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Hilbert-Poincaré Series of Bigraded Algebras.

Lorenzo Robbiano - Giuseppe Valla (*)

Sunto. – *Lo scopo di questo lavoro è la descrizione di alcune nuove tecniche per calcolare serie di Hilbert-Poincaré (HP-serie) di algebre standard, che possono essere viste come sottoalgebre di algebre bigraduate. In particolare mostriamo come calcolare in modo uniforme le HP-serie delle potenze di un idele omogeneo. Mostriamo anche come calcolare le HP-serie di prodotti di Segre e di alcune algebre di Blow-up, che sono di interesse in Geometria Algebrica. Per alcune classi siamo in grado di descrivere formule esplicite, mentre per altre proviamo che c'è un algoritmo che calcola le HP-serie direttamente senza precalcolare le equazioni generatrici.*

1. – Introduction.

The starting point of this paper is the idea of investigating finitely generated *bigraded* commutative k-algebras which arise, for instance, from certain constructions in Algebraic Geometry. To name a few of them, let us remind the tensor products of graded algebras and the graded ring, the Rees ring and the symmetric algebra associated to a homogeneous ideal in a graded algebra. Some of these algebras have bivariate Hilbert-Poincaré series (HP-series) which can easily be computed, and recently several papers (see [CHTV], [CV], [STV]) pointed out that many other interesting graded algebras sit inside these bigraded algebras as *straight-line subalgebras* (see Definition 2.1). Suppose that B is a straight-line subalgebra of some bigraded algebra S; a natural question is the following: is it possible to *compute* the univariate HP-series of B from the knowledge of the bivariate HP-series of S? This problem is too general to have a universal answer applicable in every case, so we are naturally led to restrict ourselves to some special families.

The main results that we obtain are the following. After the preliminaries of Section 2, we review the notion of *Hadamard product*, and in Section 3 we prove that there is an algorithm which computes the HP-series of the Segre product $S_1 * S_2$ of two standard algebras S_1, S_2, without computing the equa-

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tions of $S_1 * S_2$ (see Theorem 3.7). Then we generalize that result in the following way. We introduce the important notion of *separated series* and show that if we are given a bivariate separated series, then there is an algorithm which computes its univariate *diagonal* (see Theorem 4.8).

Section 5 is devoted to bigraded Rees algebras; first we show (see Theorem 5.3) that knowledge of the bivariate HP-series of the Rees algebra associated to a homogeneous ideal I allows one to construct a simple algorithm which computes the HP-series of the powers of I directly in a uniform way.

Then, as a consequence of the results on separated and almost separated HP-series, we prove that there is an algorithm which computes the HP-series of some Blow-up algebras (see Theorem 5.10) and, for a very important subclass, we can derive an explicit formula (Theorem 5.11).

Finally Section 6 is devoted to the computation of some particular HP-series. For instance we show that also in the non separated case it is sometimes possible to obtain the result by developing some computational tricks. All the computations related to the paper were carried on with CoCoA (see [CNR]).

The main conclusion is that the paper indicates a new approach to the computation of some HP-series. Some solutions are given and many questions are left open. We believe that the ideas presented here represent a first step; hence they should be further investigated to yield a deeper insight into the theoretical as well as computational problems related to graded algebras.

A detailed description and an implementation of the algorithms sketched in the paper are described in [BCNR].

2. – Bigraded algebras and straight-line subalgebras.

In this section we introduce the notion of straight-line subalgebras of a bigraded algebra.

DEFINITION 2.1. – *Let S be a $\mathbb{N}^2$-bigraded k-algebra. Let L be a straight half-line in $\mathbb{N}^2$. Then we denote by $S_L := \bigoplus_{(a,b)\in L} S_{(a,b)}$ and we call it the* straight-line submodule *of S along L. In particular, if c and e are two positive integers, we denote by $\Delta(c, e) := \{(cs, es) \mid s \in \mathbb{N}\}$ and by $S_{\Delta(c,e)}$ the corresponding subalgebra. In the special case where $c = e = 1$ we get the* diagonal subalgebra *of S, which we denote by S_Δ.*

It is clear that every finitely generated bigraded k-algebra S can be written as a quotient $S = k[X_1, \ldots, X_n]/J$ where $\deg(X_i) = (u_i, v_i)$ and J is a bihomogeneous ideal. In particular we have the following straightforward fact

LEMMA 2.2. – Let $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{N}$ *be two sets of weights which are* *linearly independent i.e. such that the matrix* $\begin{pmatrix} u_1 \dots u_n \\ v_1 \dots v_n \end{pmatrix}$ *has rank two. Let* $R := k[X_1, \dots, X_n]$ *with the bigrading defined by* $\deg(X_i) := (u_i, v_i)$ *and let* $H := \left\{ X^A \,\middle|\, \sum_{i=1}^{n} u_i a_i = cs, \ \sum_{i=1}^{n} v_i a_i = es, \text{ for some } s \in \mathbb{N} \right\}$. *Then* $R_{\Delta(c,e)} = k\langle H \rangle$, *the monoid k-algebra generated over k by the monoid H.*

REMARK 2.3. – *It is well-known that in the case described before* $R_{\Delta(c,e)}$ *is a normal Cohen-Macaulay domain with* $\dim(R_{\Delta(c,e)}) = n - 1$.

An important class of bigraded k-algebras is described below

DEFINITION 2.4. – *Let* $R := k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ *with the bigrading which is defined by* $\deg(X_i) := (u_i, 0)$, $\deg(Y_i) := (0, v_i)$. *Let J be a bihomogeneous ideal; then R/J is called a* separated (bigraded) k-algebra. *If moreover all the* u_i*'s and the* v_j*'s are equal to 1, then R/J is said to be a* separated standard (bigraded) k-algebra

Now we introduce the Hilbert-Poincaré series of a bigraded algebra.

DEFINITION 2.5. – *Let* $S := \bigoplus_{(i,j) \in \mathbb{N}^2} S_{(i,j)}$ *be a finitely generated bigraded k-algebra. Then we call* Hilbert-Poincaré series (*shortly* HP-series) *of S, the bivariate series*

$$\mathcal{P}_S(a, b) := \sum_{(i,j) \in \mathbb{N}^2} \dim(S_{(i,j)})\, a^i b^j.$$

DEFINITION 2.6. – *Let* $\mathcal{P}(a, b) := \sum_{i,j} p_{(i,j)} a^i b^j$ *be a bivariate series; then we define the univariate series* $\Delta(c, e)(\mathcal{P})(z) := \sum_{i \in \mathbb{N}} p_{(ic, ie)} z^i$. *We define* $\Delta(\mathcal{P})(z) := \Delta(1, 1)(\mathcal{P})(z) = \sum_i p_{(i,i)} z^i$.

LEMMA 2.7. – *Let* $\mathcal{P}(a, b)$ *be a bivariate series*

1) *If* $\mathcal{P}(a, b) = \mathcal{A}(a, b) + \mathcal{B}(a, b)$ *then* $\Delta(\mathcal{P})(z) = \Delta(\mathcal{A})(z) + \Delta(\mathcal{B})(z)$.
2) *If* $\mathcal{P}(a, b) = \mathcal{A}(ab) \cdot \mathcal{B}(a, b)$ *then* $\Delta(\mathcal{P})(z) = \mathcal{A}(z) \cdot \Delta(\mathcal{B})(z)$.

PROOF. – The easy proof is left to the reader. ∎

LEMMA 2.8. – *Let S be a bigraded k-algebra. Then* $\mathcal{P}_{S_{\Delta(c,e)}} = \Delta(c, e)(\mathcal{P}_S)$. *In particular* $\mathcal{P}_{S_\Delta} = \Delta \mathcal{P}_S$.

PROOF. – It is an easy consequence of the definitions. ∎

3. – Segre products and Hadamard products.

A special subclass of separated k-algebras comes from the tensor products of graded k-algebras. More precisely we have

DEFINITION 3.1. – *Let S_1 and S_2 be two $\mathbb{N}$-graded k-algebras and $S := S_1 \otimes S_2$. It is naturally bigraded by $S_{ij} := (S_1)_i \otimes (S_2)_j$, from which we get $S_{\Delta(c,\, e)} = \bigoplus_{s \in \mathbb{N}} (S_1)_{cs} \bigotimes_k (S_2)_{es}$. This is called the* Segre product *of S_1 and S_2* of order (c, e). *If $c = e = 1$, then the diagonal subalgebra S_Δ is $S_1 * S_2$, the ordinary Segre product of S_1 and S_2.*

To see that S is a separated k-algebra, we argue as follows.
Let $S_1 := k[X_1, \ldots, X_n]/I$, $S_2 := k[Y_1, \ldots, Y_m]/J$, where $\deg(X_i) = u_i$ and $\deg(Y_j) = v_j$. It follows that $S \cong k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(I, J)$, which is bigraded by $\deg(X_i) = (u_i, 0)$, $\deg(Y_j) = (0, v_j)$. This yields the conclusion.

We show an interesting example of this class.

EXAMPLE 3.2. – *Let $S_1 := k[X]$ with* $\deg(X) = m$, $S_2 := k[Y_0, Y_1]$ *with* $\deg(Y_0) = \deg(Y_1) := 1$ *and let $S := S_1 \bigotimes_k S_2$. It is easy to see that its diagonal, i.e. the Segre product $S_1 * S_2$, is generated by $XY_0^m, XY_0^{m-1}Y_1, \ldots, XY_1^m$; therefore $S_1 * S_2$ is isomorphic to the projective coordinate ring of the rational normal curve of $\mathbb{P}^m$.*

We want to show how to compute the HP Series of the Segre product of two standard k-algebras.

DEFINITION 3.3. – *Let $\mathcal{A}(z) := \sum_i p_i z^i$ and $\mathcal{B}(z) := \sum_i q_i z^i$ be two power series in $\mathbb{Z}[\![z]\!]$. Then we define the* Hadamard product *of $\mathcal{A}$ and $\mathcal{B}$ and we denote it by* $\mathrm{Had}(\mathcal{A}, \mathcal{B}) := \sum_i (p_i q_i)\, z^i$.

LEMMA 3.4. – *Let $\mathcal{A}(a) \in \mathbb{Z}[\![a]\!]$ and $\mathcal{B}(b) \in \mathbb{Z}[\![b]\!]$ be two power series. Then the product $\mathcal{A}(a)\, \mathcal{B}(b)$ is a bivariate series such that*

1) $\Delta(\mathcal{A}(a)\, \mathcal{B}(b)) = \mathrm{Had}(\mathcal{A}, \mathcal{B})$.

In particular, if S_1 and S_2 are two N-graded k-algebras, and $S := S_1 \otimes S_2$ then

2) $\mathcal{P}_S(a, b) = \mathcal{P}_{S_1}(a) \cdot \mathcal{P}_{S_2}(b)$;

3) $\mathcal{P}_{S_1 * S_2} = \mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})$.

PROOF. – The first and second assertion are easy consequences of the definitions. As for the third one, we see that $\mathcal{P}_{S_1 * S_2} = \mathcal{P}_{S_\Delta} = \Delta(\mathcal{P}_S) = \Delta(\mathcal{P}_{S_1}(a)\, \mathcal{P}_{S_2}(b)) = \mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})$. ■

This Lemma implies that our task is completed if we are able to compute the Hadamard product of the Hilbert-Poincaré series of the two algebras. We recall:

DEFINITION 3.5. – *Let* $\mathcal{A}(z)$ *be the HP-series of a standard k-algebra S. Then we denote by* $\mathrm{ri}(\mathcal{A})$ (*or* $\mathrm{ri}(S)$) *the* regularity index *of* $\mathcal{A}$ (*or of* S), *i.e. the first integer r such that for every* $s \geq r$ *the Hilbert function of S takes the same values as a polynomial, called the Hilbert polynomial of S. It is worth mentioning that* $\mathrm{ri}(S) = a(S) + 1$, *where* $a(S)$ *is the a-invariant of S as defined in* [BH], *Definition* 4.3.6.

PROPOSITION 3.6. – *Let* $\mathcal{A}(z) := P(z)/(1-z)^a$ *and* $\mathcal{B}(z) := Q(z)/(1-z)^b$, *where* $s := \deg(P)$, $t := \deg(Q)$, $P(1) \neq 0$, $Q(1) \neq = 0$, *and assume that* $\mathcal{A}(z)$ *and* $\mathcal{B}(z)$ *are the HP-series of standard k-algebras. Then*

1) $\mathrm{ri}(\mathcal{A}) = s - a + 1$ *and* $\mathrm{ri}(\mathcal{B}) = t - b + 1$;
2) $\mathrm{ri}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) \leq \mathrm{Max}(\mathrm{ri}(\mathcal{A}), \mathrm{ri}(\mathcal{B}))$;
3) $\mathrm{Had}(\mathcal{A}, \mathcal{B}) = R(z)/(1-z)^{a+b-1}$ *with* $R(1) \neq 0$;
4) $\deg(R) \leq \mathrm{Max}(\mathrm{ri}(\mathcal{A}), \mathrm{ri}(\mathcal{B})) + (a+b-1) - 1$.

PROOF. – 1) is well-known (see for instance [BH], Theorem 4.3.5). It is clear that the Hilbert polynomial of $\mathrm{Had}(\mathcal{A}, \mathcal{B})$ is the product of the Hilbert polynomial of A and the Hilbert polynomial of B. This proves 2). The assumption on $\mathcal{A}(z)$ and $\mathcal{B}(z)$ is that there exist two standard k-algebras A and B, such that $\mathcal{A} = \mathcal{P}_A$ and $\mathcal{B} = \mathcal{P}_B$ and such that $\dim(A) = a$ and $\dim(B) = b$. We deduce from Lemma 3.4 that $\mathrm{Had}(\mathcal{A}, \mathcal{B}) = \mathcal{P}_{A * B}$, therefore to prove 3) it suffices to show that $\dim(A * B) = a + b - 1$. Let $P(A)$ be the Hilbert polynomial of A and $P(B)$ be the Hilbert polynomial of B. Then it is well-known that $\deg P(A) = a - 1$, $\deg P(B) = b - 1$ and we have already observed that $P(A * B) = P(A) \cdot P(B)$, hence $\deg P(A * B) = a + b - 2$ and $\dim(A * B) = a + b - 1$. To conclude, we observe that 4) is a consequence of 1), 2) and 3). ■

THEOREM 3.7. – *Let* S_1 *and* S_2 *be two standard k-algebras and assume that we know their HP-series,* $\mathcal{P}_{S_1}$ *and* $\mathcal{P}_{S_2}$. *Then there is an algorithm which computes* $\mathcal{P}_{S_1 * S_2}$ *without computing the equations of* $S_1 * S_2$.

PROOF. – By definition it is clear that $S_1 * S_2$ is a standard k-algebra. Moreover we know from Lemma 3.4 that $\mathcal{P}_{S_1 * S_2} = \mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})$. Then we use Proposition 3.6 to get the dimension and an upper bound for the regularity of $\mathcal{P}_{S_1 * S_2}$. We may say that $\mathcal{P}_{S_1 * S_2}$ has the shape $R(z)/(1-z)^d$ where $\deg(R(z)) = \delta$.

If $R(z) := \sum_{i=1}^{\delta} r_i z^i$, then we need to find the coefficients r_i's. We know $\mathcal{P}_{S_1}$ and $\mathcal{P}_{S_2}$ hence we may compute the first $\delta + 1$ values of $\mathcal{P}_{S_1 * S_2}$. Then it suffices to take the d^{th} difference of these first $\delta + 1$ values and we get the required r_i's. ■

We show how the computation goes with an explicit example.

EXAMPLE 3.8. – *Let* $S_1 := k[U, V, W]/(UV, UW)$, $S_2 := k[A, B, C, D]/(D^5)$. *We want to compute* $\mathcal{P}_{S_1 * S_2}$ *i.e.* $\mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_1})$.

We may compute a presentation of the algebra $S_1 * S_2$ in the following way.

Let $R := k[X_1 \dots X_{12}, U, V, W, A, B, C, D]$ and let $I := \mathrm{Ideal}(X_1 - UA, X_2 - UB, X_3 - UC, X_4 - UD, X_5 - VA, X_6 - VB, X_7 - VC, X_8 - VD, X_9 - WA, X_{10} - WB, X_{11} - WC, X_{12} - WD, UV, UW, D^5)$. Then let J be the ideal obtained by eliminating U, V, W, A, B, C, D from the ideal I. The ring $S_1 * S_2$ turns out to be isomorphic to $k[X_1 \dots X_{12}]/J$ and its Hilbert Series is

$$\mathcal{P}_{S_1 * S_2}(t) = \frac{(1 + 8t - 2t^2 + 8t^3 + 3t^4 - 3t^5)}{(1-t)^4} .$$

This is a non trivial computation. Let us see how to proceed in an efficient way following Proposition 3.6. We have

$$\mathcal{P}_{S_1}(t) = \frac{(1 + t - t^2)}{(1-t)^2} \quad \text{and} \quad \mathcal{P}_{S_2}(t) = \frac{(1 + t + t^2 + t^3 + t^4)}{(1-t)^3} .$$

Consequently $\mathrm{ri}(\mathcal{P}_{S_1}) = 2 - 2 + 1 = 1$ and $\mathrm{ri}(\mathcal{P}_{S_2}) = 4 - 3 + 1 = 2$. Therefore $\mathrm{ri}(\mathcal{P}_{S_1 * S_2}) \leqslant 2$, by Proposition 3.6. If we represent the Hilbert Series of $S_1 * S_2$ as $\mathcal{P}_{S_1 * S_2}(t) = R(t)/(1-t)^4$, the degree d of $R(t)$ has to satisfy $\mathrm{ri}(S_1 * S_2) = d - 4 + 1$, hence $d \leqslant 2 + 3 = 5$.

We consider the values of the Hilbert Series of S_1 and S_2 up to degree 5. Then we get the values of the Hilbert Series of $S_1 * S_2$ up to degree 5.

Degrees	0	1	2	3	4	5
$H_{S_1}(n)$	1	3	4	5	6	7
$H_{S_2}(n)$	1	4	10	20	35	55
$H_{S_1 * S_2}(n)$	1	12	40	100	210	385

It is sufficient to compute the *fourth difference* of the first 6 values. We get

Degrees	0	1	2	3	4	5
First	1	11	28	60	110	175
Second	1	10	17	32	50	65
Third	1	9	7	15	18	15
Fourth	1	8	-2	8	3	-3

In conclusion we get

$$\mathcal{P}_{S_1 * S_2}(t) = \frac{(1+8t-2t^2+8t^3+3t^4-3t^5)}{(1-t)^4}$$

which is the correct result. ■

REMARK 3.9. – *It may happen that* $\mathrm{ri}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) < \mathrm{Max}(\mathrm{ri}(\mathcal{A}), \mathrm{ri}(\mathcal{B}))$ *as the following examples show*

EXAMPLE 3.10. – *If* $\mathcal{A}(z) := z/(1-z)^2$, $\mathcal{B}(z) := 1/(1-z)^3$ *then* $\mathrm{ri}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) \leqslant \mathrm{Max}(0, -2)$. *For* $-2 \leqslant n < 0$ *the values of the function associated to* $\mathcal{A}$ *are* 0, *hence* $\mathrm{ri}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) = -2$.

EXAMPLE 3.11. – *Let*

$$S_1 := k[V, W]/(W^3), \qquad \text{and} \qquad B := k[A, B, C, D]/(A^2, AB, AC, BC).$$

Then $\mathcal{P}_{S_1}(z) = (1+z+z^2)/(1-z)$ *and* $\mathcal{P}_{S_2}(z) = (1+2z-z^2)/(1-z)$. *Therefore* $\mathrm{ri}(\mathcal{P}_{S_1}) = 2-1+1 = 2$ *and* $\mathrm{ri}(\mathcal{P}_{S_2}) = 2-1+1 = 2$. *By Proposition* 3.6 *we have* $\mathrm{ri}(\mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})) \leqslant 2$. *Now the first values of the Hilbert Series of* S_1 *are* 1, 2, 3, 3, … *and the first values of the Hilbert Series of* S_2 *are* 1, 3, 2, 2, … *consequently the first values of the Hilbert Series of* $\mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})$ *are* 1, 6, 6, 6, …, *hence* $\mathrm{ri}(\mathrm{Had}(\mathcal{P}_{S_1}, \mathcal{P}_{S_2})) = 1$. *The Hilbert Series of the Segre product* $S_1 * S_2$ is $(1+5z)/(1-z)$.

4. – Separated and almost separated series.

The algorithm sketched in Theorem 3.7 shows how to compute the diagonal of the Hilbert-Poincaré series of a product. In this section we extend such result to the *separated and almost separated series.*

DEFINITION 4.1. – *Let* $\mathcal{A}(z) := \sum a_n z^n$ *and* $\mathcal{B}(z) := \sum b_n z^n$ *and let* r *be an integer. Then we define* $\mathcal{A}(-r)(z) := \sum a_{n-r} z^n$.

We also define $\mathrm{Diff}(\mathcal{A}) := \sum (a_n - a_{n-1}) z^n$

LEMMA 4.2. – *Let r, s be two integers; then*

1) $\mathcal{A}(-r)(z) = z^r \mathcal{A}(z)$,

2) $\mathrm{Had}(\mathcal{A}(-r), \mathcal{B}(-s)) = z^r \cdot \mathrm{Had}(\mathcal{A}, \mathcal{B}(r-s)) = z^s \cdot \mathrm{Had}(\mathcal{A}(s-r), \mathcal{B})$,

3) $\mathrm{Had}(\mathcal{A}, \mathcal{B}) = \mathrm{Had}(\mathcal{A}, \mathcal{B}(-1)) + \mathrm{Had}(\mathcal{A}, \mathrm{Diff}(\mathcal{B}))$,

4) $\mathrm{Diff}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) = \mathrm{Had}(\mathrm{Diff}(\mathcal{A}), \mathcal{B}) + \mathrm{Had}(\mathcal{A}(-1), \mathrm{Diff}(\mathcal{B}))$,

5) $\mathrm{Diff}(\mathrm{Had}(\mathcal{A}, \mathcal{B})) =$
$$\mathrm{Had}(\mathrm{Diff}(\mathcal{A}), \mathcal{B}) + \mathrm{Had}(\mathcal{A}, \mathrm{Diff}(\mathcal{B})) - \mathrm{Had}(\mathrm{Diff}(\mathcal{A}), \mathrm{Diff}(\mathcal{B})).$$

PROOF. – It is an easy exercise. For instance 5) follows from the identity

$$a_n b_n - a_{n-1} b_{n-1} = (a_n - a_{n-1}) b_n + a_n (b_n - b_{n-1}) - (a_n - a_{n-1})(b_n - b_{n-1}). \qquad \blacksquare$$

DEFINITION 4.3. – *Let* $R := k[X_1, \ldots, X_n]$ *endowed with the natural grading. Then we define* $\mathcal{P}_n := \mathcal{P}_R = 1/(1-z)^n$. *In particular we put* $\mathcal{P} := \mathcal{P}_1 = 1/(1-z)$.

Now we are ready to prove the following formula for the Hilbert-Poincarè series of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{(n+1)(m+1)-1}$. The formula is known, but we want to include it in the paper for the sake of completeness and for later use.

PROPOSITION 4.4. – *It holds*

$$\mathrm{Had}(\mathcal{P}_{n+1}, \mathcal{P}_{m+1}) = \frac{\sum_{i=0}^{\infty} \binom{n}{i}\binom{m}{i} z^i}{(1-z)^{n+m+1}}\ .$$

PROOF. – The formula is clearly true for $n = 0$, any m and it is symmetric with respect to n and m. We make double induction. We know that both sides of the desired formula represent Laurent series, hence it suffices to show that

$$\mathrm{Diff}(\mathrm{Had}(\mathcal{P}_{n+1}, \mathcal{P}_{m+1})) = \frac{\sum_{i=0}^{\infty} \binom{n}{i}\binom{m}{i} z^i}{(1-z)^{n+m}}\ .$$

By Lemma 4.2 5) we have

$$\mathrm{Diff}(\mathrm{Had}(\mathcal{P}_{n+1}, \mathcal{P}_{m+1})) = \frac{\sum_{i=0}^{\infty}\binom{n-1}{i}\binom{m}{i} z^i}{(1-z)^{n+m}} + \frac{\sum_{i=0}^{\infty}\binom{n}{i}\binom{m-1}{i} z^i}{(1-z)^{n+m}} -$$

$$\frac{\sum_{i=0}^{\infty}\binom{n-1}{i}\binom{m-1}{i} z^i}{(1-z)^{n+m-1}} = \frac{\sum_{i=0}^{\infty}\binom{n-1}{i}\binom{m}{i} z^i}{(1-z)^{n+m}} +$$

$$\frac{\sum_{i=0}^{\infty}\binom{n}{i}\binom{m-1}{i} z^i}{(1-z)^{n+m}} - \frac{\sum_{i=0}^{\infty}\left[\binom{n-1}{i}\binom{m-1}{i} - \binom{n-1}{i-1}\binom{m-1}{i-1}\right] z^i}{(1-z)^{n+m-1}}.$$

We conclude by using the identity

$$\binom{n}{i}\binom{m}{i} = \binom{n-1}{i}\binom{m}{i} + \binom{n}{i}\binom{m-1}{i} - \binom{n-1}{i}\binom{m-1}{i} + \binom{n-1}{i-1}\binom{m-1}{i-1}.$$

which is a direct consequence of the Pascal triangle. ■

LEMMA 4.5. – *Let t be a positive integer; then*

1) $\mathcal{P}_t = \mathcal{P}^t$,

2) $\frac{1}{z} \cdot \mathcal{P} = \mathcal{P} + \frac{1}{z}$,

3) $\frac{1}{z^s} \cdot \mathcal{P} = \mathcal{P} + \sum_{i=1}^{s} \frac{1}{z^i}$,

4) $\frac{1}{z} \cdot \mathcal{P}_t = \sum_{k=1}^{t} \mathcal{P}_k + \frac{1}{z}$,

5) $\frac{1}{z^s} \cdot \mathcal{P}_t \cong \sum_{k=1}^{t} \binom{s+t-k-1}{s-1} \mathcal{P}_k$,

where $\cong$ *means* equal modulo $(1/z \cdot \mathbb{Z}[1/z])$.

PROOF. – 1) is obvious and 2) can be checked immediately. Let us prove 3). Clearly

$$\frac{1}{z^s} \cdot \mathcal{P} = \mathcal{P} + \frac{1}{z^s}(\mathcal{P} - z^s \cdot \mathcal{P}) = \mathcal{P} + \frac{1}{z^s}(1 + z + \dots z^{s-1}),$$

hence we conclude. To prove 4) we make induction on t. We know from 2) that the formula is true for $t=1$. So

$$\frac{1}{z}\cdot\mathcal{P}_t=\mathcal{P}\left(\frac{1}{z}\cdot\mathcal{P}_{t-1}\right)=\mathcal{P}\left(\sum_{k=1}^{t-1}\mathcal{P}_k+\frac{1}{z}\right)\overset{\text{by } 2)}{=}\sum_{k=2}^{t}\mathcal{P}_k+\mathcal{P}+\frac{1}{z}=\sum_{k=1}^{t}\mathcal{P}_k+\frac{1}{z}\,.$$

Let us prove 5). The formula is true for $s=1$, hence we make induction on s. We have

$$\frac{1}{z^s}\cdot\mathcal{P}_t=\frac{1}{z}\left(\frac{1}{z^{s-1}}\cdot\mathcal{P}_t\right)\cong\frac{1}{z}\cdot\sum_{i=1}^{t}\binom{s+t-i-2}{s-2}\mathcal{P}_i\overset{\text{by } d)}{=}$$

$$\sum_{i=1}^{t}\binom{s+t-i-2}{s-2}\cdot\left(\sum_{k=1}^{i}\mathcal{P}_i\right)=\sum_{k=1}^{t}\left(\sum_{i=k}^{t}\binom{s+t-i-2}{s-2}\right)\mathcal{P}_k\,.$$

We make the change $j:=t-i$ and get

$$=\sum_{k=1}^{t}\left(\sum_{j=t-k}^{0}\binom{s+j-2}{s-2}\right)\mathcal{P}_k=\sum_{k=1}^{t}\binom{s+t-k-1}{s-1}\mathcal{P}_k\,.$$

DEFINITION 4.6. – *Let $\mathcal{P}(a,b)$ be a bivariate series. We say that it is* separated standard *if it can be expressed as*

$$\mathcal{P}(a,b)=(P(a,b))/((1-a)^n\cdot(1-b)^t)$$

where $P(a,b)$ is a polynomial.

The connection between Definition 4.6 and Definition 2.4 is explained below

PROPOSITION 4.7. – *Let $R:=k[X_1,\dots,X_n,Y_1,\dots,Y_m]/J$ be a bigraded separated standard k-algebra.* Then *$\mathcal{P}_R(a,b)$ is a separated standard bivariate series.*

PROOF. – The easy proof is left to the reader. ■

Now we are ready to generalize Theorem 3.7.

THEOREM 4.2. – *Let $\mathcal{P}$ be a separated standard bivariate series. Then there is an algorithm which computes $\Delta(\mathcal{P})$.*

PROOF. – We have seen in Proposition 4.4 that

$$1)\qquad \mathrm{Had}(\mathcal{P}_r, \mathcal{P}_s) = \left(\sum_{i=0}^{\infty} \binom{r-1}{i}\binom{s-1}{i} z^i \right) \cdot \mathcal{P}_{r+s-1}\,.$$

Then we have seen in Lemma 4.5) that

$$2)\qquad \frac{1}{z^s} \cdot \mathcal{P}_t \cong \sum_{i=1}^{t} \binom{s+t-i-1}{s-1} \mathcal{P}_i\,.$$

Let $\mathcal{P}(a, b) := P(a, b)/((1-a)^n \cdot (1-b)^t)$. By the additivity of Δ (see Lemma 2.7), we may assume that $P(a, b) = a^m b^{m'}$ and there is no loss of generality in assuming $m \geqslant m'$. Let $s := m - m'$. Then

$$\mathcal{P}(a, b) = (ab)^m / ((1-a)^n \cdot (1-b)^t b^s)\,.$$

Again by Lemma 2.7 we have

$$\Delta(\mathcal{P})(z) = z^m \cdot \Delta\left(\frac{1}{(1-a)^n \cdot (1-b)^t b^s} \right) = z^m \cdot \mathrm{Had}\left(\mathcal{P}_n, \frac{1}{b^s} \cdot \mathcal{P}_t \right)$$

where we extend here the operations Δ and Had to Laurent series in an obvious way. It is clearly true that if $\mathcal{A} \cong \mathcal{B}$ and $\mathcal{C}$ is a non negative series, then $\mathrm{Had}(\mathcal{C}, \mathcal{A}) = \mathrm{Had}(\mathcal{C}, \mathcal{B})$. So we apply Formula 2) and then Formula 1) and we conclude. ■

REMARK 4.9. – *Theorem 4.8 says that there is an algorithm which computes $\Delta(\mathcal{P})$. In fact it would be possible to write a formula. But what we got was too complicated and useless in practice.*

Now we want to generalize Theorem 4.8 and produce another result, which will be useful in the subsequent section. We need the following

LEMMA 4.10. – *Let $\mathcal{P}(a, b)$ and $\mathcal{P}'(a, u)$ be bivariate series such that the identity $\mathcal{P}(a, b) = \mathcal{P}'(a, a^d b)$ holds for some positive integer d. Then $\Delta(d+1, 1)(\mathcal{P}) = \Delta(\mathcal{P}')$.*

PROOF. – Let $\mathcal{P}(a, b) = \sum_{(i,j)} c_{(i,j)} a^i b^j$ and $\mathcal{P}'(a, u) = \sum_{(r,s)} d_{(r,s)} a^r u^s$. By assumption for every (r, s) we have $d_{(r,s)} = c_{(r+ds, s)}$. Hence by Definition 2.6 we get

$$\Delta(d+1, 1)(\mathcal{P}) = \sum_i c_{(i(d+1), i)} z^i = \sum_i d_{(i,i)} z^i = \Delta(\mathcal{P}')\,. \qquad ■$$

DEFINITION 4.11. – *Let $\mathcal{P}(a, b)$ be a bivariate series. We say that* *it is* almost separated (of type d) *if there exists a separated series* $\mathcal{P}'(a, u)$ *such that* $\mathcal{P}(a, b) = \mathcal{P}'(a, a^d b)$.

COROLLARY 4.12. – *Let* $\mathcal{P}(a, b)$ *be a bivariate almost separated series of type d. Then there is an algorithm which computes* $\Delta(d+1, 1)(\mathcal{P})$.

PROOF. – By the above Lemma $\Delta(d+1, 1)(\mathcal{P}) = \Delta(\mathcal{P}')$ for some separated series $\mathcal{P}'$. The conclusion follows from Theorem 4.8. ■

5. – Bigraded Rees algebras.

Let $R = k[X_1, \dots, X_n]$ be a polynomial ring over a field k graded in the standard way and I a homogeneous ideal in R. Consider the Rees algebra of I, namely the subalgebra of the polynomial ring $R[T]$ defined by $\mathcal{R}(I) := R[IT] = \bigoplus_{j \geq 0} I^j T^j$. The bigrading on $R[T]$ defined by $R[T]_{(i,j)} := R_i T^j$ induces on $\mathcal{R}(I)$ the bigrading $\mathcal{R}(I)_{(i,j)} = (I^j)_i T^j$. It is then possible to consider its straight-line subalgebras.

PROPOSITION 5.1. – *Let* $R = k[X_1, \dots, X_n]$, $I := (F_1, \dots, F_r)$ *a homogeneous ideal in R, where* $d_i := \deg(F_i)$ *and let* $\mathcal{R}(I) := R[IT]$ *be the associated bigraded Rees algebra. Then*

1) $\mathcal{R}(I) = k[X_1, \dots, X_n, Y_1, \dots, Y_r]/\mathfrak{I}$, *where* $\deg(X_i) = (1, 0)$, $\deg(Y_i) = (d_i, 1)$ *and* $\mathfrak{I}$ *is a bihomogeneous ideal.*

2) $\mathcal{P}_{\mathcal{R}(I)}(a, b) := \dfrac{P(a, b)}{(1-a)^n \prod_{i=1}^{r} (1 - a^{d_i} b)}$, *where* $P(a, b)$ *is a polynomial.*

PROOF. – It follows from the definition. ■

First we study a special straight-line submodule of $\mathcal{R}(I)$; namely, if we fix an integer $t \in \mathbb{N}$ and the horizontal half-line $L := \{(i, t) \mid i \in \mathbb{N}\}$, then $\mathcal{R}(I)_L = \bigoplus_{i \geq 0} (I^t)_i T^t = I^t T^t$. Therefore

LEMMA 5.2. – *Let* $R = k[X_1, \dots, X_n]$, $I := (F_1, \dots, F_r)$ *a homogeneous ideal in R, where* $d_i := \deg(F_i)$ *and let* $\mathcal{R}(I) := R[IT]$ *be the associated bigraded Rees algebra. Then*

1) $\mathcal{P}_{R/I^t}(a) = \dfrac{1}{(1-a)^n} - \mathcal{P}_{I^t}(a)$,

2) $\mathcal{P}_{I^t}(a)$ *is the coefficient of* b^t *in* $\mathcal{P}_{\mathcal{R}(I)}(a, b)$.

PROOF. – Assertion 1) is clear, while assertion 2) comes from the fact that if $L := \{(i, t) \mid i \in \mathbb{N}\}$, then $\mathcal{R}(I)_L = \bigoplus_{i \geq 0} (I^t)_i T^t = I^t T^t$. ■

The next result is quite interesting from the computational point of view.

THEOREM 5.3. – *Let I be an homogeneous ideal of the polynomial ring $R = k[X_1, \ldots, X_n]$ and t a positive integer. Then there is an algorithm which computes $\mathcal{P}_{I^t}$ uniformly.*

PROOF. – We know from Proposition 5.1, 2) that

$$\mathcal{P}_{\mathcal{R}(I)}(a, b) := \frac{P(a, b)}{(1-a)^n \prod_{i=1}^{r} (1 - a^{d_i} b)}$$

where $d_1, \ldots, d_r$ are the degrees of a system of generators of I. Therefore $\mathcal{P}_{I^t}$ can be represented as $Q(a)/(1-a)^n$, where $Q(a)$ is the coefficient of b^t in the bivariate series $P(a, b) \cdot \prod_{i=1}^{r} (1 + a^{d_i} b + a^{2d_i} b^2 + \ldots)$, hence in the polynomial $P(a, b) \cdot \prod_{i=1}^{r} (1 + a^{d_i} b + a^{2d_i} b^2 + \ldots + a^{td_i} b^t)$. The conclusion follows. ■

COROLLARY 5.4. – *Let I be an homogeneous ideal of the polynomial ring $R = k[X_1, \ldots, X_n]$ and t a positive integer. If D is the maximum of the degrees of a system of generators of I, then there exists a constant δ such that*

$$\mathrm{ri}(R/I^t) \leqslant Dt + \delta \, .$$

PROOF. – We use the notation of the above theorem. To prove the claim it suffices to note that the degree of $Q(a)$ is bounded above by $Dt + \delta$, where δ is the total degree of $P(a, b)$. ■

The bound given in the above Corollary can also be obtained from recent results on upper bounds for the Castelnuovo-Mumford regularity of a homogeneous ideals (see [CHT] and [K]).

We shall see at the beginning of the next section that under special circumstances we can compute an explicit formula for $\mathcal{P}_{I^t}$ hence for $\mathcal{P}_{R/I^t}$.

Now we are going to discuss another important class of straight-line subalgebras of Rees algebras.

DEFINITION 5.5. – *Let $R := k[X_1, \ldots, X_n]$ be a polynomial ring over a field k graded in the standard way and I a homogeneous ideal in R. Let $c \in \mathbb{N}$. We define*

$$\mathcal{B}(c, I) := \mathcal{R}(I)_{\Delta(c,1)} = \bigoplus_{s \geqslant 0} \mathcal{R}(I)_{(cs, s)} = \bigoplus_{s \geqslant 0} (I^s)_{cs} T^s \, .$$

The strong interest in studying such algebras is explained by the following facts

LEMMA 5.6. – *With the notation as above, assume that I is generated in degree $\leqslant c$. Then $\mathcal{B}(c, I) = k[I_c]$, the k-algebra generated by I_c and graded by $(k[I_c])_s := (I_c)^s$. In this way $\mathcal{B}(c, I)$ is a standard k-algebra.*

PROOF. – The degree s part of $k[I_c]$ is given by $(I_c)^s$; on the other hand I is generated in degree $\leqslant c$, hence $(I_c)^s$ is the degree cs part of I^s. ■

PROPOSITION 5.7. – *Let $R := k[X_1, \dots, X_n]$ be a polynomial ring over a field k graded in the standard way and I a homogeneous ideal in R generated in degree d. If $c \gg d$, then $\mathcal{B}(c, I) = k[I_c]$ is the coordinate ring of the Blow-up of $\mathbb{P}_k^{n-1}$ along the projective scheme defined by I.*

PROOF. – This is a classical fact, which is now considered as folklore; however, we have not been able to find a reference in the literature. ■

A very important class to study arises when I is a *complete intersection* i.e. I is generated by a *homogeneous regular sequence*. Then we have

PROPOSITION 5.8. – *Let $F_1, \dots, F_r$ be a homogeneous regular sequence in R, $d_i := \deg(F_i)$ and $I := (F_1, \dots, F_r)$. Let $Y_1, \dots, Y_r$ be indeterminates and $M := \begin{pmatrix} Y_1 & Y_2 \dots Y_r \\ F_1 & F_2 \dots F_r \end{pmatrix}$. Then*

1) *$\mathcal{R}(I) = k[X_1, \dots, X_n, Y_1, \dots, Y_r]/I_2(M)$, where $I_2(M)$ is the homogeneous ideal generated by the 2×2 minors of M.*
2) *The free resolution of $\mathcal{R}(I)$ is the Eagon-Northcott complex associated to M.*

PROOF. – These are also classical results (see [BH]). ■

PROPOSITION 5.9. – *Let $F_1, \dots, F_r$ be a homogeneous regular sequence in R of elements of the same degree d. Then the bivariate HP-series of $\mathcal{R}(I)$ is*

$$\mathcal{P}_{\mathcal{R}(I)}(a, b) := \frac{1 + \sum_{p=2}^{r} (-1)^{p-1} \left(\binom{r}{p} a^{pd} \sum_{m=1}^{p-1} b^m \right)}{(1-a)^n (1 - a^d b)^r} .$$

PROOF. – We deduce from Proposition 5.8 that the minimal free resolution of $\mathcal{R}(I)$ as a module over $S := k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ is the Eagon-Northcott complex

$$0 \to D_{r-1} \to D_{r-2} \dots \to D_1 \to D_0 = S \to \mathcal{R}(I) \to 0 .$$

In [CHTV], Lemma 4.1. it has been observed that

$$D_j = \bigoplus_{m=1}^{j} \binom{r}{j+1} S(-d(j+1), -m).$$

We deduce

$$\mathcal{P}_{\mathcal{R}(I)}(a, b) = \frac{1 + \sum_{j=1}^{r-1} (-1)^j \mathcal{P}_{D_j}(a, b)}{(1-a)^n (1-a^d b)^r} = \frac{1 + \sum_{j=1}^{r-1} (-1)^j \left(\binom{r}{j+1} a^{d(j+1)} \sum_{m=1}^{j} b^m \right)}{(1-a)^n (1-a^d b)^r}.$$

The conclusion follows. ■

THEOREM 5.10. – *Let $F_1, \ldots, F_r$ be a homogeneous regular sequence in R of elements of the same degree d and $I := (F_1, \ldots, F_r)$, Then there is an algorithm which computes $\mathcal{P}_{\mathcal{B}(d+1, I)}$, without computing the equations of $\mathcal{B}(d+1, I)$.*

PROOF. – By Definition 5.5 wee have to show how to compute $\mathcal{P}_{\mathcal{R}(I)_{\Delta(d+1,1)}}(a, b)$, which is $\Delta(d+1, 1)(\mathcal{P}_{\mathcal{R}(I)}(a, b))$ by Lemma 2.8. Now Proposition 5.9 tells us that $\mathcal{P}_{\mathcal{R}(I)}(a, b)$ is an almost separated series of type d, hence the conclusion follows from Corollary 4.12. ■

THEOREM 5.11. – *Let $R := k[X_1, \ldots, X_n]$, $\{L_{ij}\}$ a set of $d \times (d+1)$ homogeneous linear forms, $i := 1, \ldots, d$; $j := 1, \ldots, d+1$, M the matrix (L_{ij}). Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M and assume that $\operatorname{ht}(I_t(M)) \geq d - t + 2$ for $1 \leq t \leq d$. If $I := I_d(M)$, then*

1) $$\mathcal{P}_{\mathcal{R}(I)}(a, b) = \frac{(1-a^{d+1} b)^d}{(1-a)^n (1-a^d b)^{d+1}},$$

2) $$\mathcal{P}_{\mathcal{B}(d+1, I)} = \frac{\sum_{i=0}^{\infty} \binom{n-1}{i} \binom{d}{i} z^i}{(1-z)^n}.$$

PROOF. – The condition about the height of $I_t(M)$ implies, by a result of Huneke (see [H] Theorem 1.1), that I is of linear type. This implies that

$$\mathcal{R}(I) = k[X_1, \ldots, X_n][T_1, \ldots, T_{d+1}]/(\Phi_1, \ldots, \Phi_d),$$

where

$$\Phi_i := \sum_{k=1}^{d+1} L_{ik} T_k.$$

We have $\dim(\mathcal{R}(I)) = n+1$, hence $\operatorname{codim}(\Phi_1, \ldots, \Phi_d) = n+d+1-(n+1) = d$. This means that $\mathcal{R}(I)$ is a complete intersection of d forms of degree $(d+1, 1)$, which proves 1).

If we consider the bivariate series

$$\mathcal{P}'(a, u) := \frac{(1-au)^d}{(1-a)^n (1-u)^d}$$

it is clear that $\mathcal{P}'(a, u)$ is a separated bivariate series such that $\mathcal{P}_{\mathcal{R}(I)}(a, b) = \mathcal{P}'(a, a^d b)$. Hence, according to Definition 4, $\mathcal{P}_{\mathcal{R}(I)}$ is a bivariate almost separated series of type d. From this we get

$$\mathcal{P}_{\mathcal{B}(d+1, I)} = \mathcal{P}_{\mathcal{R}(I)_{\Delta(d+1, 1)}} = \Delta(d+1, 1)(\mathcal{P}_{\mathcal{R}(I)}) = \Delta(\mathcal{P}') =$$

$$(1-z)^d \Delta\left(\frac{1}{(1-a)^n}\,\frac{1}{(1-u)^{d+1}}\right) = (1-z)^d \operatorname{Had}(\mathcal{P}_n, \mathcal{P}_{d+1}) =$$

$$(1-z)^d \frac{\sum_{i=0}^{\infty}\binom{n-1}{i}\binom{d}{i} z^i}{(1-z)^{n+d}} = \frac{\sum_{i=0}^{\infty}\binom{n-1}{i}\binom{d}{i} z^i}{(1-z)^n}.$$

The conclusion follows. ■

REMARK 5.12. – *We have seen in Proposition* 5.7 *that for c big enough $\mathcal{B}(c, I)$ is the coordinate ring of the Blow-up of $\mathbb{P}_k^{n-1}$ along the projective scheme defined by I. The importance of Theorem* 5.10 *and Theorem* 5.11 *relies on the fact that in these cases, as well as in many other cases which are relevant to Algebraic Geometry, $c := d+1$ is «big enough» for blowing-up.*

6. – Explicit computations.

In this section we carry on some explicit computations. As promised in Section 5 we show the explicit computation of the HP-series of the powers of an ideal.

EXAMPLE 6.1. – *We compute the Hilbert Series of R/I^t where $I := I_d$ as defined in Theorem* 5.11.

We have seen in Theorem 5.11 that

$$\mathcal{P}_{\mathcal{R}(I)}(a, b) = \frac{(1-a^{d+1}b)^d}{(1-a)^n(1-a^d b)^{d+1}} =$$

$$\frac{\left(\sum_{j=0}^{d}(-1)^j\binom{d}{j}a^{(d+1)j}b^j\right)\left(\sum_{k\geqslant 0}\binom{d+k}{k}a^{dk}b^k\right)}{(1-a)^n}$$

and we know from Proposition 5.2, 2) that we have to compute the coefficient of b^t. It follows that

$$\mathcal{P}_{I^t}(a) = \frac{\sum_{j=0}^{d}(-1)^j\binom{d}{j}a^{(d+1)j}\binom{d+t-j}{t-j}a^{d(t-j)}}{(1-a)^n}$$

hence (by Proposition 5.2, 1))

$$\mathcal{P}_{R/I^t}(z) = \frac{1-\sum_{j=0}^{d}(-1)^j\binom{d}{j}\binom{d+t-j}{t-j}z^{dt+j}}{(1-z)^n}.$$

Since R/I^t has dimension $n-2$ the h-vector of R/I^t is the numerator of the above fraction divided by $(1-z)^2$. It follows that the leading coefficient of this polynomial is $(-1)^{d+1}\binom{t}{d}$ while the first non trivial coefficient, namely that of z^{dt} is $dt+1-\binom{d+t}{t}$ which is equal to $-\sum_{j=2}^{t}\binom{t}{j}\binom{d}{j}<0$. This proves that $R/I^t$is *not Cohen-Macaulay for every* $t\geqslant 2$.

Now we consider a particular instance of Theorem 5.10, namely the case $r:=2$.

COROLLARY 6.2. – *Let* $R:=k[X_1, \ldots, X_n]$ *be a polynomial ring over a field* k. *Let* F_1, F_2 *be a homogeneous regular sequence in* R *with* $\deg(F_1)=\deg(F_2):=d$ *and* $I:=(F_1, F_2)$. *Then*

1) $\mathcal{P}_{\mathcal{B}(d+1, I)} = \dfrac{1+nz+\ldots+nz^{d-1}+(n-d)z^d}{(1-z)^n}$,

2) $\mathcal{B}(d+1, I)$ *is not Cohen-Macaulay if* $d>n$.

PROOF. – We have $\mathcal{P}_{\mathcal{R}(I)}(a, u) = \dfrac{1-a^d u}{(1-a)^n(1-u)^2}$. Therefore we apply

Theorem 4.8 and get

$$\mathcal{P}_{\mathcal{B}(d+1,\, I)} = \mathrm{Had}\,(\mathcal{P}_n,\, \mathcal{P}_2) - z^d \cdot \mathrm{Had}\left(\mathcal{P}_n,\ \frac{1}{(1-z^2)\, z^{d-1}}\right) =$$

$$\mathrm{Had}\,(\mathcal{P}_n,\, \mathcal{P}_2) - z^d \cdot \sum_{k=1}^{2} \binom{d-k}{d-2} \mathrm{Had}\,(\mathcal{P}_n,\, \mathcal{P}_k) =$$

$$(1-z^d) \cdot \mathrm{Had}\,(\mathcal{P}_n,\, \mathcal{P}_2) - (d-1) z^d \cdot \mathrm{Had}\,(\mathcal{P}_n,\, \mathcal{P}_1) =$$

$$(1-z^d)\ \frac{1+(n-1)z}{(1-z)^{n+1}} - (d-1)\, z^d \cdot \frac{1}{(1-z)^n} =$$

$$\frac{(1+z+\ldots+z^{d-1})(1+(n-1)\,z) - (d-1)\, z^d}{(1-z)^n} =$$

$$\frac{1+nz+\ldots+nz^{d-1}+(n-d)\, z^d}{(1-z)^n}\ .$$

If $d > n$, the h-vector has a negative component. ■

EXAMPLE 6.3. – *Let $R := k[X_1, X_2, X_3]$ be a polynomial ring over a field k. Let F_1, F_2 be a homogeneous regular sequence in R with* $\deg(F_1) = 2$, $\deg(F_2) = 3$ *and $I := (F_1, F_2)$. Then*

$$\mathcal{P}_{\mathcal{B}(4,\, I)} = \frac{1+6z+3z^2}{(1-z)^3}\ .$$

PROOF. – The Rees algebra $\mathcal{R}(I)$ is isomorphic to $k[X_1, X_2, X_3, Y_1, Y_2]/(Y_1 F_2 - Y_2 F_1)$, with the bigrading given by $\deg(X_i) = (1, 0)$ for $i := 1, \ldots, 3$ and $\deg(Y_1) = (2, 1)$, $\deg(Y_2) = (3, 1)$. Then $\mathcal{P}_{\mathcal{R}(I)}(a, b) = (1-a^5 b)/N(a, b)$, where $N(a, b) := (1-a)^3 (1-a^2 b)(1-a^3 b)$.

In this case we cannot apply the technique explained in Theorem 4.8, since there is no way of transforming $\mathcal{P}_{\mathcal{R}(I)}(a, b)$ into a separated series.

We need to compute $\Delta(4, 1)(\mathcal{P}_{\mathcal{R}(I)}(a, b))$. We get

$$\Delta(4, 1)(\mathcal{P}_{\mathcal{R}(I)}(a, b)) =$$

$$\Delta(4, 1)\left(\frac{1-a^5 b}{N(a, b)}\right) = \Delta(4, 1)\left(\frac{1}{N(a, b)}\right) - z \cdot \Delta(4, 1)\left(\frac{a}{N(a, b)}\right).$$

Now $\dfrac{1}{N(a, b)} = \left(\sum_{i \geqslant 0} \binom{i+2}{2} a^i\right)\left(\sum_{j \geqslant 0} a^{2j} b^j\right)\left(\sum_{k \geqslant 0} a^{3k} b^k\right)$, hence we see that the coefficients of $a^{4p} b^p$ correspond to the solutions of the system $\begin{cases} i+2j+3k=4p \\ j+k=p \end{cases}$ or equivalently $\begin{cases} i+2j+3(p-j)=4p \\ 0 \leqslant j \leqslant p \end{cases}$ which is $\begin{cases} 0 \leqslant j \leqslant p \\ i=j+p \end{cases}$. So we have proved that

$$\Delta(4, 1)\left(\frac{1}{D(a, b)}\right) = \sum_{p \geqslant 0} \left[\binom{p+2}{2} + \ldots + \binom{2p+2}{2}\right] z^p =$$

$$\sum_{p \geqslant 0} \left[\binom{2p+3}{3} - \binom{p+2}{3}\right] z^p .$$

At this point we have the Hilbert polynomial, from which we deduce the dimension and the regularity, which turn out to be 4 and -1 respectively. Now

$$\sum_{p \geqslant 0} \left[\binom{2p+3}{3} - \binom{p+2}{3}\right] z^p = 1 + 9z + 31z^2 \bmod (z^3)$$

so that, by taking the 4^{th} difference we get $\Delta(4, 1)(1/D(a, b)) = (1 + 5z + z^2)/(1-z)^4$.

To compute $\Delta(4, 1)(a/D(a, b))$ we proceed as before and get $\Delta(4, 1)(a/D(a, b)) = (4z + 3z^2)/(1-z)^4$. Then the computation goes as follows

$$\Delta(4, 1)(\mathcal{P}_{\mathcal{R}(I)}(a, b)) = \frac{1 + 5z + z^2}{(1-z)^4} - z\frac{4z + 3z^2}{(1-z)^4} =$$

$$\frac{1 + 5z - 3z^2 - 3z^3}{(1-z)^4} = \frac{1 + 6z + 3z^2}{(1-z)^3}$$

as we wanted to show. ■

REMARK 6.4. – *The case $n = 4$ can be carried over in the same way and we get*

$$\mathcal{P}_{\mathcal{B}(4, I)} = \frac{1 + 10z + 10z^2 + z^3}{(1-z)^4} .$$

Since it can be proved that $\mathcal{B}(4, I)$ is Cohen-Macaulay, the symmetry of the h-vector tells us that it is Gorenstein.

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