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On the Localization of the Vortices (*).

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Sunto. – *Studiamo l'evoluzione temporale di un fluido bidimensionale incomprimibile non viscoso quando la vorticità iniziale è concentrata in N regioni di diametro ε e mostriamo che la vorticità evoluta temporalmente è anche lei concentrata in N piccole regioni di diametro d, $d \leq \text{const}\, \varepsilon^\alpha$ per qualunque $\alpha < 1/3$. Noi chiamiamo questa proprietà "localizzazione". Come conseguenza abbiamo una connessione rigorosa tra il modello dei vortici puntiformi e l'Equazione di Eulero.*

1. – Introduction and main result.

In this paper we discuss the time evolution of a two-dimensional inviscid incompressible fluid of unitary density, when the initial vorticity is concentrated in N small disjoint regions $\Lambda_i(0)$ of diameter ε. It is known that, for any fixed time t, the time evolved vorticity remains concentrated in N small regions $\Lambda_i(t)$ of diameter $d(\varepsilon)$ and $d(\varepsilon) \to 0$ as $\varepsilon \to 0$. We call this property "localization". Furthermore if z_i, $i = 1, \ldots, N$, is the solution of the ordinary differential system (called point vortex system [Hel67], [Kir76], [Poi93], [Kel10])

$$\frac{d}{dt} z_i(t) = -\nabla_i^\perp \frac{1}{2\pi} \sum_{j=1; j \neq i} a_j \ln |z_i(t) - z_j(t)| \qquad z_i(0) = z_i, \qquad z_i \in \mathbb{R}^2 \tag{1.1}$$

where

$$\nabla^\perp \equiv (\partial_2, -\partial_1) \tag{1.2}$$

it can be proved that if $z_i(0) \in \Lambda_i(0)$ then $z_i(t) \in \Lambda_i(t)$. (This proves a rigorous connection between the Euler equation and the point vortex model: [MaP83], [MaP84], [MaP86], [Tur87], [Mar88], [MaP93]. For a review on the topic see [MaP94]).

In the present paper we investigate in more details the property of "locali-

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zation" ($d(\varepsilon)\to 0$ as $\varepsilon\to 0$). In fact in ref. [MaP93] it was given an estimate on $d(\varepsilon)$ like ε^β with β very small (say $<(100)^{-1}$). Here we improves this result and we give the "best" estimates of the size of the support of the vorticity $d(\varepsilon)$ compatible with the present mathematical technique. It turns out that $d(\varepsilon)$ must vanishes as $\varepsilon\to 0$ with a law more observable in a real fluid. Moreover, as a technical point, we use in the proof a weaker assumption on the initial data. We will return on this point after the statement of Theorem 1.1.

For simplicity we formulate the result when the fluid moves in the whole $\mathbb{R}^2$. The generalization to a generic domain is straightforward.

Consider the Euler equation in $\mathbb{R}^2$ in terms of vorticity:

$$\partial_t \omega(x,t) + (u\cdot\nabla)\,\omega(x,t) = 0 \tag{1.3}$$

$$\nabla\cdot u(x,t) = 0 \tag{1.4}$$

$$\begin{cases} \omega \equiv \operatorname{curl} u \equiv \partial_1 u_2 - \partial_2 u_1 \\ \omega(x,t) = \omega_0\,, \qquad x=(x_1,x_2)\,. \end{cases} \tag{1.5}$$

Here $u=(u_1,u_2)$ denotes the velocity field.

We assume that u decays at infinity and so we can reconstruct the velocity field by means of ω as

$$u(x,t) \equiv \int K(x-y)\,\omega(y,t)\,dy\,, \tag{1.6}$$

$$K = \nabla^\perp G\,, \tag{1.7}$$

$$G(x) = -\frac{1}{2\pi}\ln|x|\,. \tag{1.8}$$

As well known, eq. (1.2) means that the vorticity is constant along the particle paths which are the characteristics of the Euler equations. Therefore

$$\omega\big(x(x_0(t),t\big) = \omega(x_0,0) \tag{1.9}$$

where the trajectory $x(x_0,t)$ of the fluid particle, initially in x_0, satisfies:

$$\frac{d}{dt}x(x_0,t) = u(x(x_0,t),t)\,, \qquad x(x_0,0)=x_0\,, \tag{1.10}$$

where the velocity field $u(x,t)$ is given by eq. (1.5).

We want to study the Euler equation when no strong properties of regularity on the initial vorticity are supposed. So we need a weak formulation of the Euler equation, which is meaningful in this case. As well known eqs. (1.9), (1.10), (1.11) imply the weak form of the Euler equation:

$$\frac{d}{dt}\omega[f] = \omega[u\cdot\nabla f] + \omega[\partial_t f], \tag{1.11}$$

where $f(x, t)$ is a bounded smooth function and

$$\omega[f] \equiv \int dx\, \omega(x, t) f(x, t). \tag{1.12}$$

It is well known that there exists a unique solution $\omega(x, t) \in L_1 \cap l_\infty$ to the initial value problem associated to (1.11) provided that $\omega(x, t) \in L_1 \cap l_\infty$. Moreover the divergence-free condition (1.3) implies that the time evolution (1.10) preserves the Lebesgue measure on $\mathbb{R}^2$.

We consider an initial datum of the form:

$$\omega_\varepsilon(x, 0) = \sum_{i=1}^{N} \omega_{\varepsilon;i}(x, 0) \tag{1.13}$$

where $\omega_{\varepsilon;i}(x, 0)$ is a function with a definite sign supported in a region $\Lambda_{\varepsilon;i}$ such that

$$\Lambda_{\varepsilon;i} \equiv \operatorname{supp}\omega_{\varepsilon;i} \subset \Sigma(z_i\,|\,\varepsilon); \qquad \Sigma(z_i\,|\,\varepsilon) \cap \Sigma(z_j\,|\,\varepsilon) = 0 \qquad \text{if } i \neq j \tag{1.14}$$

for ε small enough. Here $\Sigma(z|r)$ denotes the circle of center z and radius r.

We put

$$\int dx\, \omega_{\varepsilon;i}(x, 0) \equiv a_i \in \mathbb{R} \tag{1.15}$$

the vortex intensity (independent of ε) and assume

$$|\omega_{\varepsilon;i}(x, 0)| \leqslant M\varepsilon^{-\gamma}, \qquad M > 0, \quad \gamma > 0. \tag{1.16}$$

We remark that in previous paper [MaP93] we assumed that

$$|\omega_{\varepsilon;i}(x, 0)| \leqslant \text{const}\,\varepsilon^{-\eta}, \qquad \eta < 8/3 \tag{1.17}$$

while here γ is arbitrary, so that condition (1.16) is an improvement of the known results.

The main result of the present paper is the following

THEOREM 1.1. – Denote by $\omega_\varepsilon(x, t)$ the time evolution of $\omega_\varepsilon(x, 0)$ according the Euler equation. Then for any fixed $T > 0$ for any $\alpha < 1/3$

i) there exists $C(\alpha, T)$ such that for $0 \leqslant t \leqslant T$

$$\operatorname{supp}\omega_{\varepsilon;i}(x, t) \subset \Sigma(z_i(t)\,|\,d) \tag{1.18}$$

where

$$d = C(\alpha,\, T)\, \varepsilon^{a} \tag{1.19}$$

and $z_i(t)$ is the solution of the ordinary differential system (1.1), provided that such a solution exists up to the time T.

ii) for any continuous bounded function $f(x)$

$$\lim_{\varepsilon \to 0} \int dx\, \omega_{\varepsilon}(x,\, t) f(t) = \sum_{i=1}^{N} a_i f(z_i(t))\,. \tag{1.20}$$

Proposition i) states that the blobs of vorticity remain localized until time the T. Position ii) states that

$$\omega_{\varepsilon}(x,\, t) \xrightarrow[\varepsilon \to 0]{} \sum_{i=1}^{N} a_i \delta(z_i(t)) \tag{1.21}$$

weakly in the sense of measures, where $\delta(\cdot)$ denotes the Dirac measure. This last statement gives a rigorous justification of the point vortex model.

We observe that the singular nature of the right hand side of eq. (1.1), diverging when two vortices are close, does not guarantee the existence of the solutions of eq. (1.1) for every time. In many cases (for instance for all $a_i > 0$) collapes are forbidden by the first integrals of the motion, but there are cases in which singularities do happen. However it can be proved that the collapses are exceptional [MaP84]. In general we can say that Theorem 1.1 holds up to the time T for which the solution of eq. (1.1) does exist.

The proof of the Theorem 1.1 will be given in the next Section. Here we discuss the meaning of the result and the improvement contained in the present paper.

Condition (1.16) means that the initial conditions could be very singular without changing the localization property.

Equation (1.19) gives a bound on the localization. It is not trivial, as we will see in the proof, but we believe that in general it is not optimal. In fact it holds for any initial condition of form (1.13). For particular initial conditions the bound might be better. A wide discussion on this point is contained in the introduction of [Mar94]. In that paper we discussed the growth of the support of a vortex patch. By changing the scale of time and length we can obtain the result of the present paper for a single vortex. In this case a can be equal to 1/3 while in Theorem 1.1, that is a generalization to the case in which many vortices are present, a can be close as we want but smaller than 1/3. Finally, we remark that the result in [Mar94], given for a bounded vortex patch, has been generalized for unbounded initial vorticity in [LNL98], i.e. in our language by a changing of scale from $\gamma = 2$ to any γ. Of course these results refer to a single vortex, while here we investigate the case of many disjoint blobs of vorticity.

2. – Proof of Theorem 1.1.

The proof is a mixture of the proofs of [MaP93] and [Mar94] with some new ideas. First we study the motion of a single vortex in a Lipschitz external field and prove that this vortex remains localized. Then we observe that the other vortices produce a Lipschitz external field.

We consider a single blob of unitary vorticity moving in an external,divergence-free, uniformly bounded, time dependent, vector field $F(x, t)$, satisfying the Lipschitz condition

$$|F(x,t) - F(y,t)| \leq L|x-y|, \qquad L > 0. \tag{2.1}$$

Equation (1.10) becomes

$$\frac{d}{dt} x(t) = u(x,t) + F(x,t) \tag{2.2}$$

while eqs. (1.9), (1.11) remain unchanged. The Euler equation in weak form reads as

$$\frac{d}{dt}\omega[f] = \omega[(u+F)\cdot\nabla f] + \omega[\partial_t f]. \tag{2.3}$$

Then we prove proposition i) of Theorem 1.1 for this particular evolution. Define the center of vorticity as

$$B_\varepsilon(t) \equiv \int x\,\omega_\varepsilon(x,t)\,dt. \tag{2.4}$$

THEOREM 2.1. – Suppose that

$$\operatorname{supp}\omega_\varepsilon(x,0) \subset \Sigma(x^* \,|\, \varepsilon) \tag{2.5}$$

and

$$|\omega_\varepsilon(x,0)| \leq \mathrm{const}\,\varepsilon^{-\gamma} \qquad \gamma > 0 \tag{2.6}$$

(from now on const. denotes a constant independent of ε).

$$\int dx\,\omega_\varepsilon(x,t) \equiv 1. \tag{2.7}$$

Then for any $\alpha < 1/3$ there exists $C(\alpha, T)$ such that for $0 \leq t \leq T$

$$\operatorname{supp}\omega_\varepsilon(x,t) \subset \Sigma(B(t) \,|\, d) \tag{2.8}$$

where

$$d = C(\alpha, T)\, \varepsilon^{\alpha} \tag{2.9}$$

and $B(t)$ is the solution of the ordinary differential equation

$$\frac{d}{dt} B(t) = F(B(t), t)\,, \qquad B(0) = x^*\,. \tag{2.10}$$

Moreover

$$|B_\varepsilon(t) - B(t)| \underset{\varepsilon \to 0}{\longrightarrow} \text{ at least as } \varepsilon\,, \qquad \text{uniformly in } t \in [0, T]. \tag{2.11}$$

PROOF. – The difficulty of the proof arises from the singularity of the kernel K which forces a fluid particle to rotate with a very large velocity around the center of vorticity. To overcome this difficulty we study the motion of the center of vorticity which will turn out to be much more regular than the motion of a given fluid particle. Moreover,the moment of inertia is almost conserved during the motion, so that we can also control the spreading of the vorticity distribution around the center of vorticity. However, as we shall see, the control given by the moment of inertia is not enough for our purposes.

We introduce the moment of inertia Ie with respect to B_ε:

$$I_\varepsilon(t) \equiv \int \omega_\varepsilon(x, t)(x - B_\varepsilon(t))^2\, dx\,. \tag{2.12}$$

Then we study its growth in time. If F would vanish, B_ε and I_ε would be constant along the motion. For $F \neq 0$ we have

$$\frac{d}{dt} B_\varepsilon(t) = \int F(x, t)\, \omega_\varepsilon(x, t)\, dx\,, \tag{2.13}$$

$$\frac{d}{dt} I_\varepsilon(t) = 2 \int (x - B_\varepsilon(t)) \cdot F(x, t) \omega_\varepsilon(x, t)\, dx\,. \tag{2.14}$$

where we have taken into account the antisymmetry of K. Making use of the fact that

$$\int (x - B_\varepsilon(t)) \cdot F(B_\varepsilon(t), t)\, \omega_\varepsilon(x, t)\, dx \equiv 0 \tag{2.15}$$

and the Lipschitz condition on F, we have

$$\left| \frac{d}{dt} I_\varepsilon(t) \right| \leq 2L \int \omega_\varepsilon(x, t)(x - B_\varepsilon(t))^2\, dx = 2L I_\varepsilon(t) \tag{2.16}$$

from which

$$I_\varepsilon(t) \leqslant I_\varepsilon \exp(2Lt). \tag{2.17}$$

Therefore

$$\lim_{\varepsilon \to 0} I_\varepsilon(t) = 0 \quad \text{at least as } \varepsilon^2, \quad \text{uniformly in } t \in [0, T]. \tag{2.18}$$

The next steps to obtain eq. (2.11) are similar to that used in [MaP93] to obtain eq. (3.16) and we omit them.

We have obtained eq. (2.18) which says that the main part of the vorticity is concentrated around the center of vorticity. A priori small filaments of vorticity could go far away. We want to prove that is not the case and also that the support of the vorticity remains concentrated around the center. To this purpose we study the radial part of the velocity field near the boundary of the support of vorticity and prove that the difference between this field and the velocity field acting on the center of the vortex vanishes as $\varepsilon \to 0$. So the particle paths cannot go far apart from B_ε. This field is due essentially to three terms: the velocity produced by the external field, the velocity produced by the particles near the center of the vortex and the velocity produced by the particles near the boundary. The first contribution is easily controled by the Lipschitz condition, the second contribution gives a radial part which vanishes as the initial vorticity is sharply concentrated and the third contribution needs more care and vanishes after an iterative procedure. The proof of this fact, given in the sequel, is rather technical.

We study the growth of the distance of a fluid particle in $x \in \operatorname{supp} \omega_\varepsilon(x, t)$ farest from $B_\varepsilon(t)$:

$$\left| \left(u(x, t) + F(x, t) - \frac{d}{dt} B_\varepsilon(t)\right) \cdot \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \right| \leqslant \tag{2.19}$$

(by using eqs. (2.2), (1.11) and (2.13)) $\leqslant$

$$\left| F(x, t) - \int dy\, \omega_\varepsilon(y, t)\, F(y, t) \right| + \left| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int dy\, K(x - y)\, \omega_\varepsilon(y, t) \right| =$$

(by using eq. (2. 7)) $=$

$$\left| \int dy\, \omega_\varepsilon(y, t)[F(x, t) - F(y, t)] \right| + \left| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int dy\, K(x - y)\, \omega_\varepsilon(y, t) \right| .$$

The first contribution due to the external field is trivial. Using the Lipschitz condition (2.1):

$$\leqslant \text{const}\, R\,, \qquad R \equiv |x - B_\varepsilon(t)|\,. \tag{2.20}$$

Now we study the term

$$\left| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int dy\, K(x-y)\, \omega_\varepsilon(y, t) \right| \tag{2.21}$$

The contribution due to the vorticity near the center of the vortex can be computed as in [Mar.94]. We divide the circle $\Sigma(B_\varepsilon(t)\,|\,R)$ into many different annulii:

$$\Sigma(B_\varepsilon(t)\,|\,R) = \sum_{k=1}^{k^*} [\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1}) \cup [\Sigma(B_\varepsilon(t)\,|\,R) - \Sigma(B_\varepsilon(t)\,|\,a_{k^*})] \tag{2.22}$$

where

$$a_0 = 0\,, \qquad a_1 = \varepsilon\,, \qquad a_k = 2a_{k-1}\,. \tag{2.23}$$

We choose k^* such that $a_{k^*+1} \leqslant R$ and $a_{k^*+2} > R$.

The radial velocity can be expressed by the sum of the contribution obtained when the particles are contained in each annulus:

$$\begin{aligned} &\frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} dy\, \omega_\varepsilon(y, t)\, K(x-y) = \\ &\frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} dy\, \omega_\varepsilon(y, t)\, K(x - B_\varepsilon(t)) + \\ &\frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} dy\, \omega_\varepsilon(y, t)[K(x-y) - K(x - B_\varepsilon(t))]\,. \end{aligned} \tag{2.24}$$

The first term in the right-hand side of eq. (2.24) vanishes because of $x \cdot K(x) = 0$. Moreover by the explicit form of $K(x)$, we have

$$|K(x-y) - K(x)| < \text{const}\, \frac{\varrho}{|x|(|x| - \varrho)} \qquad \text{if } |y| < \varrho < |x|\,. \tag{2.25}$$

Hence

$$\begin{aligned} &\left| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} dy\, \omega_\varepsilon(y, t)[K(x-y) - K(x - B_\varepsilon(t))] \right| \leqslant \\ &\qquad \text{const}\, \frac{a_k}{R(R - a_k)} \int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} \omega_\varepsilon(y, t)\, dy\,. \end{aligned} \tag{2.26}$$

The last integral describes the vorticity mass contained in the annulus

$[\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})]$. It can be bounded by I_ε. It is obvious that

$$I_\varepsilon \geqslant r^2 m_t(r)\,, \tag{2.27}$$

where

$$m_t(r) \equiv 1 - \int_{\Sigma(B_\varepsilon(t)\,|\,r)} \omega_\varepsilon(y,\,t)\,dy \tag{2.28}$$

is the vorticity mass outside $\Sigma(B_\varepsilon(t)\,|\,r)$. Equation (2.27) and eq. (2.18) imply

$$m_t(r) \leqslant \text{const}\,\frac{\varepsilon^2}{r^2}\,. \tag{2.29}$$

Hence

$$\int_{\Sigma(B_\varepsilon(t)\,|\,a_k) - \Sigma(B_\varepsilon(t)\,|\,a_{k-1})} \omega_\varepsilon(y,\,t)\,dy \leqslant \text{const}\,\frac{\varepsilon^2}{a_{k-1^2}} \qquad k > 1\,. \tag{2.30}$$

We use this estimate in eq. (2.26), sum on k and obtain that the radial velocity field produced by the fluid particles far from the boundary is bounded by

$$\text{const}\,\frac{\varepsilon}{R^2}\,. \tag{2.31}$$

Now we prove that the vorticity mass near the boundary of the support is very small and so it can produce only a very weak velocity field.

To control the vorticity flux we introduce, for $R > 0$, the following nonnegative function $W_R \in C^\infty(\mathbb{R}^2)$, $r \to W_R(r)$ depending only on $|r|$, defined as:

$$W_R(r) = \begin{cases} 1\,, & \text{if } |r| < R\,, \\ 0\,, & \text{if } |r| > 2R\,, \end{cases} \tag{2.32}$$

such that, for some $C_1 > 0$:

$$|\nabla W_R(r)| < \frac{C_1}{R}\,, \tag{2.33}$$

$$|\nabla W_R(r) - \nabla W_R(r')| < \frac{C_1}{R^2}\,|r - r'|\,. \tag{2.34}$$

Define the quantity:

$$\mu_t(R) = 1 - \int dx\, W_R(x - B_\varepsilon(t))\, \omega_\varepsilon(x, t). \tag{2.35}$$

Notice that, if $\operatorname{supp} \omega_\varepsilon(x, t) \subset \Sigma(B_\varepsilon(t) \,|\, R)$ then $\mu_t(R) = 0$. Thus we choose $\mu_t(R)$ as a measure of the localization of $\omega_\varepsilon(x, t)$ around B_ε. Then we evaluate the time derivative by using eq. (2.3):

$$\frac{d\mu_t(R)}{dt} = \tag{2.36}$$

$$- \int dx\, \nabla W_R(x - B_\varepsilon(t)) \cdot \left\{ u(x, t) + F(x, t) - \frac{d}{dt} B_\varepsilon(t) \right\} \omega_\varepsilon(x, t) =$$

$$- \int dx\, \omega_\varepsilon(x, t)\, \nabla W_R(x - B_\varepsilon(t)) \int dy\, K(x - y)\, \omega_\varepsilon(y, t) -$$

$$\int dx\, \omega_\varepsilon(x, t)\, \nabla W_R(x - B_\varepsilon(t)) \cdot \int dy\, \omega_\varepsilon(y, t) [F(x, t) - F(y, t)]$$

where we have used eqs. (2.7) and (2.13).

We now estimate the first term in the right hand side of eq. (2.36). By the antisymmetry of K, it can be written as:

$$-\frac{1}{2} \int dx \int dy\, \omega_\varepsilon(x, t)\, \omega_\varepsilon(y, t) \{ \nabla W_R(x - B_\varepsilon(t)) - \nabla W_R(y - B_\varepsilon(t)) \} \cdot K(x - y). \tag{2.37}$$

To estimate this term for $R = \varepsilon 2^{n-1}$, we split the integration domain in the following sets

$$T_h = \{ (x, y) \,|\, x \notin \Sigma(B_\varepsilon(t) \,|\, R)\, y \in [\Sigma(B_\varepsilon(t) \,|\, a_h) - \Sigma(B_\varepsilon(t) \,|\, a_{h-1})] \} \quad \text{if } h < n, \tag{2.38}$$

$$T_h = \{ (x, y) \,|\, x \notin \Sigma(B_\varepsilon(t) \,|\, R)\, y \notin \Sigma(B_\varepsilon(t) \,|\, a_{n-1}) \} \quad \text{if } h = n, \tag{2.39}$$

$$S_h = \{ (x, y) \,|\, y \notin \Sigma(B_\varepsilon(t) \,|\, R)\, x \in [\Sigma(B_\varepsilon(t) \,|\, a_h) - \Sigma(B_\varepsilon(t) \,|\, a_{h-1})] \} \quad \text{if } h < n, \tag{2.40}$$

$$S_h = \{ (x, y) \,|\, y \notin \Sigma(B_\varepsilon(t) \,|\, R)\, x \notin \Sigma(B_\varepsilon(t) \,|\, a_{n-1}) \} \quad \text{if } h = n, \tag{2.41}$$

where n is a positive integer number and a_h is defined as in eq. (2.23).

Notice that the integrand in eq. (2.37) vanishes in the complement of $\bigcup_{h=1}^{n} (T_h \cup S_h)$.

Thanks to the identities $\nabla W_R(x - B_\varepsilon(t)) \cdot K(x - B_\varepsilon(t)) = 0$ and $\nabla W_R \cdot$

$(y - B_\varepsilon(t)) = 0$ if $y \in [\Sigma(B_\varepsilon(t) \,|\, a_h) - \Sigma(B_\varepsilon(t) \,|\, a_{h-1})]$, $h < n$ the contribution to the integral (2.37) due to T_h, $h < n$ is bounded by

$$(2.42) \qquad \left| \int_{\Sigma(B_\varepsilon(t) \,|\, a_h) - \Sigma(B_\varepsilon(t) \,|\, a_{h-1})} dy\, \omega_\varepsilon(x, t)\, \omega_\varepsilon(y, t) \cdot \nabla W_R(x - B_\varepsilon(t)) \cdot \{K(x-y) - K(x - B_\varepsilon(t)\} \right| .$$

We now use eq. (2.33), the fact that $\nabla W_R(x - B_\varepsilon(t)) = 0$ if $|x - B_\varepsilon(t)| > R$, and we obtain the bound:

$$(2.43) \qquad (2.37) \leqslant \text{const}\, \frac{m_t(R)}{R} \left\{ \text{const}\, \frac{\varepsilon}{R^2} + \sum_{h=2}^{n-1} \frac{a_h}{R(R - a_h)} \frac{\varepsilon^2}{a_{h-1}^2} \right\} \leqslant \text{const}\, \frac{\varepsilon}{R^3} m_t(R) .$$

To estimate the contribution due to T_n, we use the obvious inequality $|K(x)| \leqslant \text{const}\, |x|^{-1}$, eq. (2.34) and the bound

$$(2.44) \qquad |\{\nabla W_R(x) - \nabla W_R(y)\} \cdot K(x - y)| \leqslant \frac{\text{const}}{R^2} .$$

We have that this contribution is smaller than $\text{const}\,(\varepsilon^2 / R^4)\, m_t(R)$.

We can handle in the same way the term with S_h.

Finally we study the last term in eq. (2.36).

We consider two cases: either $|y - B_\varepsilon(t)| > R$ or $|y - B_\varepsilon(t)| \leqslant R$. In the first case

$$(2.45) \qquad \left| \int dx\, \omega_\varepsilon(x, t)\, \nabla W_R(x - B_\varepsilon(t)) \int dy\, \omega_\varepsilon(y, t) [F(x, t) - F(y, t)] \right| \leqslant \text{const}\, \|F\|_\infty \frac{m_t(r)\, \varepsilon^2}{R^3} .$$

In the second one, by using the Lipschitz condition (2.1),

$$(2.46) \qquad \left| \int dx\, \omega_\varepsilon(x, t)\, \nabla W_R(x - B_\varepsilon(t)) \int dy\, \omega_\varepsilon(y, t) [F(x, t) - F(y, t)] \right| \leqslant \text{const}\, m_t(R) .$$

In conclusion we have:

$$\left| \frac{d}{dt} \mu_t(R) \right| \leqslant \left[\text{const} \frac{\varepsilon}{R^3} + \text{const} \frac{\varepsilon^2}{R^4} + \text{const} \right] m_t(R). \tag{2.47}$$

We observe now that

$$m_t(R) \leqslant \mu_t\left(\frac{R}{2}\right). \tag{2.48}$$

Putting eq. (2.48) in the integral form of eq. (2.47), we obtain:

$$\mu_t(R) \leqslant \mu_0(R) + A(R) \int_0^t \mu_\tau\left(\frac{R}{2}\right) d\tau \tag{2.49}$$

where

$$A(R) = \left[\text{const} \frac{\varepsilon}{R^3} + \text{const} \frac{\varepsilon^2}{R^4} + \text{const} \right]. \tag{2.50}$$

We start now an iterative procedure

$$\mu_t(R) \leqslant \mu_0(R) + A(R) \int_0^t \mu_\tau\left(\frac{R}{2}\right) d\tau \leqslant \tag{2.51}$$

$$\mu_0(R) + \mu_0\left(\frac{R}{2}\right) A(R) \int_0^t d\tau + A(R)\, A\left(\frac{R}{2}\right) \int_0^t dt_1 \int_0^{t_1} \mu_\tau\left(\frac{R}{4}\right) d\tau$$

and so on.

We start from $R = \text{const}\, \varepsilon^\alpha$, $\alpha < 1/3$, and we iterate eq. (2.49) n times, where n is chosen such that $n \to \infty$ as $\varepsilon \to 0$ and in the same time $A(R2^{-k})$ is bounded for any positive integer $k \leqslant n$ and $\mu_0(R2^{-n}) = 0$. We choose

$$n = \text{Integer part of } \left[-\frac{1-3\alpha}{4} \log_2 \varepsilon \right], \qquad (\varepsilon < 1). \tag{2.52}$$

Then

$$R2^{-n} = \text{const}\, \varepsilon^{(1+\alpha)/4} \tag{2.53}$$

and

$$A(R2^{-k}) \leqslant \text{const} \tag{2.54}$$

for any positive integer $k \leqslant n$. So after the n iterations we have

$$(2.55) \quad m_t(R) \leqslant \frac{(\text{const})^n}{n!} \to 0 \text{ as } \varepsilon \to 0 \text{ faster than any power in } \varepsilon .$$

In conclusion we have proved that the vorticity mass becomes very small near the boundary of the support. It is easy to bound the velocity field produced by it:

$$(2.56) \quad \left| \int_{\Sigma(B_\varepsilon(t)|R) - \Sigma(B_\varepsilon(t)|a_{k^*})} dy\, K(x-y)\, \omega_\varepsilon(y,t) \right| \leqslant \frac{1}{2\pi} \left| \int_{\Sigma(B_\varepsilon(t)|R) - \Sigma(B_\varepsilon(t)|a_{k^*})} dy\, \omega_\varepsilon(x,t)\, |y|^{-1} \right| .$$

The integrand is monotonically unbounded as $y \to x$, and so the maximum of the integral is obtained when we rearrange the vorticity mass as close as possible to the singularity:

$$(2.57) \quad \left| \int_{\Sigma(B_\varepsilon(t)|R) - \Sigma(B_\varepsilon(t)|a_{k^*})} dy\, \omega_\varepsilon(x,t)\, |y|^{-1} \right| \leqslant \text{const}\, \varepsilon^{-\gamma} \left| \int_{\Sigma(O|\eta)} dy\, |y|^{-1} \right|$$

where O denotes the origin and η is such that

$$(2.58) \quad M\varepsilon^{-\gamma}\pi\eta^2 = m_t(a_{k^*}) .$$

By using eq. (2.55) we have that

$$(2.59) \quad \left| \int_{\Sigma(B_\varepsilon(t)|R) - \Sigma(B_\varepsilon(t)|a_{k^*})} dy\, \omega_\varepsilon(x,t)\, |y|^{-1} \right| \to 0$$

as $\varepsilon \to 0$ faster than any power.

We are now able to bound the radial velocity of a particle at a distance R from $B_\varepsilon(t)$. By using eqs. (2.20), (2.31) (2.59) we have:

$$(2.60) \quad \left| \frac{d}{dt} R \right| \leqslant \text{const}\, R + \text{const}\, \frac{\varepsilon}{R^2} +$$

terms smaller than any power in ε when $R > \text{const}\, \varepsilon^\alpha \quad \alpha \leqslant 1/3$.

Hence for $R > \text{const}\, \varepsilon^\alpha$ the last two terms of the right hand side of eq. (2.60) are neglectable and inequality (2.60) by using Gronwall Lemma gives bound (2.9). ■

We return to the proof of Theorem 1.1. It is easy and we only sketch it. Denoting by R_m the minimal distance between point vortices evolving via (1.1), we chooce $\varepsilon \ll R_m$. Initially the vortices are separated and we simulate the influence of other vortices as an external field. Theorem 2.1 states that the vortices remain separated. We observe that actually the other vortices produce a velocity field depending on ε but this dependence is very small. Then, it is easy to prove the convergences stated in Theorem 1.1. ∎

Some further generalization are possible: we can consider more singular initial data, with the only bound on the singularity to require that eq. (2.59) remains valid (this would be a very weak result). Moreover we can consider non-compact initial data or vortex-wave system as in [MaP93].

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