E_i$ is empty if i does not belong to J.

2) For each $x_0 \in E_J$, $p_J^{-1}(x_0)$ is an analytic submanifold with boundary of W.

3) $(\pm h_{i_1}^{1/m(i_1)}, \ldots, \pm h_{i_j}^{1/m(i_j)})$, for suitable $\pm$, makes analytic coordinates on $p_J^{-1}(x_0)$.

Using it, we construct an analytic normal retraction $r_J \colon U_J \to E \cap U_J$. We want to past these partial retractions toghether. For each $k = 0, 1, 2, 3$, let E^k denote the union of strata of dimension k of the natural stratification of E, and let, with the obvious meaning, $p^k \colon U^k \to E^k$, $r^k \colon U^k \to E \cap U^k$ be defined by means of p_J and r_J. Replace U^0 with $(r^0)^{-1}(E \cap \{$the closed ε-neighbourhood of $E^0\}$, for a suitable small ε, and U^1 with $(r^1)^{-1}(E \cap U^1 \setminus \{$the open ε-neighbourhood of $E^0\}$, respectively. Regard $E \cap U^0 \cap U^1$ included in U^0 and in U^1 as disjoint pairs, paste them at $E \cap U^0 \cap U^1$ by the identity map, and then paste $(r^0)^{-1}(E \cap U^0 \cap U^1$ and $(r^1)^{-1}(E \cap U^0 \cap U^1)$ so that r^0 and r^1 are compatible. Repeating it for the other strata, we get, abstractly, an analytic manifold, with an analytic retraction to E; it can be easily smoothly embedded onto an open neighbourhood of E in W, keeping E pointwise fixed. Using Cartan Theorem A we can take in fact an analytic diffeomorphism, and, eventually shinking the neighbourhood of E, we can assume that it is of the form $W' = h^{-1}([0, \varepsilon])$ for some small ε, the level surfaces of h being transversal to the foliation by arcs induced by r. Finally, using the fact that $W \setminus W'$ is an analytic collar of ∂W, and the previous lemma, we can lift what we have done along E to the whole W. Thus we have converted r to an analytic normal retraction, and eventually modifying the initial h_i, the required properties are satisfied. We want to define now the analytic involutions. On each U_J we define these analytic $\iota_{i_1}, \ldots, \iota_{i_j}$ by the conditions that they are invariant on each fibre of p_J, and $h_{i_s} \iota_{i_t} = h_{i_s}$ for every s and t. Then we have $r_J \iota_{i_s} = \iota_{i_s} r_J$. Moreover, by the above method of pasting U^1 and U^2, and so on, we see that the definition of ι_i does not depend on J, namely, each ι_i is defined on some neighbourhood of E_i in W, of the form U_i as stated above. By construction we have the conditions $\iota_i \iota_j = \iota_j \iota_i$; moreover, up to eventual isotopy of f, $g = (-1)^{\varrho(i)} g\iota_i$ (for a suitable r-cone of f) is satisfied.

STEP 2. – Up to approximation, we can replace f by an analytic map f'.

Apply the above step 1 imposing that $m(i) = 4$ if $\varrho(i) = 0$ and $m(i) = 2$ if $\varrho(i) = 1$.

A stratification of ∂W. Let Q^0 be the union of all $U_J = U_{i_1} \cap \dots U_{i_4}$ for any set of 4 distinct indices; let Q^1 be the union of all $U_J \setminus r^{-1}(E^0)$, for any set J of 3 distinct indices, and so on. Then each Q^j is a neighbourhood of E^j and $r^{-1}(Q^j \cap E) = Q^j$. Set $R^j = Q^j \cap \partial W$, denote R^j_s its connected components, $S^j_s = r^{-1}(E^j) \cap R^j_s$ and similarly for S^j. Then R^j_s is an open neighbourhood of the analytic j-submanifold S^j_s in ∂W and the involutions ι_i induce analytic diffeomorphisms $v^j_{st}: (R^j_s, S^j_s) \to (R^j_t, S^j_t)$ for some triples j, s, t. Then

1) $v^j_{sq} = v^j_{pq} v^j_{sp}$ and $v^j_{ii} = id$.

2) v^j_{st} is defined iff v^j_{ts} is defined.

3) v^j_{st} is defined iff $r(S^j_s) = r(S^j_t)$, in such a case $v^j_{st}(S^j_s) = S^j_t$.

4) If v^j_{st} is defined, $v^j_{st}(R^j_s \cap R^k_q) = R^j_s \cap R^k_p$, for k smaller than j, then it is defined v^k_{qp} that equals v^j_{st} on $R^j_s \cap R^k_q$.

5) ϱ induces numbers ϱ^j_{st}, equal to either 0 or 1, such that

$$f = (-1)^{\varrho^j_{st}} f v^j_{st}$$

on R^j_{st}. In the sequel we will refer to this last condition as *condition* (*).

In the sequel we shall work up to firstly shrinking and then restoring the whole W similarly to the above step 1. We tacitely do it. During the construction the functions h_i and h shall be modified but keeping their qualitative properties.

Putting f in normal form. At this point we use peculiar, well known (see [11]), facts in low dimensions, that is that *generic* smooth maps between low-dimensional manifolds make a *dense* subset, are *stable* and have simple (local) *normal forms*. More precisely here is the normal forms we are going to use:

let m be the source dimension and n the target dimension.

If $m = 3$ and $n = 2$ we have the following three possibilities:

$$(x_1, x_2, x_3) \to (x_1, \pm x_2^2 + x_3^3 + x_3 x_1), \quad (x_1, \pm x_2^2 \pm x_3^2), \quad (x_1, x_3).$$

If $m = n = 2$ we have the three possibilities:

$$(x_1, x_2) \to (x_1, x_2^3 + x_2 x_1), \qquad (x_1, x_2^2), \qquad (x_1, x_2).$$

If $m = 1$ and $n = 2$ the generic maps are the regular ones.

When $m = 3$ and $n = 2$; in the first case 0 is a singular point of type 1 and in the second case a singular point of type 2. If q is a generic map, its singular set is $\mathrm{Sing}\, q = \mathrm{Sing}^1 q \cup \mathrm{Sing}^2 q$, with the obvious meaning; $\mathrm{Sing}\, q$ is a closed submanifold of the source manifold, $\mathrm{Sing}^j q$ are, in general not closed, submanifolds of dimension $j-1$. If q is analytic these manifolds are analytic; a similar fact holds when $m = n = 2$.

In case the source manifold M has boundary (or corners) we consider maps defined on bigger manifold M' containing the first one in the interior. Thus a generic map on M is, by definition, generic on M' and with singular sets in general position w.r.t. the boundary.

Claim. – We can assume that the smooth map f is generic on ∂W and on each $\partial W \cap r^{-1}(E_{i_1} \cap \ldots \cap E_{i_j})$; moreover all the curves $f(\mathrm{Sing}^2\, f | S_s^j)$, $j = 2, 3$, $f(S_s^1)$, and all the points $f(\mathrm{Sing}^1\, f | S_s^j)$ $j = 2, 3$, $f(S_s^0)$ are in general position (*multi-transversality condition*; briefly we call such a map good).

To realize it, we note first that for every x_0 of ∂W there are open neighbourhoods $N' \subset N$ such that f can be perturbed with support on N in order to get f_N that is good map N'. We have also the global problem to preserve the *condition* (*). To do it we choose the above N so that

$$N \subset R_s^j, \qquad N \cap v_{st}^j = \emptyset,$$

for suitable j and all possible $s \neq t$. Then define a perturbation of f_N on each v_{st}^j to be $(-1)^{\varrho_{st}^j} f_N v_{st}^j$ in order to restore the *condition* (*). Then perform this type of local perturbation in sequence on $S^0, \ldots, S^3$ (here we use also the stability of good maps, that is a small perturbation of a good map is again good); as ∂W is compact, this process ends and we finally obtain the required good version of f.

Making f analytic. The idea is to firstly get the analitycity up to a modification of the analytic structure of W, and then to restore the initial setting by the use of Cartan theorem A (in the same spirit of what we have done in step 1, to produce the analytic retraction). As in the previuos point we use the same trick in order to first perturb and then to restore the *condition* (*).

Eventually adding to S^0 any finite set, we can assume (up to a perturbation with small support) that f is analytic on R^0; the reason to increase S^0 is to gain further regularity conditions for the restriction of f to S^j, $0 \leq j$; for example, considering the finite set made by the union of the sets $\mathrm{Sing}^1\, f | S^j$, $j = 2, 3$, and the sets of inverse image in S^1 of double points of f, we can assume also that the restriction of f to S^1 is a smooth covering map onto the curve $f(S^1)$. In the sequel we will modify several time in the same spirit S^0.

We can assume that $f(S^1)$ is an analytic curve by the following reason: give $f(R^0 \cup R^1)$ a suitable analytic structure that coincides with the old one on $f(R^0)$ and such that $f(S^1)$ is an analytic curve. Paste analytically $f(R^0 \cup R^1)$ with $\mathbb{R}^2 \backslash f(S^0 \cup S^1)$ to get an analytic manifold Θ and a smooth diffeomorphism $\gamma: \mathbb{R}^2 \to \Theta$ that is analytic on $f(R^0)$ and such that $\gamma f(S^1)$ is an analytic curve. Approximate γ by an analytic α and replace f with $\alpha^{-1}\gamma f$; then we can assume that $f(S^1)$ is an analytic curve.

By modifying the analytic structure on ∂W we can assume that $f|S^1$ is analytic. The idea is to pull-back on S^1 the analytic structure on $f(S^1)$ using the fact that the restriction of f to S^1 is a covering map, and then make it compatible with a global change of analytic structure on W (preserving all the other structures: analytic retraction etc.), that is possible by the previous lemma.

Up to modify S^0 as said before, we can assume that $f|R^1 \cap r^{-1}(E_i \cap E_j)$ is a regular map for any $i \neq j$. Let p and q analytic projections of tubular neighbourhoods of S^1 and $f(S^1)$, such that $qf = fp$. We want to modify the analytic structure on a neighbourhood of $S^1 \backslash R^0$, so that f becomes analytic on it. Using p, q we can assume that

$$R^1 = \mathbb{R}^2 \times \mathbb{R}\,, \qquad S^1 = \{(\pm t, t), (0, -t);\, 0 \leqslant t\} \times \mathbb{R}\,,$$

$$f(x, y, z) = (a(x, y, z), z)\,,$$

$a(0, 0, z) = 0$; a is analytic outside $\mathbb{R}^2 \times [0, 1]$; for each z a is regular on $\mathbb{R}^2 \times \{z\}$ and on $S^1 \cup (\mathbb{R}^2 \times \{z\}$.

Set $A = \{(t, t);\, t \geqslant 0\}$, $B = \{(-t, t);\, t \geqslant 0\}$, $C = \{(0, -t);\, t \geqslant 0\}$. We have a smooth diffeomorphism of $A \times \mathbb{R}$ of the form $(?, z)$ which is analytic outside $A \times [0, 1]$ and whose composition with a is $\pm$(the projection to the y-axis) on a neighbourhood of $\{0\} \times \mathbb{R}$. Similar properties holds for $B \times \mathbb{R}$ and $C \times \mathbb{R}$. Hence, via a smooth diffeomorphism of $\mathbb{R}^2 \times \mathbb{R}$ we can change S^1 and a so that $a(x, y, z) = y$ and $S^1 = (A \cup B) \times \mathbb{R} \cup C'$, where C' is a smooth 2-dimensional manifold with boundary $\{0\} \times \mathbb{R}$, contained in $(\mathbb{R}_+)^2 \times \mathbb{R}$ and analytic outside $\mathbb{R}^2 \times [0, 1]$. Moreover it is easy to find a smooth diffeomorphism δ of $\mathbb{R}^2 \times \mathbb{R}$ of the form $\delta(x, y, z) = (b(x, y, z), y, z)$, such that $\delta(A \times \mathbb{R}) = A \times \mathbb{R}$, similarly for $\delta(B \times \mathbb{R}$, and such that $\delta(C') \cap (\mathbb{R}^2 \times \{z_0\})$ is a segment for each $z_0 \in \mathbb{R}$; δ is analytic outside $\mathbb{R}^2 \times [0, 1]$. Note that $a\delta = a$. Hence we can assume that $(C' \cap \mathbb{R}^2) \times \{z_0\}$ is a segment for each z_0. Define a smooth function Θ on $\mathbb{R}$ so that

$$(C' \cap \mathbb{R}^2) \times \{z_0\} = \{(x, y, z_0);\, x = \Theta(z_0)\, y,\, y \leqslant 0\}\,.$$

Θ is analytic outside $\mathbb{R}^2 \times [0, 1]$. We can assume that Θ is regular. Change the analytic structure on neighbourhoods of $\{0\} \times \mathbb{R}$ and of $f(\{0\} \times \mathbb{R})$. Thus we make Θ globally analytic. Note that by the *condition* (*) such a modification of analytic structures can be made compatible w.r.t. all R_s^1 because $f(S_s^1) \cap f(S_t^1)$ is non empty iff $f(S_s^1) = f(S_t^1)$, that is iff v_{st}^1 is defined. In this way we can assume that f is analytic on $R^0 \cup R^1$.

We repeat this procedure of modifying the analytic structures firstly to make f analytic on S^2, without changing it where it is already analytic, and then on $R^0 \cup R^1 \cup R^2$. And so on, till f becomes analytic on a modified analytic structure on W. The new analytic W is smoothly diffeomorphic to the old one, and the normal retraction r keeps meaning, i.e. it was invariant under the above changements. So using Cartan theorem A we can construct an analytic diffeomorphism between the old and new W preserving $r^{-1}(x)$ for each $x \in E$. So we can assume that the conclusions of both step 1 and 2 hold at the same time.

FINAL STEP. – *An analytic r-cone.*

This step contains, in a sense, the core of the construction. Set

$$g(x) = \pm f\big(r^{-1}(r(x))\, h^{1/2}, \qquad x \in W\,.$$

Here the $\pm$ are chosen so that g is a continuos extension of f. We claim that g is analytic. The problem is local. Consider g around a point of, say, $E_1 \cap \ldots \cap E_4$. Then we can assume $W = V_4'$ (with the notation of the definition of divisorial spines), r is defined by π_4, and there exist analytic functions a on V_4' and f^* on $\mathbb{R}^3$ such that

$$g(x) = \pm\, x_1^{2-\varrho(1)} \ldots x_4^{2-\varrho(4)}\, a(x)(f^* \pi_4)(x)\,.$$

Here $\pm$ is independent of the points of V_4' because of *condition* (*), hence, finally, g is analytic.

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