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The Ornstein-Uhlenbeck Generator Perturbed by the Gradient of a Potential.

GIUSEPPE DA PRATO [1]

Sunto. – *Si considera, in uno spazio di Hilbert H l'operatore lineare $\mathcal{N}_0 \varphi = 1/2 \operatorname{Tr}[D^2 \varphi] + \langle x, AD\varphi \rangle - \langle DU(x), D\varphi \rangle$, dove A è un operatore negative autoaggiiunto e U è un potenziale che soddisfa a opportune condizioni di integrabilità. Si dimostra con un metodo analitico che $\mathcal{N}_0$ è essenzialmente autoaggiunto in uno spazio $L^2(H, \nu)$ e si caratterizza il dominio della sua chiusura $\mathcal{N}$ come sottospazio di $W^{2,2}(H, \nu)$. Si studia inoltre la «spectral gap property» del semigruppo generato da $\mathcal{N}$.*

1. – Introduction and setting of the problem.

Let H be a separable Hilbert space, $A: D(A) \subset H \to H$ a self-adjoint negative operator such that A^{-1} is of trace class. We denote by μ the Gaussian measure of mean 0 and covariance operator $Q = -(1/2) A^{-1}$. We are concerned with the following linear operator on $L^2(H, \mu)$:

$$\mathcal{N}_0 \varphi(x) = \frac{1}{2} \operatorname{Tr}[D^2 \varphi] + \langle x, AD\varphi \rangle - \langle DU(x), D\varphi \rangle, \qquad \varphi \in \mathcal{E}_A(H), \tag{1.1}$$

where U is a nonlinear real function in H, and $\mathcal{E}_A(H)$ is the linear subspace of $L^2(H, \mu)$ spanned by all exponential functions

$$\psi_h(x) = e^{\langle h, x \rangle}, \qquad x \in H,$$

where $h \in D(A)$. Notice that $\mathcal{E}_A(H)$ is dense in $L^2(H, \mu)$.

The goal of this paper is to show that, under suitable assumptions, $\mathcal{N}_0$ is essentially self-adjoint on the space $L^2(H, \nu)$, where ν is the probability measure

$$\nu(dx) = c e^{-2U(x)} \mu(dx), \qquad c = \left[\int_H e^{-2U(x)} \mu(dx) \right]^{-1}.$$

This problem has a long story, see the recent paper [1] and the references therein for an approach based on the theory of Dirichlet forms. Another ap-

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proach consists in solving the differential stochastic equation

$$dX = (AX - DU(X))\,dt + dW(t), \qquad X(0) = x\,,$$

where W is a cylindrical Wiener process on H, see e.g. [7], and then by identifying the closure $\mathcal{N}$ of $\mathcal{N}_0$ with the infinitesimal generator of the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \qquad \varphi \in L^2(H, \nu)\,.$$

In this paper we follow a purely analytic approach, different of that based on Dirichlet forms. The advantage is that we require weaker assumptions on U and that we are able to characterize the domain of $\mathcal{N}$ as a subspace of the Sobolev space $W^{2,2}(H, \nu)$ instead of $W^{1,2}(H, \nu)$, as in the case of Dirichlet forms. Moreover we believe that similar ideas could be applied to more general situations when $\mathcal{N}_0$ is not symmetric.

Let us briefly explain our method. We first consider the linear operator

$$\mathcal{A}_0\varphi(x) = \frac{1}{2}\mathrm{Tr}\,[D^2\varphi] + \langle x, AD\varphi\rangle, \qquad \varphi \in \mathcal{E}_A(H)\,. \tag{1.2}$$

It well known see e.g. [7], that $\mathcal{A}_0$ is essentially self-adjoint. Moreover the domain of the closure $\mathcal{A}$ of $\mathcal{A}_0$ is given by, see [5] and § 2 below,

$$D(\mathcal{A}) = \{\varphi \in W^{2,2}(H, \mu) \colon |(-A)^{1/2}D\varphi| \in L^2(H;\mu)\}\,. \tag{1.3}$$

We first study the operator $\mathcal{N}_0$ under the assumption that U is of class C^2 and DU and D^2U are bounded, see § 3. In this case we prove that $\mathcal{N}_0$ is symmetric on $L^2(H, \nu)$ and the following identity holds for any $\varphi \in \mathcal{E}_A(H)$,

$$\frac{1}{2}\int_H \mathrm{Tr}\,[(D^2\varphi)^2]\,d\nu + \int_H |(-A)^{1/2}D\varphi|^2\,d\nu + \int_H \langle D^2U D\varphi, D\varphi\rangle\,d\nu = 2\int_H (\mathcal{N}_0\varphi)^2\,d\nu\,. \tag{1.4}$$

Finally, denoting by $\mathcal{N}$ the closure of $\mathcal{N}_0$, we show, by a simple perturbation argument that for λ_0 sufficiently large we have

$$(\lambda_0 - \mathcal{N})(D(\mathcal{N})) \supset L^2(H, \mu)\,.$$

Since $L^2(H, \mu)$ is dense on $L^2(H, \nu)$, it follows that $\mathcal{N}$ is m-dissipative see e.g. [4, Corollaire II.9.3], and so it is self-adjoint.

In § 4 we consider a more general case when

$$\int_H |DU(x)|^p \nu(dx) < +\infty . \tag{1.5}$$

This condition is similar to assumptions (5) and (6) in [1], that however are required to hold for all p. Under this assumption we can again show that $\mathcal{N}_0$ is symmetric, that an estimate similar to identity (1.4) holds and that for all $\lambda > 0$, $(\lambda - \mathcal{N})(D(\mathcal{N}))$ contains the closure on $L^2(H, \nu)$ of $W^{1, 2p/(p-2)}(H, \mu)$, that is dense in $L^2(H, \nu)$. This implies, by the previous argument, that $\mathcal{N}$ is self-adjoint on $L^2(H, \nu)$. In order to prove the above inclusion we need some a-priori estimates on $W^{1, 2p/(p-2)}(H, \mu)$, that are proved in Appendix A.

Finally § 5 is devoted to ergodicity and spectral gap for the semigroup $e^{t\mathcal{N}}$. Here we generalize to the situation when (1.5) holds, some previous results due to [2], [1], and [7].

2. – Notation and preliminary results.

We are given a separable Hilbert space H, (norm $|\cdot|$, inner product $\langle \cdot, \cdot \rangle$), and a linear operator $A: D(A) \subset H \to H$. We assume

HYPOTHESIS 2.1. – (i) A is self-adjoint and there exists $\omega > 0$ such that

$$\langle Ax, x \rangle \le -\omega |x|^2, \qquad x \in D(A). \tag{2.1}$$

(ii) A^{-1} *is of trace-class.*

There exists a complete orthonormal system $\{e_k\}$ in H and a sequence of positive numbers $\{\mu_k\}$ such that

$$Ae_k = -\mu_k e_k, \qquad k \in \mathbb{N}. \tag{2.2}$$

We denote by μ the Gaussian measure on $(H, \mathcal{B}(H))$ (2) with mean 0 and covariance operator $Q = -(1/2) A^{-1}$, and we set $\lambda_k = 1/2\mu_k$, $k \in \mathbb{N}$.

(2) $\mathcal{B}(H)$ is the σ-algebra of all Borel subsets of H.

We consider the Ornstein-Uhlenbeck semigroup R_t, $t \geq 0$, defined by

$$R_t \varphi(x) = \int_H \varphi(y) \, \mathcal{N}(e^{tA} x, Q_t)(dy), \qquad \varphi \in L^2(H, \mu), \tag{2.3}$$

where

$$Q_t = \frac{1}{2} A^{-1}(e^{2tA} - 1), \qquad t \geq 0. \tag{2.4}$$

One can show, see [7], that R_t, $t \geq 0$, is a strongly continuous contraction semigroup on $L^2(H, \mu)$, having as infinitesimal generator $\mathcal{A}$ the closure of the linear operator $\mathcal{A}_0$ defined as

$$\mathcal{A}_0 \varphi(x) = \frac{1}{2} \mathrm{Tr}\,[D^2 \varphi(x)] + \langle x, AD\varphi(x) \rangle, \qquad \varphi \in \mathcal{E}_A(H), \tag{2.5}$$

where

$$\mathcal{E}_A(H) = \mathrm{span}\,\{x \to e^{\langle h, x \rangle}, \, h \in D(A)\}. \tag{2.6}$$

We finally recall two identities, valid for any $\varphi, \psi \in \mathcal{E}_A(H)$, that we shall use later, see [5] and the references therein

$$\int_H \mathcal{A}\varphi(x) \, \varphi(x) \, \mu(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \mu(dx), \tag{2.7}$$

$$\frac{1}{2} \int_H \mathrm{Tr}\,[(D^2 \varphi)^2] \, \mu(dx) + \int_H |(-A)^{1/2} D\varphi(x)|^2 \mu(dx) = 2 \int_H |\mathcal{A}\varphi(x)|^2 \mu(dx). \tag{2.8}$$

The following result is an easy consequence of estimates (2.7) and (2.8), see [5].

Proposition 2.2. – *We have*

(i) $D((-\mathcal{A})^{1/2}) = W^{1,2}(H, \mu)$ [3];

(ii) $D(\mathcal{A}) = \{\varphi \in W^{2,2}(H, \mu) \colon |(-A)^{1/2} D\varphi| \in L^2(H; \mu)\}$ [4].

(3) $W^{1,2}(H, \mu)$ is the space of all $\varphi \in L^2(H; \mu)$ such that $\sum_{k=1}^{\infty} \int_H |D_k \varphi(x)|^2 \mu(dx) < +\infty$, where D_k is the derivative in the direction e_k.

(4) $W^{2,2}(H, \mu)$ is the space of all $\varphi \in W^{1,2}(H; \mu)$ such that $\sum_{h,k=1}^{\infty} \int_H |D_h D_k \varphi(x)|^2 \cdot \mu(dx) < +\infty$.

Moreover, for all $\lambda > 0$, $\varphi \in D(\mathcal{A})$, *we have, setting* $f = \lambda\varphi - \mathcal{A}$,

$$\|\varphi\|_{L^2(H,\mu)} \leq \frac{1}{\lambda}\|f\|_{L^2(H,\mu)}, \tag{2.9}$$

$$\|D\varphi\|_{L^2(H,\mu)} \leq \sqrt{\frac{2}{\lambda}}\|f\|_{L^2(H,\mu)}, \tag{2.10}$$

$$\|\mathrm{Tr}\,[(D^2\varphi)^2]\|_{L^1(H,\mu)} \leq 4\|f\|_{L^2(H,\mu)}, \tag{2.11}$$

$$\|(-A)^{1/2}D\varphi\|_{L^2(H,\mu)} \leq 2\|f\|_{L^2(H,\mu)}. \tag{2.12}$$

In the following we shall write

$$D(\mathcal{A}) = W^{2,2}(H,\mu) \cap W_A^{1,2}(H,\mu),$$

where

$$W_A^{1,2}(H,\mu) = \{\varphi \in L^2(H,\mu):\ |(-A)^{1/2}D\varphi| \in L^2(H;\mu)\}.$$

3. – The case when U is regular.

We are given here a mapping $U: H \to \mathbb{R}$ such that

HYPOTHESIS 3.1. – (i) U is nonnegative and twice Gateaux differentiable.

(ii) *There exists* $\kappa > 0$ *such that*

$$\sup_{x\in H}|DU(x)| + \sup_{x\in H}\|D^2U(x)\| \leq \kappa.$$

We define a linear operator

$$\mathcal{N}_0\varphi = \mathcal{A}\varphi - \langle DU(x), D\varphi\rangle, \qquad \varphi \in \mathcal{E}_A(H), \tag{3.1}$$

and a measure ν on $(H, \mathcal{B}(H))$, by setting

$$\nu(dx) = ce^{-2U(x)}\mu(dx),$$

where $c = \left[\int_H e^{-2U(x)}\mu(dx)\right]^{-1}$.

Our goal is to prove that $\mathcal{N}_0$ is essentially self-adjoint. To do this we will prove that $\mathcal{N}_0$ is symmetric and that for some $\lambda_0 > 0$ the set

$$(\lambda_0 - \mathcal{N})(D(\mathcal{N})),$$

where $\mathcal{N}$ is the closure of $\mathcal{N}_0$, is dense on $L^2(H,\mu)$. This will imply that $\mathcal{N}$ is m-dissipative, and thus self-adjoint, see [4].

To carry out this program we need some preliminary results: an integration by parts formula, and some a-priori estimates.

LEMMA 3.2. – *Assume that Hypotheses* 2.1, *and* 3.1 *hold. Let* $\varphi, \psi \in \mathcal{E}_A(H)$, *and let* $h \in \mathbb{N}$. *Then we have*

$$\int_H [D_h \varphi\psi + \varphi D_h \psi]\, d\nu = \int_H \left(\frac{x_h}{\lambda_h} + 2 D_h U \right) \varphi\psi\, d\nu \,, \tag{3.2}$$

where $x_h = \langle x, e_h \rangle$ *and* D_h *denotes the derivative in the direction* e_h.

PROOF. – We recall a well known formula, see e.g. [3], [8],

$$\int_H [D_h \alpha\beta + \alpha D_h \beta]\, d\mu = \int_H \frac{x_h}{\lambda_h} \alpha\beta\, d\mu, \qquad \alpha, \beta \in \mathcal{E}_A(H).$$

Using this formula we find

$$\int_H D_h \varphi\psi\, d\nu = c \int_H D_h \varphi\psi\, e^{-2U} d\mu =$$

$$-c \int_H \varphi D_h(\psi e^{-2U})\, d\mu + \int_H \frac{x_h}{\lambda_h} \varphi\psi\, e^{-2U} d\mu =$$

$$-\int_H \varphi D_h \psi\, d\nu + 2 \int_H \varphi\psi D_h U\, d\nu + \int_H \frac{x_h}{\lambda_h} \varphi\psi\, d\nu \,. \qquad \blacksquare$$

PROPOSITION 3.3. – *Let* $\varphi, \psi \in \mathcal{E}_A(H)$. *Then*

(i) *We have*

$$\int_H \mathcal{N}_0 \varphi\psi\, d\nu = -\frac{1}{2} \int_H \langle D\varphi, D\psi \rangle\, d\nu \,, \tag{3.3}$$

so that $\mathcal{N}_0$ *is symmetric.*

(ii) *We have*

$$\frac{1}{2} \int_H \mathrm{Tr}\,[(D^2 \varphi)^2]\, d\nu + \int_H |(-A)^{1/2} D\varphi|^2\, d\nu + \int_H \langle D^2 U D\varphi, D\varphi \rangle\, d\nu = 2 \int_H (\mathcal{N}_0 \varphi)^2\, d\nu \,. \tag{3.4}$$

PROOF. – We first compute, following [8],

$$\int_H \langle Ax, D\varphi\rangle\, \psi\, d\nu = -\frac{1}{2}\sum_{h=1}^{\infty}\int_H \frac{x_h}{\lambda_h} D_h\varphi\psi\, d\nu .$$

By (3.19) we have

$$\int_H \langle Ax, D\varphi\rangle\, \psi\, d\nu =$$

$$-\frac{1}{2}\sum_{h=1}^{\infty}\int_H [D_h^2\varphi\psi + D_h\varphi\, D_h\psi]\, d\nu + \sum_{h=1}^{\infty}\int_H D_h U\, D_h\varphi\psi\, d\nu =$$

$$-\frac{1}{2}\int_H \mathrm{Tr}\,[D^2\varphi]\,\psi\, d\nu - \frac{1}{2}\langle D\varphi, D\psi\rangle\, d\nu + \int_H \langle DU, D\varphi\rangle\,\psi\, d\nu .$$

Now (3.3) follows easily. Let us prove (3.4). Set $\mathcal{A}\varphi = f$, and

$$\mathcal{N}_0\varphi = \frac{1}{2}\sum_{k=1}^{\infty} D_k^2\varphi - \sum_{k=1}^{\infty}\mu_k x_k D_k\varphi - \sum_{k=1}^{\infty} D_k U\, D_k\varphi = f .$$

Differentiating with respect to e_h gives

$$\mathcal{N}_0 D_h\varphi - \mu_h D_h\varphi - \sum_{k=1}^{\infty} D_h D_k U D_k\varphi = D_h\, f .$$

Multiplying both sides for $D_h\varphi$, integrating in H with respect to ν, and taking into account (3.19), we find

$$\frac{1}{2}\int_H |DD_h\varphi|^2 d\nu + \int_H \mu_h\, |D_h\varphi|^2 d\nu + \sum_{k=1}^{\infty}\int_H D_h D_k U\, D_h\varphi\, D_k\varphi\, d\nu =$$

$$-\int_H D_h\, f D_h\varphi\, d\nu = \int_H D_h^2\varphi f\, d\nu - \int_H \frac{x_h}{\lambda_h} D_h\varphi f\, d\nu - 2\int_H D_h U\, D_h\varphi f\, d\nu ,$$

where we have used again the integration by parts formula (3.2). Summing up on h gives (3.4). ■

We are now able to prove the main result of this section.

THEOREM 3.4. – *Assume that Hypotheses* 2.1 *and* 3.1 *hold. Then the operator* $\mathcal{N}_0$, *defined by* (3.1) *is essentially self-adjoint. Denoting by* $\mathcal{N}$ *its closure we have*

(3.5) $$D((-\mathcal{N})^{1/2}) = W^{1,2}(H, \nu),$$

and

$$D(\mathcal{N}) = \{\varphi \in W^{2,2}(H, \nu) \colon |(-A)^{1/2} D\varphi| \in L^2(H, \nu)\}. \tag{3.6}$$

Moreover the measure ν is invariant for the semigroup $e^{t\mathcal{N}}$.

PROOF. – We first notice that, since $\mathcal{N}_0$ is symmetric by (3.3), then it is closable. Let us denote by $\mathcal{N}$ its closure. We now proceed in three steps.

STEP 1. – We have

$$D(\mathcal{A}) = W^{2,2}(H, \mu) \cap W_A^{1,2}(H, \mu) \subset D(\mathcal{N}), \tag{3.7}$$

and

$$\mathcal{N}\varphi = \mathcal{A}\varphi - \langle DU, D\varphi \rangle, \qquad \varphi \in D(\mathcal{A}). \tag{3.8}$$

Let in fact $\varphi \in D(\mathcal{A})$. Since $\mathcal{E}_A(H)$ is a core for $\mathcal{A}$ there exists a sequence $\{\varphi_n\} \subset \mathcal{E}_A(H)$ such that

$$\varphi_n \to \varphi, \qquad \mathcal{A}\varphi_n \to \mathcal{A}\varphi \quad \text{in} \quad L^2(H, \mu), \qquad \text{and so in} \quad L^2(H, \nu).$$

Recalling the well known estimate see e.g. [5],

$$\int_H |x|^2 \, |D\varphi(x)|^2 \mu(dx) \leq C \|\varphi\|^2_{W^{2,2}(H, \mu)}, \qquad \varphi \in W^{2,2}(H, \mu),$$

we see that

$$\langle DU, D\varphi_n \rangle \to \langle DU, D\varphi \rangle \quad \text{in} \quad L^2(H, \mu), \qquad \text{and so in} \quad L^2(H, \nu).$$

Consequently $\mathcal{N}_0 \varphi_n \to \mathcal{A}\varphi - \langle DU, D\varphi \rangle$, and the claim is proved.

STEP 2. – There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ and all $f \in L^2(H, \mu)$, the equation

$$\lambda\varphi - \mathcal{N}\varphi = \lambda\varphi - \mathcal{A}\varphi + \langle DU, D\varphi \rangle = f, \tag{3.9}$$

has a unique solution $\varphi \in D(\mathcal{A})$.

In fact, setting $\lambda\varphi - \mathcal{A}\varphi = \psi$, equation (3.9) is equivalent to

$$\psi - T\psi = f, \tag{3.10}$$

where $T\psi = \langle DU, DR(\lambda, \mathcal{A})\, \psi \rangle$. Now, taking into account (2.7), we see that

$$\|T\psi\|_{L^2(H, \mu)} \leq \kappa \sqrt{\frac{2}{\lambda}} \|\psi\|_{L^2(H, \mu)},$$

and the conclusion follows with $\lambda_0 = 8\kappa^2$.

STEP 3. – Conclusion.
By step 2 we have

$$(\lambda_0 - \mathcal{N})(D(\mathcal{N})) \supset L^2(H, \mu).$$

Since $L^2(H, \mu)$ is dense in $L^2(H, \nu)$ it follows that $\mathcal{N}$ is m-dissipative and so self-adjoint see e.g. [4, Corollaire II.9.3]. Now it follows by approximation that identities (3.3) and (3.4) hold for any $\varphi \in D(\mathcal{N})$. Then (3.5) and (3.6) follow easily. ■

4. – The general case.

We are given a mapping $U: H \to [0, +\infty]$ such that

HYPOTHESIS 4.1. – (i) U is convex, lower semi-continuous, not identically $+\infty$.

(ii) There exists $p > 2$ such that

$$\int_H |DU(x)|^p \nu(dx) < +\infty,$$

where $DU(x)$ is the sub-differential of $U(x)$, $\nu(dx) = ce^{-2U(x)}\mu(dx)$, and $c = [\int_H e^{-2U(x)}\mu(dx)]^{-1}$.

(iii) There exists a sequence $\{U_n\}$ of functions fulfilling Hypothesis 3.1 such that $U_n(x) \uparrow U(x)$ and

$$\lim_{n \to \infty} \int_H |DU(x) - DU_n(x)|^p \nu(dx) = 0.$$

We denote by ν_n the measure $\nu_n(dx) = c_n e^{-2U_n(x)}\mu(dx)$, where $c_n = [\int_H e^{-2U_n(x)}\mu(dx)]^{-1}$. We have the following continuous and dense inclusions

$$L^p(H, \mu) \subset L^p(H, \nu_n) \subset L^p(H, \nu), \qquad p > 1,$$

and, for all $\varphi \in L^p(H, \mu)$,

$$\int_H |\varphi|^p d\nu \leq \frac{c}{c_n}\int_H |\varphi|^p d\nu_n \leq c\int_H |\varphi|^p d\mu. \tag{4.1}$$

We define a linear operator $\mathcal{N}_0$ on $L^2(H, \nu)$ with domain $\mathcal{E}_A(H)$ by setting

$$\mathcal{N}_0 \varphi = \mathcal{A}\varphi - \langle DU, D\varphi \rangle, \qquad \varphi \in \mathcal{E}_A(H). \tag{4.2}$$

This definition is meaningful in virtue of Hypothesis 4.1-(ii). We also set

$$\mathcal{N}_{0,n}\varphi = \mathcal{A}\varphi - \langle DU_n, D\varphi\rangle, \qquad \varphi \in \mathcal{E}_A(H), \tag{4.3}$$

and denote by $\mathcal{N}_n$ the closure of $\mathcal{N}_{0,n}$ on $L^2(H, \nu_n)$. Clearly for any $\varphi \in \mathcal{E}_A(H)$ we have

$$\lim_{n\to\infty} \mathcal{N}_{0,n}\varphi = \mathcal{N}_0\varphi \quad \text{in} \quad L^2(H,\nu). \tag{4.4}$$

PROPOSITION 4.2. – *Let $\varphi, \psi \in \mathcal{E}_A(H)$. Then*

(i) *We have*

$$\int_H \mathcal{N}_0\varphi\psi\, d\nu = -\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, d\nu\,,$$

so that $\mathcal{N}_0$ is symmetric.

(ii) *We have*

$$\frac{1}{2}\int_H \mathrm{Tr}\,[(D^2\varphi)^2]\, d\nu + \int_H |(-A)^{1/2}D\varphi|^2\, d\nu \le 2\int_H (\mathcal{N}_0\varphi)^2\, d\nu\,. \tag{4.6}$$

PROOF. – Let us prove (4.5). For any $\varphi, \psi \in \mathcal{E}_A(H)$ we have by (3.3)

$$\int_H \mathcal{N}_{0,n}\varphi\psi\, d\nu_n = -\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, d\nu_n,$$

which is equivalent to

$$c_n\int_H \mathcal{N}_{0,n}\varphi\psi\, e^{-2U_n}\, d\mu = -\,c_n\,\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, e^{-2U_n}\, d\mu\,.$$

As $n\to\infty$, (4.5) follows.

Let us finally prove (4.6). For any $\varphi, \psi \in \mathcal{E}_A(H)$ we have by (3.4), recalling that U_n is convex

$$\frac{1}{2}\int_H \mathrm{Tr}\,[(D^2\varphi)^2]\, d\nu_n + \int_H |(-A)^{1/2}D\varphi|^2\, d\nu_n \le 2\int_H (\mathcal{N}_{0,n}\varphi)^2\, d\nu_n.$$

As $n\to\infty$, (4.6) follows. ■

We need now a technical lemma whose proof is given in Appendix A.

LEMMA 4.3. – *Let* $\varphi \in \mathcal{E}_A(H)$, $\lambda > 0$, $p \geq 2$, *and* $f = \lambda\varphi - \mathcal{N}_0\varphi$. *The following estimate holds*

$$\|\varphi\|_{W^{1,p}(H,\nu_n)} \leq \frac{1}{\lambda}\|f\|_{W^{1,p}(H,\nu_n)}. \tag{4.7}$$

Now we can prove the result

THEOREM 4.4. – *Assume that Hypotheses* 2.1 *and* 4.1 *hold. Then the operator* $\mathcal{N}_0$, *defined by* (4.3) *is essentially self-adjoint. Denoting by* $\mathcal{N}$ *its closure we have*

$$D((-\mathcal{N})^{1/2}) = W^{1,2}(H,\nu), \tag{4.8}$$

and

$$D(\mathcal{N}) \subset \{\varphi \in W^{2,2}(H,\nu) \colon |(-A)^{1/2}D\varphi| \in L^2(H,\nu)\}. \tag{4.9}$$

Moreover the measure ν *is invariant for the semigroup* $e^{t\mathcal{N}}$.

PROOF. – We set $q = 2p/(p-2)$. By proceding as in the proof of Step 1 of Theorem 3.4 we see that $W^{1,q}(H,\mu) \subset D(\mathcal{N})$. Now let $f \in W^{1,q}(H,\mu)$. Then for any $n \in \mathbb{N}$ there exists $\varphi_n \in D(\mathcal{N}_n)$ such that

$$\lambda\varphi_n - \mathcal{A}\varphi_n + \langle DU_n, D\varphi_n\rangle = f. \tag{4.10}$$

Moreover, by Lemma 4.3 we have

$$\|\varphi_n\|_{W^{1,q}(H,\nu_n)} \leq \frac{c^{1/q}}{\lambda}\|f\|_{W^{1,q}(H,\nu_n)}.$$

It follows

$$\|\varphi_n\|_{W^{1,q}(H,\nu)} \leq \left(\frac{c}{c_n}\right)^{1/q}\|\varphi_n\|_{W^{1,q}(H,\nu_n)} \leq \frac{1}{\lambda}\left(\frac{c}{c_n}\right)^{1/q}\|f\|_{W^{1,q}(H,\nu_n)} \leq \frac{1}{\lambda}c^{1/q}\|f\|_{W^{1,q}(H,\nu)}.$$

Thus we have proved that

$$\|\varphi_n\|_{W^{1,q}(H,\nu)} \leq \frac{c^{1/q}}{\lambda}\|f\|_{W^{1,q}(H,\mu)}. \tag{4.11}$$

Now we can conclude the proof. We have

$$\lambda\varphi_n - \mathcal{N}_0\varphi_n = f + \langle DU - DU_n, D\varphi_n\rangle. \tag{4.12}$$

But

$$\int_H |\langle DU - DU_n, D\varphi_n\rangle|^2 d\nu \leq \int_H |DU - DU_n|^2 |D\varphi_n|^2 d\nu \leq$$

$$\left(\int_H |DU - DU_n|^p d\nu\right)^{2/p} \left(\int_H |D\varphi_n|^q d\nu\right)^{2/q} \leq$$

$$\frac{1}{\lambda} c^{1/p} \left(\int_H |DU - DU_n|^p d\nu\right)^{2p} \|f\|_{W^{1,q}(H,\mu)}.$$

Consequently

$$\lim_{n\to\infty} \langle DU - DU_n, D\varphi_n\rangle = 0 \qquad \text{in } L^2(H, \nu),$$

and so $(\lambda - \mathcal{N})(D(\mathcal{N}))$ contains the closure on $L^2(H, \nu)$ of $W^{1,q}(H, \mu)$. Since $W^{1,q}(H, \mu)$ is dense on $L^2(H, \nu)$. As in the proof of Theorem 3.4 this implies that $\mathcal{N}$ is self-adjoint. ■

REMARK 4.5. – If $D^2U(x)$ exists for ν almost $x \in H$ and it is Borel, then we have the following characterization of $D(\mathcal{N})$:

$$D(\mathcal{N}) = \tag{4.13}$$

$$\{\varphi \in W^{2,2}(H, \nu): |(-A)^{1/2} D\varphi| \in L^2(H, \nu), \langle D^2 U D\varphi, D\varphi\rangle \in L^1(H, \nu)\}.$$

EXAMPLE 4.6. – Let $H = L^2(0, \pi)$, $Ax = D_\xi^2 x$, $x \in D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$. Set moreover

$$e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin k\xi, \qquad f_k(\xi) = \sqrt{\frac{2}{\pi}} \cos k\xi, \quad k \in \mathbb{N},$$

and denote by T the isometry on H:

$$T\left(\sum_{k=1}^{\infty} x_k e_k\right) = \left(\sum_{k=1}^{\infty} x_k f_k\right), \qquad x \in H, \quad x_k = \langle x, e_k \rangle .$$

Let moreover Q be the trace class operator on H such that $Qe_k = (1/2k^2)\, e_k$, $k \in \mathbb{N}$, and let $\mu = \mathcal{N}(0, Q)$.

Let finally

$$U(x) = \begin{cases} \dfrac{1}{4}\langle x^4, 1\rangle & \text{if } x \in L^4(0, \pi), \\ +\infty & \text{if } x \notin L^4(0, \pi). \end{cases}$$

Then we have

$$DU(x) = -x^3 \qquad \text{if } x \in L^6(0, \pi).$$

It is easy to check that for all $x \in H$,

$$x(\xi) = \langle Q^{-1/2} x, T^* \chi_{[0, \xi]} \rangle, \qquad \xi \in [0, \pi].$$

For any $m \geq 1$ there exists a constant $C_m > 0$ such that

$$\int_H |DU(x)|^{2m} \mu(dx) = \int_H \left(\int_0^\pi |x(\xi)|^{6m} d\xi \right) \mu(dx) =$$

$$= \int_0^\pi \left[\int_H |\langle Q^{-1/2} x, T^* \chi_{[0, \xi]} \rangle|^{6m} \mu(dx) \right] d\xi = C_m \int_0^\pi |T^* \chi_{[0, \xi]}|_H^{6m} d\xi = C_m \int_0^\pi \xi^{3m} d\xi .$$

Thus all assumptions of Theorem 4.4 are fulfilled.

5. – Ergodicity and spectral gap.

We set $P_t \varphi = e^{t\mathcal{N}} \varphi$, for all $\varphi \in L^2(H, \nu)$, where $\mathcal{N}$ is the self-adjoint operator defined in Theorem 4.4. We first prove that ν is ergodic and strongly mixing.

For this we need a lemma.

LEMMA 5.1. – *For any $\varphi \in W^{1,2}(H, \nu)$ we have*

$$\|DP_t \varphi\|^2_{L^2(H, \nu)} \leq e^{-2\omega t} \|D\varphi\|^2_{L^2(H, \nu)} \text{(5)} . \tag{5.1}$$

(5) Recall that $\langle Ax, x \rangle \leq -\omega |x|^2$, $x \in D(A)$.

PROOF. – It is enough to show (5.1) for all $\varphi \in \mathcal{E}_A(H)$. In this case we have

$$\frac{d}{dt} D_h u(t, x) = \mathcal{N} D_h u(t, x) - \mu_h D_h u(t, x) + \sum_{k=1}^{\infty} D_h D_k U D_h u(t, x),$$

from which

$$\frac{1}{2} \frac{d}{dt} \int_H |D_h u(t, x)|^2 d\nu = -\frac{1}{2} \int_H |DD_h u(t, x)|^2 d\nu -$$
$$-\mu_h \int_H |D_h u(t, x)|^2 d\nu + \sum_{k=1}^{\infty} \int_H D_h D_k U(x) D_k u(t, x)\, d\nu .$$

Summing up on h it follows

$$\frac{1}{2} \frac{d}{dt} \int_H |Du(t, x)|^2 d\nu + \frac{1}{2} \int_H \mathrm{Tr}\,[(D^2 u(t, x))^2]\, d\nu \leqslant$$
$$-\omega \int_H |Du(t, x)|^2 d\nu + \int_H \langle D^2 U(x)\, u(t, x), u(t, x)\rangle\, d\nu ,$$

and the conclusion follows. ■

THEOREM 5.2. – *Assume that Hypotheses* 2.1 *and* 4.1 *hold. Then we have*

$$\lim_{n \to \infty} P_t \varphi(x) = \int_H \varphi(y)\, \nu(dy). \tag{5.2}$$

PROOF. – It is enough to prove (5.2) for $\varphi \in \mathcal{E}_A(H)$. In this case, setting $u(t, x) = P_t \varphi(x)$ we have

$$u(t, x) = e^{t\mathcal{C}} \varphi(x) - \int_H e^{(t-s)\mathcal{C}} \langle DU(x), Du(s, x)\rangle\, ds . \tag{5.3}$$

In virtue of (5.1) we can pass to the limit as $t \to +\infty$ in (5.3). Recalling that

$$\lim_{t \to \infty} e^{t\mathcal{C}} \varphi(x) = \int_H \varphi(y)\, \mu(dy),$$

we find

$$\lim_{t \to \infty} u(t, x) = \int_H \varphi(y)\, \mu(dy) - \int_0^{+\infty} ds \int_H \langle DU(x), Du(s, y)\rangle\, \mu(dy).$$

Now the conclusion follows from the Von Neumann ergodic theorem. ■

To prove spectral gap we need a Poincaré inequality.

PROPOSITION 5.3. – *For any* $\varphi \in W^{1,2}(H, \nu)$ *we have*

$$\int_H |\varphi - \overline{\varphi}|^2 d\nu \leq \frac{1}{2\omega} \int_H |D\varphi|^2 d\nu \,, \tag{5.4}$$

where

$$\overline{\varphi} = \int_H \varphi(y)\, \nu(dy)\,.$$

PROOF. – It is enough to prove (5.2) for $\varphi \in \mathcal{E}_A(H)$. In this case we have

$$\frac{1}{2}\frac{d}{dt}\int_H |P_t \varphi|^2 d\nu = \int_H \mathcal{N} P_t \varphi\, P_t \varphi\, d\nu = -\frac{1}{2}\int_H |DP_t \varphi|^2 d\nu\,.$$

By Lemma 5.1 it follows

$$\frac{1}{2}\frac{d}{dt}\int_H |P_t \varphi|^2 d\nu \geq -\frac{1}{2} e^{-2\omega t}\int_H |D\varphi|^2 d\nu\,.$$

Integrating in t we have

$$\int_H |P_t \varphi|^2 d\nu \geq \int_H \varphi^2 d\nu - \frac{1}{2\omega}(1 - e^{-2\omega t})\int_H |D\varphi|^2 d\nu\,.$$

Letting n tend to ∞ it follows by Theorem 5.3

$$(\overline{\varphi})^2 \geq \int_H \varphi^2 d\nu - \frac{1}{2\omega}\int_H |D\varphi|^2 d\nu\,,$$

that it is equivalent to (5.4). ■

We can now prove the result

THEOREM 5.4. – *Assume that Hypotheses* 2.1 *and* 4.1 *hold. Then we have*

$$\int_H |P_t \varphi(x) - \overline{\varphi}|^2 d\nu \leq C e^{-2\omega t}\int_H |\varphi|^2 d\nu\,. \tag{5.5}$$

PROOF. – By (5.4) it follows

$$\int_H |P_t\varphi - \overline{\varphi}|^2 d\nu \leq \frac{1}{2\omega}\int_H |DP_t\varphi|^2 d\nu .$$

Moreover by (5.1) we have

$$\int_H |P_t\varphi - \overline{\varphi}|^2 d\nu \leq \frac{e^{-2\omega t}}{2\omega}\int_H |D\varphi|^2 d\nu .$$

Thus for any $\varepsilon > 0$ it follows

$$\int_H |P_{t+\varepsilon}\varphi - \overline{\varphi}|^2 d\nu \leq \frac{e^{-2\omega t}}{2\omega}\int_H |DP_\varepsilon\varphi|^2 d\nu \leq \frac{e^{-2\omega t}}{\varepsilon\omega e}\int_H |D\varphi|^2 d\nu ,$$

since

$$\int_H |DP_\varepsilon\varphi|^2 d\nu = 2\int_H |(-\mathcal{N})^{1/2} P_\varepsilon\varphi|^2 d\nu \leq \frac{2}{\varepsilon e}\int_H |D\varphi|^2 d\nu .$$

The conclusion follows. ■

A. – L^p estimates.

We assume here that Hypotheses 2.1 and 3.1 hold, and consider the equation

$$\lambda\varphi - \mathcal{N}\varphi = \lambda\varphi - \mathcal{A}\varphi + \langle DU, D\varphi\rangle = f, \tag{A.1}$$

where $\mathcal{N}$ is defined by (3.1), $\lambda > 0$, and $f \in L^2(H, \nu)$.

PROPOSITION A.1. – *For all $\varphi \in \mathcal{E}_A(H)$ and $p \geq 2$, the following identity holds.*

$$\lambda\int_H |\varphi|^p d\nu + \frac{p-1}{2}\int_H |D\varphi|^2 |\varphi|^{p-2} d\nu = \int_H f|\varphi|^{p-2}\varphi\, d\nu . \tag{A.2}$$

Moreover

$$\|\varphi\|_{L^p(H,\nu)} \leq \frac{1}{\lambda}\|f\|_{L^p(H,\nu)}. \tag{A.3}$$

PROOF. – We have

$$\int_H \langle Ax, D\varphi\rangle |\varphi|^{p-2}\varphi d\nu = -\frac{1}{2}\sum_{h=1}^{\infty}\int_H \frac{x_h}{\lambda_h} D_h\varphi|\varphi|^{p-2}\varphi\, d\nu =$$

$$-\frac{1}{2}\sum_{h=1}^{\infty}\int_H D_h^2\varphi|\varphi|^{p-2}d\nu - \frac{p-1}{2}\sum_{h=1}^{\infty}\int_H |D_h\varphi|^2|\varphi|^{p-2}d\nu +$$

$$\sum_{h=1}^{\infty}\int_H D_h U D_h\varphi|\varphi|^{p-2}\varphi\, d\nu =$$

$$-\frac{1}{2}\int_H \mathrm{Tr}[D^2\varphi]|\varphi|^{p-2}\varphi\, d\nu - \frac{p-1}{2}\int_H |D\varphi|^2|\varphi|^{p-2}d\nu +$$

$$\int_H \langle DU, D\varphi\rangle |\varphi|^{p-2}\varphi\, d\nu .$$

Now the conclusion follows easily. ■

LEMMA A.2. – *Let* $\varphi, \psi_1, \dots, \psi_n \in \mathcal{E}_A(H)$. *Then we have*

$$\text{(A.4)}\qquad \int_H \mathcal{N}\varphi\psi_1^2, \dots, \psi_n^2 d\nu = -\frac{1}{2}\int_H |D\varphi|^2\psi_1^2, \dots, \psi_n^2 d\nu -$$

$$\sum_{k=1}^{n}\int_H \langle D\varphi, D\psi_k\rangle \varphi\psi k\psi_1^2, \dots, \psi_{k-1}^2\psi_k^2, \dots, \psi_n^2 d\nu .$$

PROOF. – We have

$$\int_H \langle Ax, D\varphi\rangle \varphi\psi_1^2, \dots, \psi_n^2 d\nu = -\frac{1}{2}\sum_{h=1}^{\infty}\int_H \frac{x_h}{\lambda_h} D_h\varphi\varphi\psi_1^2, \dots, \psi_n^2 d\nu -$$

$$\frac{1}{2}\sum_{h=1}^{n}\int_H D_h^2\varphi\varphi\psi_1^2, \dots, \psi_{k-1}^2\psi_k^2, \dots, \psi_n^2 d\nu -$$

$$\frac{1}{2}\sum_{h=1}^{\infty}\int_H |D_h\varphi|^2\varphi\psi_1^2, \dots, \psi_{k-1}^2\psi_k^2, \dots, \psi_n^2 d\nu -$$

$$\sum_{h=1}^{\infty}\sum_{k=1}^{n}\int_H D_h\varphi\varphi D_h\psi_k\psi_k\psi_1^2, \dots, \psi_{k-1}^2\psi_k^2, \dots, \psi_n^2 d\nu +$$

$$\sum_{h=1}^{\infty}\int_H D_h U D_h\varphi\varphi\psi_1^2, \dots, \psi_n^2 d\nu . \qquad ■$$

PROPOSITION A.3. – *Let $\varphi \in \mathcal{E}_A(H)$, $\lambda > 0$, $\lambda\varphi - \mathcal{N}\varphi = f$. Then the following identity holds*

$$\text{(A.5)} \qquad \lambda \int_H |D\varphi|^{2m} d\nu + \frac{1}{2} \int_H \operatorname{Tr}[(D^2\varphi)^2] |D\varphi|^{2m-2} d\nu +$$

$$(m-1) \int_H \langle (D^2\varphi)^2 D\varphi, D\varphi \rangle |D\varphi|^{2m-1} d\nu + \int_H |(-A)^{1/2} D\varphi|^2 |D\varphi|^{2m-2} d\nu +$$

$$\int_H |\langle D^2 U D\varphi, D\varphi \rangle |D\varphi|^{2m-2} d\nu = \int_H \langle D\varphi, D\psi \rangle |D\varphi|^{2m-2} d\nu .$$

Moreover

$$\text{(A.6)} \qquad \|D\varphi\|_{L^p(H,\mu)} \leq \frac{1}{\lambda} \|Df\|_{L^p(H,\mu)}.$$

PROOF. – For any $h \in \mathbb{N}$ we have

$$\lambda D_h \varphi - \mathcal{N} D_h \varphi + \mu_h D_h \varphi + \sum_{k=1}^{\infty} D_h D_k U D_k \varphi = D_h \varphi .$$

Multiplying both sides for

$$D_h \varphi (D_{\alpha_1}\varphi)^2 \dots (D_{\alpha_{m-1}}\varphi)^2,$$

and using Lemma A.2 we obtain

$$\lambda \int_H |D_h\varphi|^2 |D_{\alpha_1}\varphi|^2 \dots |D_{\alpha_{m-1}}\varphi|^2 d\nu + \frac{1}{2} \int_H |DD_h\varphi|^2 |D_{\alpha_1}\varphi|^2 \dots |D_{\alpha_{m-1}}\varphi|^2 d\nu +$$

$$\sum_{j=1}^{m-1} \int_H \langle DD_h\varphi, DD_{\alpha_j}\varphi \rangle D_h\varphi D_{\alpha_j}\varphi (D_{\alpha_1}\varphi)^2 \dots (D_{\alpha_{j-1}}\varphi)^2 (D_{\alpha_{j+1}}\varphi)^2 \dots (D_{\alpha_{m-1}}\varphi)^2 d\nu +$$

$$\mu_h \int_H |D_h\varphi|^2 |D_{\alpha_1}\varphi|^2 \dots |D_{\alpha_{m-1}}\varphi|^2 d\nu +$$

$$\sum_{k=1}^{m-1} \int_H D_h D_k U D_k \varphi D_h \varphi |D_{\alpha_1}\varphi|^2 \dots |D_{\alpha_{m-1}}\varphi|^2 d\nu =$$

$$\int_H D_h f D_h \varphi |D_{\alpha_1}\varphi|^2 \dots |D_{\alpha_{m-1}}\varphi|^2 d\nu .$$

Now identity (A.4) follows summing up on $h, \alpha_1, \dots, \alpha_{m-1}$. Finally (A.5) follows from the Hölder estimate. ■

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