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DMF-Algebras: Representation and Topological Characterization.

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Sunto. – *Gli insiemi parziali sono coppie* (A, B) *di sottoinsiemi di* X, *dove* $A \cap B \neq 0$. *Gli insiemi parziali su* X *costituiscono una DMF-algebra, ossia un'algebra di De Morgan in cui la negazione ha un solo punto fisso. Dimostriamo che ogni DMF-algebra è isomorfa a un campo di insiemi parziali. Utilizzando gli insiemi parziali su* X *come aperti, introduciamo il concetto di spazio topologico parziale su* X. *Infine associamo ad ogni DMF-algebra* $\mathcal{A}$ *uno spazio topologico parziale i cui clopen compatti costituiscono un campo d'insiemi parziali isomorfo ad* $\mathcal{A}$..

1. – Partial sets.

The classical conception of property (of individuals of a given domain X) is characterized by the following aspects that we would underline: 1) from any property P we can obtain the opposite property not-P by an operation called negation, 2) for any property P and any individual a, if P does not hold for a, then the property not-P holds for a. From an extensional point of view, classical properties can be identified with subsets of X and the usual operations of conjunction, disjunction and negation can be represented by the operations of intersection, union and complement. For any set X we denote with $\mathcal{P}(X)$ the algebra of signature $\mathcal{L}_{BA} = \{\wedge, \vee, \neg, 0, 1\}$ whose domain is the power set $P(X)$ and whose operations and constants $\wedge^{\mathcal{B}}$ $\vee^{\mathcal{B}}$ $\neg^{\mathcal{B}}$, $0^{\mathcal{B}}$, $1^{\mathcal{B}}$ are respectively $\cap$, $\cup$, $-$, $\emptyset$ and X. Any algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ is called a *field of sets on* X, whereas $\mathcal{P}(X)$ is the field of *all* sets on X. The concept of a *ring of sets on* X is defined in the same way for the signature $\{\wedge, \vee, 0, 1\}$, dropping the interpretation of $\neg$. We denote with $\mathcal{R}(X)$ the ring of *all* sets on X.

The concept of partial property retains the first and rejects the second point above: there are properties P and individuals $x \in X$ such that neither P nor not-P holds for x. (A study of the concept of partial predicate is contained in [3, 7.7], where the locution 'inexact predicate' is preferred.) From an extensional point of view partial properties can be identified with partial sets. We define a *partial set* on X as an ordered couple (A, B) of subsets of X such that $A \cap B = \emptyset$. If we identify P with (A, B), we say that P holds for x iff $x \in A$. The first element A contains the elements of X definitely enjoying P and B contains the elements of X definitely not-enjoying P. So not-P, the negation of P, is simply obtained by exchanging A with B. As we suppose that $A \cap B = \emptyset$, there is no x such that P and not-P simultaneously hold for x. However we do not require $A \cup B = X$, so when $A \cup B$ is strictly included in X, there is some $x \in X - A \cup B$ such that neither P nor not-P holds for x: *tertium non datur* is no longer a logical truth. When $x \in X - A \cup B$ we say that P is undefined for x. Classical properties arise when A is exactly the complement of B and in this case the information carried by the couple (A, B) is the same of the single A or B. If P and Q are properties represented by partial sets (A, B) and (C, D), then the conjunction «P and Q» can be represented by $(A \cap C, B \cup D)$. If P is «red» and Q is «hot», then the partial predicate «P and Q» has a positive part containing individuals being simultaneously red and hot, the intersection $A \cap C$ of the positive parts of P and Q, and a negative part containing individuals failing to be red or failing to be hot or failing to be «red and hot», the union of the negative parts of P and Q. So a logical operation «and» is interpreted by two different operations on positive and negative parts that are reciprocally dual. The same can be said about the operation «or».

As an analog to the classical notion of field of sets we introduce the concept of partial field of sets on X. Firstly we define $D(X) = \{(A, B): A, B \subseteq X, A \cap B = \emptyset\}$, then for any set X we denote with $\mathcal{D}(X)$ the algebra of signature $\mathcal{L}_{DMF} = \{\wedge, \vee, \neg, 0, n, 1\}$ whose domain is $D(X)$ and whose operations and constants are defined as follows:

$$(A, B) \wedge^{\mathcal{D}} (A', B') = (A \cap A', B \cup B'),$$

$$(A, B) \vee^{\mathcal{D}} (A', B') = (A \cup A', B \cap B'),$$

$$\neg^{\mathcal{D}}(A, B) = (B, A),$$

$$0^{\mathcal{D}} = (\emptyset, X),$$

$$1^{\mathcal{D}} = (X, \emptyset),$$

$$n^{\mathcal{D}} = (\emptyset, \emptyset).$$

One can easily see that $D(X)$ is closed with respect to these operations We say that an algebra $\mathcal{A}$ of signature L_{DMF} is a *field of partial sets on* X if $\mathcal{A} \subseteq \mathcal{D}(X)$. In particular, $\mathcal{D}(X)$ is the field of *all* partial sets on X.

To simplify notation we shall write $\cap$ instead of $\wedge^{\mathcal{D}}$, being clear that in a context like $(A, B) \cap (A', B')$ the symbol $\cap$ stands for $\wedge^{\mathcal{D}}$, whereas in a context like $A \cap B$ it denotes the usual intersection. The same holds for the other operations. We introduce as usual a partial order relation on partial sets setting

$$(A, B) \subseteq (A', B') \quad \text{iff } (A, B) \cap (A', B') = (A, B).$$

It can be easily verified that $(A, B) \subseteq (A', B')$ iff $A \subseteq B$ and $B' \subseteq A'$.

When in a partial set (A, B) we have $B = -A$, then we say that it is a classic (bivalent) set. (We write $-A$ instead of $X - A$ when X is clear from the context.) The set of all classical sets in $\mathcal{D}(X)$ is a Boolean algebra, but it is not a field of partial sets because it is not closed with respect to the constant n.

Partial sets on X can equally well be represented by another structure that we call $\mathcal{E}(X)$. The domain of $\mathcal{E}(X)$ is the set of couples (A, B) of subsets of X such that $A \cup B = X$. The operations $\wedge$, $\vee$, $\neg$ and constants 0, 1 are defined as above, the only difference being $n = (X, X)$. We can define an isomorphism φ between $\mathcal{D}(X)$ and $\mathcal{E}(X)$ setting $\varphi(A, B) = (-B, -A)$, where $-A = X - A$ and $-B = X - B$, so these two ways of viewing partial sets are equivalent.

It is well known that fields of sets are representative of Boolean algebras (BA) and ring of sets are representative of bounded distributive lattices ($DL_{0,1}$). The aim of this work is to show that partial sets play with DMF-algebras (DMF) the same role that classical sets play with Boolean algebras, DM-algebras and distributive lattices. We can shortly describe DMF-algebras as follows. De Morgan algebras (DM) are structures of signature $\mathcal{L}_{BA}$ satisfying the axioms of bounded distributive lattices plus De Morgan laws

$$\neg(x \wedge y) = \neg x \vee \neg y \quad \text{and} \quad \neg(x \vee y) = \neg x \wedge \neg y$$

and double negation law

$$\neg\neg x = x.$$

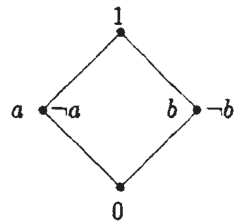
DMF-algebras are structures of signature $\mathcal{L}_{\mathrm{DMF}}$ satisfying the axioms of DM-algebras plus normality axiom

$$x \wedge \neg x \leqslant y \vee \neg y$$

and the fixed point axiom

$$\neg n = n.$$

DMF-algebras can also be seen as DM-algebras with a single fixed point for negation, because any DM-algebra satisfying $\exists! x(\neg x = x)$ is normal (see [3.3] [4]) and then can be expanded to a DMF-algebra. In paragraph 3 we shall show that any DMF-algebra is isomorphic to a field of partial sets. If we drop the interpretation of n, any DMF-algebra becomes a DM-algebra and can be represented as a quasi field of sets following [1]. (They define in the ring of all subsets $\mathcal{R}(X)$ an operation $\sim A = X - g[A]$, for any $A \subseteq X$, where $g\colon X \to X$ is a fixed involution on X, and call any subalgebra of $\mathcal{R}(X)$ expanded with $\sim$ a quasi-field of sets.) Our representation theorem is more specific because there are DM-algebras that cannot be represented by fields of partial sets, because every field of partial sets has a single fixed point $(\emptyset, \emptyset)$ for negation, whereas DM-algebras can have more than one fixed point, or none at all: the following four-element DM-algebra is an instance of the first case,



and any BA is an instance of the second.

In paragraph 4 we develop the concept of partial topological space in close analogy with the classical concept of topological space: we only take as open sets partial sets instead of classical sets. The resulting partial topologies preserve some aspects of classical topologies, but the different nature of the complement in the context of partial sets causes a different behaviour of closed sets. For instance, a closed subset of a compact space is generally no longer compact. In paragraph 5 we associate a partial topological space with every DMF-algebra $\mathcal{A}$ and prove that $\mathcal{A}$ can be characterized as the set of compact clopens of the associated topological space. We leave open the problem of showing duality between DMF-algebras and some class of partial spaces.

2. – Construction of DMF-algebras.

If $\mathcal{A}$ is a lattice, we define the *dual* of $\mathcal{A}$ as a structure $\mathcal{B} = \langle A, \vee^{\mathcal{A}}, \wedge^{\mathcal{A}} \rangle$, i.e. we set $\wedge^{\mathcal{B}} = \vee^{\mathcal{A}}$ and $\vee^{\mathcal{B}} = \wedge^{\mathcal{A}}$. We denote with $\mathcal{A}^\circ$ the dual of $\mathcal{A}$. If $\mathcal{A}$ is a bounded distributive lattice, then so is $\mathcal{A}^\circ$. We say that $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a *dual isomorphism* iff $\varphi\colon \mathcal{A} \to \mathcal{B}^\circ$ is an isomorphism.

The following theorem shows a general method to construct DMF-algebras from bounded distributive lattices. If $\mathcal{A}$ is a bounded distributive lattice, then the product $\mathcal{B} = \mathcal{A} \times \mathcal{A}^\circ$ is a bounded distributive lattice too. If we denote with $\wedge$ the operation $\wedge^{\mathcal{A}}$ and with $\wedge^\circ$ the operation $\wedge^{\mathcal{A}^\circ}$ (and adopt the same convention with $\vee$, 0, 1), then we have

$$(x, y) \wedge^{\mathcal{B}} (z, w) = ((x \wedge z), (y \wedge^\circ w)) = ((x \wedge z), (y \vee w)),$$

$$(x, y) \vee^{\mathcal{B}} (z, w) = ((x \vee z), (y \vee^\circ w)) = ((x \vee z), (y \wedge w)),$$

$$0^{\mathcal{B}} = (0, 0^\circ) = (0, 1),$$

$$1^{\mathcal{B}} = (1, 1^\circ) = (1, 0).$$

For any bounded distributive lattice $\mathcal{A}$ we define

$$\pi(A) = A \times A^\circ | \{(a, b): a \wedge b = 0\}$$

and

$$\varrho(A) = A \times A^\circ | \{(a, b): a \vee b = 1\}.$$

We denote with $\pi(\mathcal{A})$ the structure with signature $\mathcal{L}_{DMF}$ defined as follows: the domain of $\pi(\mathcal{A})$ is $\pi(A)$, the lattice operations and constants are defined as in $\mathcal{A} \times \mathcal{A}^\circ$, $\neg(a, b) = (b, a)$ and $n = (0, 0)$. We define in the same way $\varrho(\mathcal{A})$ on $\varrho(A)$, the only difference being $n = (1, 1)$. The next theorem shows that π-construction takes bounded distributive lattices in DMF-algebras

TEOREM 2.1. – *If* $\mathcal{A} \in DL_{0,1}$ *then* $\pi(\mathcal{A})$ *and* $\varrho(\mathcal{A})$ *are DMF-algebras.*

Firstly we prove that $\pi(A)$ is the domain of a substructure of $\mathcal{A} \times \mathcal{A}^\circ$. If (a, b) and (c, d) are in $\pi(A)$, then (a, b) and (c, d) are in $A \times A$ and $a \wedge b = c \wedge d = 0$. Then we have, setting $\mathcal{A} \times \mathcal{A}^\circ = \mathcal{B}$ to simplify notation,

$$(a, b) \wedge^{\mathcal{B}} (c, d) = (a \wedge c, b \vee d)$$

where

$$(a \wedge c) \wedge (b \vee d) = (a \wedge c \wedge b) \vee (a \wedge c \wedge d) = 0,$$

and

$$(a, b) \vee^{\mathcal{B}} (c, d) = (a \vee c, b \vee d)$$

where

$$(a \vee c) \wedge (b \wedge d) = (b \wedge d \wedge a) \vee (b \wedge d \wedge c) = 0.$$

So $\pi(A)$ is closed with respect to $\wedge^{\mathcal{B}}$ and $\vee^{\mathcal{B}}$. As $\pi(A)$ is also closed with respect to $0^{\mathcal{B}} = (0, 1)$ and $1^{\mathcal{B}} = (1, 0)$, we have $\pi(\mathcal{A}) \subseteq \mathcal{A} \times \mathcal{A}^\circ$. As bounded distributive lattices have equational axioms, this proves that $\pi(\mathcal{A})$, forgetting n and $\neg$ is a bounded distributive lattice. (The same can be proved for $\varrho(\mathcal{A})$).

Now we show that $\pi(\mathcal{A})$ is a DMF-algebra: in fact, $\neg$ is an involutive dual automorphism, because

$$\neg\neg(a, b) = (a, b)$$

and

$$\neg((a, b) \wedge^{\mathcal{B}} (c, d)) = \neg(a \wedge c, b \vee d) = (b \vee d, a \wedge c) =$$

$$(b, d) \vee^{\mathcal{B}} (a, c) = \neg(a, b) \vee^{\mathcal{B}} \neg(c, d).$$

Of course we have $\neg n = \neg(0, 0) = (0, 0) = n$. As for normality axiom $x \wedge \neg x \leqslant y \vee \neg y$, we must show that

$$(a, b) \wedge^{\mathcal{B}} (b, a) \leqslant (c, d) \vee^{\mathcal{B}} (d, c).$$

If we remember that $a \wedge b = c \wedge d = 0$, we have

$$(a \wedge b, b \vee a) \wedge^{\mathcal{B}} (c \vee d, d \wedge c) = (0, b \vee a) \wedge^{\mathcal{B}} (c \vee d, 0) =$$

$$((c \vee d) \wedge 0, (b \vee a) \vee 0) = (0, b \vee a) = (a \wedge b, b \vee a).$$

(In the same way we can prove that $\varrho(\mathcal{A})$ is a DMF-algebra.) The following theorem shows that π-construction preserves morphisms.

THEOREM 2.2. – *Let* $\mathcal{A}, \mathcal{B} \in DL_{0,1}$. *If* $\varphi: \mathcal{A} \to \mathcal{B}$ *is a* $DL_{0,1}$*-monomorphism (isomorphism), then there is a DMF-monomorphism (isomorphism)* $\psi: \pi(\mathcal{A}) \to \pi(\mathcal{B})$. *If* φ *is* id_A *then* ψ *is* $id_{\pi(A)}$.

We set $\psi(x, y) = (\varphi(x), \varphi(y))$. Firstly we verify that $\psi(x, y)$ belongs to $\pi(B)$, i.e. $\varphi(x) \wedge^{\mathcal{B}} \varphi(y) = 0^{\mathcal{B}}$. As $(x, y) \in \pi(A)$, we have $x \wedge^{\mathcal{A}} y = 0^{\mathcal{A}}$, so $\varphi(x) \wedge^{\mathcal{B}} \varphi(y) = \varphi(x \wedge^{\mathcal{A}} y) = \varphi(0^{\mathcal{A}}) = 0^{\mathcal{B}}$, because φ is a morphism. ψ is injective because φ is injective. Let $\wedge^{\pi_{\mathcal{A}}}$ and $\wedge^{\pi_{\mathcal{B}}}$ denote respectively meet in $\pi(\mathcal{A})$ and $\pi(\mathcal{B})$. ψ preserves meets:

$$\psi((x, y) \wedge^{\pi_{\mathcal{A}}} (z, w)) = \psi(x \wedge^{\mathcal{A}} z, y \vee^{\mathcal{A}} w) = (\varphi(x \wedge^{\mathcal{A}} z), \varphi(y \vee^{\mathcal{A}} w)) =$$

$$(\varphi(x) \wedge^{\mathcal{B}} \varphi(z), \varphi(y) \vee^{\mathcal{B}} \varphi(w)) = (\varphi(x), \varphi(y)) \wedge^{\pi_{\mathcal{B}}} (\varphi(z), \varphi(w)) =$$

$$\psi(x, y) \wedge^{\pi_{\mathcal{B}}} \psi(z, w).$$

In the same way we can prove that ψ preserves the other operations and constants.

In every DMF-algebra $\mathcal{A}$ we can define two sublattices $\Delta_{\mathcal{A}}$ and $\nabla_{\mathcal{A}}$ as follows:

$$\Delta_{\mathcal{A}} = \{x \wedge \neg x : x \in A\} \quad \text{and} \quad \nabla_{\mathcal{A}} = \{x \vee \neg x : x \in A\}.$$

From [4, 3.3, 3.4] we know that, for all $x \in \Delta_{\mathcal{A}}$, $x \leqslant n$, and for all $x \in \nabla_{\mathcal{A}}$, $n \leqslant x$. So $[0, n] = \Delta_{\mathcal{A}}$ and $[n, 1] = \nabla_{\mathcal{A}}$, then $\Delta_{\mathcal{A}}$ and $\nabla_{\mathcal{A}}$ are bounded distributive lattices. However they are only sublattices of $\mathcal{A}$, and not bounded sublattices, because $0^{\nabla_{\mathcal{A}}} = n$ and $1^{\Delta_{\mathcal{A}}} = n$. We observe that $\nabla_{\mathcal{A}} \cup \Delta_{\mathcal{A}}$ is a subalgebra of $\mathcal{A}$ and then a DMF-algebra.

We can classify DMF-algebras in equivalence classes setting $\mathcal{A} \sim \mathcal{B}$ iff $\nabla_{\mathcal{A}}$ and $\nabla_{\mathcal{B}}$ are isomorphic as bounded distributive lattices. In other words, DMF-algebras sharing the same $[n, 1]$ interval are equivalent. In every class $|\mathcal{A}_{\sim}|$

there is an «initial» element $\nabla_{\mathcal{A}} \cup \Delta_{\mathcal{A}}$ that can be embedded in every other element and there is a «final» element $\pi(\nabla_{\mathcal{A}})$ in which any other element can be embedded. The first assertion is clear while the second assertion is proved by the following two theorems.

THEOREM 2.3. – *If $\mathcal{A}$ is a DMF-algebra, then there is a monomorphism $\varphi\colon \mathcal{A} \to \pi(\nabla_{\mathcal{A}})$.*

We define $\varphi\colon \mathcal{A} \to \pi(\nabla_{\mathcal{A}})$ setting $\varphi(x) = (x \vee n,\ \neg x \vee n)$, where $\vee$ and $\neg$ are taken in $\mathcal{A}$. In general operations and constants without superscripts are supposed to be in $\mathcal{A}$. As $\nabla_{\mathcal{A}} \subseteq \mathcal{A}$ as lattices, we write $\wedge$ and $\vee$ instead of $\wedge^{\nabla_{\mathcal{A}}}$ and $\vee^{\nabla_{\mathcal{A}}}$ when no confusion is possible.

Firstly we show that $\varphi(x) \in \pi(\nabla_{\mathcal{A}})$. As $n \leqslant x \vee n$ and $n \leqslant \neg x \vee n$, we have $x \vee n,\ \neg x \vee n \in \nabla_{\mathcal{A}}$, so we have only to verify that

$$(x \vee n) \wedge^{\nabla_{\mathcal{A}}} (\neg x \vee n) = 0^{\nabla_{\mathcal{A}}},$$

remembering that $0^{\nabla_{\mathcal{A}}} = n$. In fact, dropping all superscripts because every operation can be taken in $\mathcal{A}$, we have

$$(x \vee n) \wedge (\neg x \vee n) = ((x \vee n) \wedge \neg x) \vee ((x \vee n) \wedge n) = ((x \vee n) \wedge \neg x) \vee n =$$
$$= (x \wedge \neg x) \vee (n \wedge \neg x) \vee n = (x \wedge \neg x) \vee n.$$

By [4, 3.3, 3.4] we have $x \wedge \neg x \leqslant n$ for every x so $(x \wedge \neg x) \vee n = n$. We show that φ is injective. If $\varphi(x) = \varphi(y)$ then

$$(x \vee n,\ \neg x \vee n) = (y \vee n,\ \neg y \vee n),$$

so

$$x \vee n = y \vee n \tag{1}$$

and $\neg x \vee n = \neg y \vee n$. As $n = \neg n$, we have $\neg x \vee \neg n = \neg y \vee \neg n$, so $\neg(x \wedge n) = \neg(y \wedge n)$ and then $\neg\neg(x \wedge n) = \neg\neg(y \wedge n)$, so

$$x \wedge n = y \wedge n. \tag{2}$$

Then we have

$$x = (x \wedge n) \vee x$$
$$\text{by (2)} = (y \wedge n) \vee x = (y \vee x) \wedge (n \vee x)$$
$$\text{by (1)} = (y \vee x) \wedge (y \vee n) = y \vee (x \wedge n)$$
$$\text{by (2)} = y \vee (y \wedge n) = y.$$

We denote with $\wedge^{\pi}$, $\vee^{\pi}$ and $\neg^{\pi}$ the operations in $\pi(\nabla_{\mathcal{A}})$ and show that φ preserves meets:

$$\varphi(x \wedge y) = ((x \wedge y) \vee n,\ \neg(x \wedge y) \vee n) = ((x \wedge y) \vee n,\ \neg x \vee \neg y \vee n) =$$
$$((x \vee n) \wedge (y \vee n),\ \neg x \vee n \vee \neg y \vee n) =$$
$$(x \vee n,\ \neg x \vee n) \wedge^{\pi} (y \vee n,\ \neg y \vee n) = \varphi(x) \wedge^{\pi} \varphi(y).$$

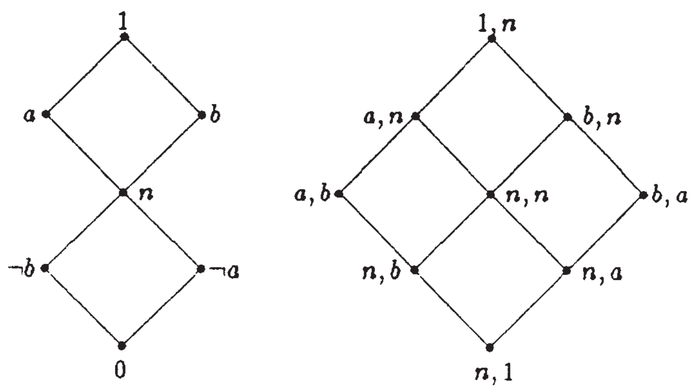
In the same way we can prove that φ preserves joins. As for negation:

$$\varphi(\neg x) = (\neg x \vee n,\ \neg\neg x \vee n) = (\neg x \vee n,\ x \vee n) =$$
$$\neg^{\pi}(x \vee n,\ \neg x \vee n) = \neg^{\pi}(\varphi(x)).$$

Finally φ preserves 0, 1 and n:

$$\varphi(0) = (n,\ 1) = (0^{\nabla_{\mathcal{A}}},\ 1^{\nabla_{\mathcal{A}}}) = 0^{\pi},$$
$$\varphi(1) = (1,\ n) = (1^{\nabla_{\mathcal{A}}},\ 0^{\nabla_{\mathcal{A}}}) = 1^{\pi},$$
$$\varphi(n) = (n,\ n) = (0^{\nabla_{\mathcal{A}}},\ 0^{\nabla_{\mathcal{A}}}) = n^{\pi}.$$

The following picture shows $\mathcal{A}$ and $\pi(\nabla_{\mathcal{A}})$ in a simple case.



THEOREM 2.4. – *If* $\mathcal{A}$ *is a DMF-algebra, then* $\pi(\nabla_{\mathcal{A}}) \in |\mathcal{A}|_{\sim}$.

Let φ be as in the above theorem. We show that φ is an isomorphism between $\nabla_{\mathcal{A}}$ and $\nabla_{\pi(\nabla_{\mathcal{A}})}$. As $\varphi\colon \mathcal{A} \to \pi(\nabla_{\mathcal{A}})$ is a monomorphism by the above theorem, we have only to show that φ takes $\nabla_{\mathcal{A}}$ onto $\nabla_{\pi(\nabla_{\mathcal{A}})}$. Let $(a, a') \in \nabla_{\pi(\nabla_{\mathcal{A}})}$, then $n^{\pi(\nabla_{\mathcal{A}})} = (n, n) \leqslant (a, a')$ because $\pi(\nabla_{\mathcal{A}})$ is a DMF-algebra and every element of $\nabla_{\pi(\nabla_{\mathcal{A}})}$ dominates $n^{\pi(\nabla_{\mathcal{A}})}$. So $n \leqslant a$ and $a' \leqslant n$. From $n \leqslant a$ we have $a = a \vee n$ and $\neg a \leqslant \neg n = n$, so $n = \neg a \vee n$. As (a, a') is an element of $\pi(\nabla_{\mathcal{A}})$, we have $a \in \nabla_{\mathcal{A}}$ and $a' \in \nabla_{\mathcal{A}}$. As $\mathcal{A}$ is a DMF-algebra we have $n \leqslant a'$, so from $a' \leqslant n$

we have $a' = n$. Then we have

$$(a, a') = (a, n) = (a \vee n, \neg a \vee n) = \varphi(a),$$

where $a \in \nabla_{\mathcal{A}}$. A theorem like 2.3 can be proved with $\varrho(\Delta_{\mathcal{A}})$ setting $\varphi(x) = (x \wedge n, \neg x \wedge n)$. This is not, however, a new embedding, because we can show that $\pi(\nabla_{\mathcal{A}})$ and $\varrho(\Delta_{\mathcal{A}})$ are isomorphic.

THEOREM 2.5. – *If $\mathcal{A}$ is a DMF-algebra, then $\pi(\nabla_{\mathcal{A}}) \simeq \varrho(\Delta_{\mathcal{A}})$.*

Firstly we observe that

$$\begin{aligned}(a, b) \in \pi(\nabla_{\mathcal{A}}) \quad & \text{iff } a, b \in \nabla_{\mathcal{A}} \text{ and } a \wedge b = 0^{\nabla_{\mathcal{A}}} = n, \\ & \text{iff } \neg a, \neg b \in \Delta_{\mathcal{A}} \text{ and } \neg a \vee \neg b = 1^{\Delta_{\mathcal{A}}} = n, \\ & \text{iff } (\neg a, \neg b) \in \varrho(\Delta_{\mathcal{A}}).\end{aligned}$$

So we can define $\varphi\colon \pi(\nabla_{\mathcal{A}}) \to \varrho(\Delta_{\mathcal{A}})$ setting $\varphi(a, b) = (\neg b, \neg a)$. As $\neg$ is a bijection between $\nabla_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}$, φ is a bijection too. φ preserves $\neg^{\pi}$:

$$\varphi(\neg^{\pi}(a, b)) = \varphi(b, a) = (\neg a, \neg b) = \neg^{\varrho}(\varphi(a, b)).$$

φ preserves $\wedge^{\pi}$:

$$\varphi((a, b) \wedge^{\pi} (c, d)) = \varphi(a \wedge c, b \vee d) = (\neg(b \vee d), \neg(a \wedge c)) =$$
$$(\neg b \wedge \neg d, \neg a \vee \neg c) = ((\neg b, \neg a) \wedge^{\varrho} (\neg d, \neg c)) = \varphi(a, b) \wedge^{\varrho} \varphi(c, d).$$

In the same way we show that φ preserves the others operations and constants.

We have shown that every DMF-algebra $\mathcal{A}$ can be embedded in $\pi(\mathcal{B})$, for some $\mathcal{B} \in DL_{0,1}$, in particular $\mathcal{A}$ can be embedded in $\pi(\nabla_{\mathcal{A}})$, but we cannot prove in general that any DMF-algebra $\mathcal{A}$ is isomorphic to $\pi(\mathcal{B})$, for some $\mathcal{B} \in DL_{0,1}$. The following axiom

$$\forall xy(x \wedge y = n \to \exists z(z \vee n = x \,\&\, \neg z \vee n = y)) \tag{2.3}$$

characterizes those DMF-algebras which can be obtained from bounded distributive lattices by a π-construction, as the following theorem shows.

THEOREM 2.6. – *If $\mathcal{A}$ is a DMF-algebra, the the following propositions are equivalent*:

1) $\mathcal{A} \simeq \pi(\mathcal{B})$, *for some* $\mathcal{B} \in DL_{0,1}$,
2) $\forall xy(x \wedge y = n \to \exists z(z \vee n = x \,\&\, \neg z \vee n = y))$,
3) $\mathcal{A} \simeq \pi(\nabla_{\mathcal{A}})$.

1) implies 2). We show that 2) holds in $\pi(\mathcal{B})$, for any $\mathcal{B} \in DL_{0,1}$. Let

$(b, b'), (c, c') \in \pi(B)$. We suppose $(b, b') \wedge (c, c') = n = (0, 0)$, where 0 is $0^{\mathcal{B}}$, so $b \wedge c = 0$ and $b' \vee c' = 0$. As $b \wedge c = 0$, we have $(b, c) \in \pi(B)$. We show that we can set $z = (b, c)$. In fact,

$$(b, c) \vee n = (b \vee 0, c \wedge 0) = (b, 0) \quad \text{and} \quad \neg (b, c) \vee n = (c \vee 0, b \wedge 0) = (c, 0).$$

But from $b' \vee c' = 0$ we have $b' = 0 = c'$, so $(b, c) \vee n = (b, b')$ and $\neg (b, c) \vee n = (c, c')$.

2) implies 3). By Theorem 2.3 there is a monomorphism $\varphi: \mathcal{A} \to \pi(\nabla_{\mathcal{A}})$, where $\varphi(x) = (x \vee n, \neg x \vee n)$. We show that if 2) holds then φ is onto. If $(a, a') \in \pi(\nabla_{\mathcal{A}})$ then $a, a' \in \nabla_{\mathcal{A}}$ and $a \wedge a' = 0^{\nabla_{\mathcal{A}}} = n$. By 2) there is a $z \in A$ such that $z \vee n = a$ and $\neg z \vee n = b'$, so $\varphi(z) = (a, a')$.

3) implies 1). We can set $\mathcal{B} = \nabla_{\mathcal{A}}$ because $\nabla_{\mathcal{A}} \in DL_{0,1}$.

If $\mathcal{A}$ is the DMF-algebra shown on the left side of the preceding figure, then there is no $\mathcal{B} \in DL_{0,1}$ such that $\mathcal{A} \simeq \pi(\mathcal{B})$, because $a \wedge b = n$ holds in $\mathcal{A}$, yet there is no z such that $a = z \vee n$ e $b = \neg z \vee n$. In particular, $\mathcal{A} \not\simeq \pi(\nabla_{\mathcal{A}})$.

We conclude this paragraph with some remarks about fields of partial sets. We can see that $\mathcal{D}(X)$, the field of all partial sets on X, arises from a particular case of π-construction, because $\mathcal{D}(X) = \pi(\mathcal{R}(X))$, where $\mathcal{R}(X)$ is the ring of all sets on X. From Theorem 2.2 we see that $\mathcal{A} \subseteq \mathcal{R}(X)$ implies $\pi(\mathcal{A}) \subseteq \pi(\mathcal{R}(X))$, so $\pi(\mathcal{A})$ is a field of partial sets whenever $\mathcal{A}$ is a ring of sets. We can ask if every field of partial sets arises from a ring of sets by a π-construction, but we can easily see that it happens iff (3) is satisfied. If $\mathcal{A}$ is a field of partial sets on X satisfying (3), then $\mathcal{A} \simeq \pi(\nabla_{\mathcal{A}})$ and $\nabla_{\mathcal{A}}$ is isomorphic to the ring of set $\{Z \subseteq X: (Z, \emptyset) \in \nabla_{\mathcal{A}}\}$. On the other side, if $\mathcal{A} \simeq \pi(\mathcal{B})$, for some ring of sets $\mathcal{B}$, then $\mathcal{A}$ satisfies (3).

3. – The representation theorem.

From Stone representation theorem we know that every $\mathcal{A} \in DL_{0,1}$ is isomorphic to a ring of sets. In particular Stone theorem gives an embedding $\theta: \mathcal{A} \to R(FP(\mathcal{A}))$, where $FP(\mathcal{A})$ is the set of prime filters in $\mathcal{A}$ and $\theta(a) = \{X \in FP(\mathcal{A}): a \in X\}$.

THEOREM 3.1. – *For any $\mathcal{A} \in DL_{0,1}$, the DMF-algebra $\pi(\mathcal{A})$ is isomorphic to a field of partial sets on $FP(\mathcal{A})$.*

If θ is the Stone embedding, by Theorem 2.2 there is an embedding $\psi: \pi(\mathcal{A}) \to \pi(R(FP(\mathcal{A})))$ where $\psi(x, y) = (\theta(x), \theta(y))$. As $\pi(R(FP(\mathcal{A}))) = \mathcal{D}(FP(\mathcal{A}))$, we have shown that $\pi(\mathcal{A})$ is isomorphic to a subalgebra of

$\mathcal{P}(FP(\mathcal{A}))$, the field of all partial sets on $FP(\mathcal{A})$, so $\pi(\nabla_{\mathcal{A}})$ is isomorphic to a field of partial sets on $FP(\mathcal{A})$.

COROLLARY 3.2. – *Every DMF-algebra $\mathcal{A}$ is isomorphic to a field of partial sets on $FP(\nabla_{\mathcal{A}})$.*

By Theorem 2.3 there is a monomorphism $\varphi\colon \mathcal{A} \to \pi(\nabla_{\mathcal{A}})$. By the preceding theorem there is a monomorphism $\psi\colon \pi(\nabla_{\mathcal{A}}) \to \mathcal{P}(FP(\mathcal{A}))$.

We take a closer look at the images of DMF-algebras as fields of partial sets, because they will play a role in what follows as bases of topological spaces. For any DMF-algebra $\mathcal{A}$, we denote with $\mathcal{A}^*$ the isomorphic image of $\mathcal{A}$ as a field of partial sets given by Corollary 3.2, so we have

$$\mathcal{A}^* = \psi \circ \varphi[\mathcal{A}] = \psi[\{(a \vee n,\ \neg a \vee n)\colon a \in A\}] = \{(\theta(a \vee n),\ \theta(\neg a \vee n))\colon a \in A\}. \tag{4}$$

When $\mathcal{A}$ is $\pi(\nabla_{\mathcal{A}})$ we have simply

$$\pi(\nabla_{\mathcal{A}})^* = \psi[\pi(\nabla_{\mathcal{A}})] = \{(\theta(x),\ \theta(y))\colon x \in \nabla_{\mathcal{A}},\ x \wedge y = n\} \tag{5}$$

by Theorem 3.1. (We can obtain the same result by applying the above corollary. In fact $\pi(\nabla_{\mathcal{A}}) \simeq \pi(\nabla_{\pi(\nabla_{\mathcal{A}})})$ by (3) of Theorem 2.6 and in particular φ is an isomorphism, so $\pi(\nabla_{\pi(\nabla_{\mathcal{A}})}) = \varphi[\pi(\nabla_{\mathcal{A}})]$. By Corollary 3.2 we have $\pi(\nabla_{\mathcal{A}})^* = \psi \circ \varphi[\pi(\nabla_{\mathcal{A}})] = \psi[\pi(\nabla_{\mathcal{A}})]$ because we can identifiy isomorphic structures.) As $\mathcal{A}$ is represented through a previous embedding in $\pi(\nabla_{\mathcal{A}})$, we have $\mathcal{A}^* \subseteq \pi(\nabla_{\mathcal{A}})^*$.

For any set A and $Z \subseteq A \times A$, we denote with $(Z)_0$ the set of all *left components*, $(Z)_0 = \{a \in A\colon (a,\ b) \in Z$, for some $b \in A\}$, and with $(Z)_1$ the set of all *right components*, $(Z)_1 = \{a \in A\colon (b,\ a) \in Z$, for some $b \in A\}$. The following theorem shows that left and right components of the representation of a DMF-algebra $\mathcal{A}$ coincide with the Stone representation $\theta[\nabla_{\mathcal{A}}]$ of $\nabla_{\mathcal{A}}$.

THEOREM 3.3. – *Let $\mathcal{F}$ be any field of partial sets on X, then*

1) $(\mathcal{F})_0 = (\mathcal{F})_1$.

2) *Both $(\mathcal{F})_0$ and $(\mathcal{F})_1$ are rings of sets on X.*

3) $(\pi(\nabla_{\mathcal{A}})^*)_0 = (\pi(\nabla_{\mathcal{A}})^*)_1 = (\mathcal{A}^*)_0 = (\mathcal{A}^*)_1 = \theta[\nabla_{\mathcal{A}}]$, *for any DMF-algebra $\mathcal{A}$.*

1) If $A \in (\mathcal{F})_0$ then there is $B \subseteq X$ such that $(A,\ B) \in \mathcal{F}$, so $\neg(A,\ B) = (B,\ A) \in \mathcal{F}$ and $A \in (\mathcal{F})_1$. In the same way we prove that $(\mathcal{F})_1$ is included in $(\mathcal{F})_0$.

2) As $(X, \emptyset)$ and $(\emptyset, X)$ belong to $\mathcal{F}$, X and $\emptyset$ belong to $(\mathcal{F})_0$. If $A, B \in (\mathcal{F})_0$ then, for some $A', B' \subseteq X$, (A, A') and (B, B') belong to $\mathcal{F}$, so $(A \cap B, A' \cup B')$ and $(A \cup B, A' \cap B')$ belong to $\mathcal{F}$ and then $(A \cap B)$ and $(A \cup B)$ belong to $(\mathcal{F})_0$. As $(\mathcal{F})_0 = (\mathcal{F})_1$ by 1), $(\mathcal{F})_1$ is a ring of sets too.

3) By 1) we have $(\pi(\nabla_{\mathcal{A}})^*)_0 = (\pi(\nabla_{\mathcal{A}})^*)_1$ and $(\mathcal{A}^*)_0 = (\mathcal{A}^*)_1$. Firstly we prove $(\pi(\nabla_{\mathcal{A}})^*)_0 = \theta[\nabla_{\mathcal{A}}]$. Obviously $(\pi(\nabla_{\mathcal{A}})^*)_0 \subseteq \theta[\nabla_{\mathcal{A}}]$ by (5) above. The inclusion $\theta[\nabla_{\mathcal{A}}] \subseteq (\pi(\nabla_{\mathcal{A}})^*)_0$ follows by observing that, for all $x \in \nabla_{\mathcal{A}}$, we have $x \wedge n = n$, so $(\theta(x), \theta(n)) \in \pi(\nabla_{\mathcal{A}})^*$ by (5) above. Finally we prove $(\mathcal{A}^*)_0 = \theta[\nabla_{\mathcal{A}}]$. On one side we have $(\mathcal{A}^*)_0 \subseteq \theta[\nabla_{\mathcal{A}}]$, because $\mathcal{A}^* \subseteq \pi(\nabla_{\mathcal{A}})^*$, so $(\mathcal{A}^*)_0 \subseteq (\pi(\nabla_{\mathcal{A}})^*)_0$ and by the first part of the proof $(\pi(\nabla_{\mathcal{A}})^*)_0 = \theta[\nabla_{\mathcal{A}}]$. On the other side, let $\theta(a) \in \theta[\nabla_{\mathcal{A}}]$, then $\theta(a) = \theta(a \vee n)$ because $a \in \nabla_{\mathcal{A}}$ and $a = a \vee n$. But $\theta(a \vee n) \in (\mathcal{A}^*)_0$ by (4), so $\theta[\nabla_{\mathcal{A}}] \subseteq (\mathcal{A}^*)_0$.

4. – Partial topologies.

In Stone representation theory, for any $\mathcal{A} \in DL_{0,1}$ a topological space $(X, \mathcal{T})$ is constructed, where $X = FP(\mathcal{A})$ and $\mathcal{T}$ is the topology generated by taking the Stone representation $\theta[\mathcal{A}]$ as a base. Next Stone showed that $\mathcal{A}$ can be characterized as the set of all compact open sets in $(X, \mathcal{T})$. In this paragraph we develop partial topology, i.e. topology based on partial sets, to prove a similar result.

Firstly we introduce infinitary joins and meets in DMF-algebras. If $\mathcal{A}$ is a complete lattice, we introduce the operation $\bigwedge^{\pi}$ in $\pi(\nabla_{\mathcal{A}})$ setting, for all $Z \subseteq \pi(A)$,

$$\bigwedge^{\pi}(Z) = \left(\bigwedge^{\mathcal{A}} ((Z)_0), \bigvee^{\mathcal{A}} ((Z)_1) \right).$$

The set Z can also be given as an indexed set $\{(a_i, b_i): i \in I\}$: in this case we write

$$\bigwedge^{\pi} (\{(a_i, b_i): i \in I\}) = \left(\bigwedge^{\mathcal{A}} \{a_i: i \in I\}, \bigvee^{\mathcal{A}} \{b_i: i \in I\} \right).$$

Infinitary joins can be introduced in the same way. When no confusion can arise, we simplify notation, dropping exponents and omitting any explicit mention of the index set I, and simply write $\{a_i\}$ instead of $\{a_i: i \in I\}$.

Infinitary intersection and union can be treated as particular cases arising when $\mathcal{A}$ is a complete set lattice. In any complete field of partial sets on X we have in particular

$$\overset{\pi}{\bigcap}\emptyset = \left(\bigcap\emptyset, \bigcup\emptyset\right) = (X, \emptyset) \quad \text{and} \quad \overset{\pi}{\bigcup}\emptyset = \left(\bigcup\emptyset, \bigcap\emptyset\right) = (\emptyset, X)$$

in analogy with classical infinitary operations.

Now we can define a *partial topology* on X as a class $\mathcal{T} \subseteq \wp(X)$ such that: 1) $\mathcal{T}$ is closed under finitary intersections, 2) $\mathcal{T}$ is closed under infinitary unions and 3) $n \in \mathcal{T}$ (i.e. $(\emptyset, \emptyset) \in \mathcal{T}$). Observe that we have $(X, \emptyset) \in \mathcal{T}$ as the result of an empty intersection and $(\emptyset, X) \in \mathcal{T}$ as the result of an empty union. A *partial topological space on* X is a couple $(X, \mathcal{T})$ where $\mathcal{T}$ is a partial topology on X. The elements of $\mathcal{T}$ are called *open* sets. A *closed* set is a (partial) set (A, B) such that $(B, A) \in \mathcal{T}$. As usual a *clopen* set is a simultaneously open and closed set: $(X, \emptyset)$, $(\emptyset, X)$ and $(\emptyset, \emptyset)$ are always clopen. It can be easily seen that the clopens of a topological space on X are a field of partial sets on X.

A *base* for a partial topology $\mathcal{T}$ on X is a $\mathcal{B} \subseteq \mathcal{T}$ such that every open set in $\mathcal{T}$ is $\bigcup\mathcal{B}'$, for some $\mathcal{B}' \subseteq \mathcal{B}$. For any class of sets $\mathcal{B}$ we set $\mathcal{T}(\mathcal{B}) = \{\bigcup\mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B}\}$. (The notation $\mathcal{T}(\mathcal{B})$ will denote a class of partial or classical sets when the elements of $\mathcal{B}$ are respectively partial or classical sets.) The following theorem gives a sufficient condition on $\mathcal{B}$ for $\mathcal{T}(\mathcal{B})$ being a partial topology.

THEOREM 4.1. – *For any $\mathcal{B} \subseteq \wp(X)$, if $\mathcal{B}$ has the fip (finite intersection property) and $n \in \mathcal{B}$, then $\mathcal{B}$ is a base of $\mathcal{T}(\mathcal{B})$.*

We only observe that $(\emptyset, X) = \bigcap\emptyset$ and $(X, \emptyset) = \bigcup\emptyset$ belong to $\mathcal{T}(\mathcal{B})$. The proof is similar to the classical case.

We can easily obtain partial topological spaces from classical ones as follows. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be classical topologies on X, we define

$$[\mathcal{T}_0, \mathcal{T}_1] = \{(A, B): A \in \mathcal{T}_0,\ -B \in \mathcal{T}_1, A \cap B = \emptyset\},$$

the set of all disjoint couples ($\mathcal{T}_0$-open, $\mathcal{T}_1$-closed).

THEOREM 4.2. – *$[\mathcal{T}_0, \mathcal{T}_1]$ is a partial topology on X.*

Obviously $(X, \emptyset)$, $(\emptyset, X)$, $(\emptyset, \emptyset) \in [\mathcal{T}_0, \mathcal{T}_1]$. Closure under finite intersections: if (A, B), $(A', B') \in [\mathcal{T}_0, \mathcal{T}_1]$ then A, $A' \in \mathcal{T}_0$ and $-B$, $-B' \in \mathcal{T}_1$, so $A \cap A' \in \mathcal{T}_0$ and $-B \cap -B' = -(B \cup B') \in \mathcal{T}_1$. So $(A \cap A', B \cup B') = (A, B) \cap (A', B') \in [\mathcal{T}_0, \mathcal{T}_1]$. Closure under infinite unions: let $\{(A_i^0, A_i^1)\}$ be a sequence where $(A_i^0, A_i^1) \in [\mathcal{T}_0, \mathcal{T}_1]$, for all $i \in I$, then $\{A_i^0\}$ is a sequence where $A_i^0 \in \mathcal{T}_0$, for all $i \in I$, and $\{A_i^1\}$ is a sequence where $-A_i^1 \in \mathcal{T}_1$, for all $i \in I$. Then $\bigcup\{A_i^0\} \in \mathcal{T}_0$ and $-\bigcap\{A_i^1\} = \bigcup\{-A_i^1\} \in \mathcal{T}_1$. So $(\bigcup\{A_i^0\}, \bigcap\{A_i^1\}) = \bigcup\{(A_i^0, A_i^1)\} \in [\mathcal{T}_0, \mathcal{T}_1]$.

In the other direction, we can associate two classical topologies on X to every partial topology $\mathcal{C}$ on X. We define the *left topology* $\mathcal{C}^{\nabla} = (\mathcal{C})_0$, whose open sets are the left components of $\mathcal{C}$, and the *right topology* $\mathcal{C}^{\Delta} = -(\mathcal{C})_1 = \{X - Z\colon Z \in (\mathcal{C})_1\}$, whose open sets are the complements (in X) of the right components of $\mathcal{C}$.

THEOREM 4.3. – *If $\mathcal{C}$ is a partial topology on X, then $\mathcal{C}^{\nabla}$ and $\mathcal{C}^{\Delta}$ are classical topologies on X.*

From $(X, \emptyset) \in \mathcal{C}$ we have $X \in \mathcal{C}^{\nabla}$ and from $(\emptyset, X) \in \mathcal{C}$ we have $\emptyset \in \mathcal{C}^{\nabla}$. Closure under finite intersections: if $Z_0, Z_1 \in \mathcal{C}^{\nabla}$, then $(Z_0, W_0), (Z_1, W_1) \in \mathcal{C}$, for some $W_0, W_1 \subseteq X$, and

$$(Z_0, W_0) \cap (Z_1, W_1) = (Z_0 \cap Z_1, W_0 \cup W_1) \in \mathcal{C}$$

so $Z_0 \cap Z_1 \in \mathcal{C}^{\nabla}$. Closure under infinite unions: let $\{Z_i\colon i \in I\}$ be a family of open sets in $\mathcal{C}^{\nabla}$, then $\{(Z_i, W_i)\colon i \in I\}$ is a family of open sets in $\mathcal{C}$, for some $\{W_i\colon i \in I\}$ with $W_i \subseteq X$, then

$$\bigcup\{(Z_i, W_i)\} = \left(\bigcup\{Z_i\}, \bigcap\{W_i\}\right) \in \mathcal{C}$$

and $\bigcup\{Z_i\} \in \mathcal{C}^{\nabla}$. This proves that $\mathcal{C}^{\nabla}$ is a classical topology on X.

As $(X, \emptyset) \in \mathcal{C}$, $\emptyset \in (\mathcal{C})_1$ so $-\emptyset = X \in \mathcal{C}^{\Delta}$ by definition of $\mathcal{C}^{\Delta}$. From $(\emptyset, X) \in \mathcal{C}$ we have $X \in (\mathcal{C})_1$ so $-X = \emptyset \in \mathcal{C}^{\Delta}$. Closure under finite intersections: if W_0, $W_1 \in \mathcal{C}^{\Delta}$, then $(Z_0, -W_0), (Z_1, -W_1) \in \mathcal{C}$, for some $Z_0, Z_1 \subseteq X$, and

$$(Z_0, -W_0) \cap (Z_1, -W_1) = (Z_0 \cap Z_1, -(W_0 \cap W_1)) \in \mathcal{C}$$

so $W_0 \cap W_1 \in \mathcal{C}^{\Delta}$. Closure under infinite unions: let $\{W_i\colon i \in I\}$ be a family of open sets in $\mathcal{C}^{\Delta}$, then $\{(Z_i, -W_i)\colon i \in I\}$ is a family of open sets in $\mathcal{C}$ and

$$\bigcup\{(Z_i, -W_i)\} = \left(\bigcup\{Z_i\}, -\bigcup\{W_i\}\right) \in \mathcal{C}$$

so $\bigcup\{W_i\} \in \mathcal{C}^{\Delta}$. This proves that $\mathcal{C}^{\Delta}$ is a classical topology on X.

The following theorem shows that we can extract classical bases for the classical topologies $\mathcal{C}(\mathcal{B})^{\nabla}$ and $\mathcal{C}(\mathcal{B})^{\Delta}$ from the partial base $\mathcal{B}$ of the partial topology $\mathcal{C}(\mathcal{B})$.

THEOREM 4.4. – *If $\mathcal{C}(\mathcal{B})$ is a partial topology on X having $\mathcal{B}$ as a base, then $\mathcal{C}(\mathcal{B})^{\nabla} = \mathcal{C}((\mathcal{B})_0)$ and $\mathcal{C}(\mathcal{B})^{\Delta} = \mathcal{C}(-(\mathcal{B})_1)$.*

If $Z \in \mathcal{C}(\mathcal{B})^{\nabla}$, then $(Z, W) \in \mathcal{C}(\mathcal{B})$, for some $W \subseteq X$. As $\mathcal{B}$ is a base, $(Z, W) = \bigcup\{(B_i^0, B_i^1)\}$ where $(B_i^0, B_i^1) \in \mathcal{B}$ for all $i \in I$. Then $Z = \bigcup\{B_i^0\}$, where $B_i^0 \in (\mathcal{B})_0$, so $Z \in \mathcal{C}((\mathcal{B})_0)$.

If $Z \in \mathcal{C}((\mathcal{B})_0)$, then $Z = \bigcup\{B_i^0\}$, where $B_i^0 \in (\mathcal{B})_0$ for all $i \in I$. Then there are $B_i^1 \subseteq X$ such that $(B_i^0, B_i^1) \in \mathcal{B}$ for all $i \in I$. So $\bigcup\{(B_i^0, B_i^1)\} \in \mathcal{T}(\mathcal{B})$ and $\bigcup\{B_i^0\} = Z$ belongs to $\mathcal{T}(\mathcal{B})^\nabla$.

If $W \in \mathcal{T}(\mathcal{B})^\Delta$, then $(Z, -W) \in \mathcal{T}(\mathcal{B})$, for some $Z \subseteq X$. Then $(Z, -W) = \bigcup\{(B_i^0, B_i^1)\}$ where $(B_i^0, B_i^1) \in \mathcal{B}$ for all $i \in I$. Then $-W = \bigcap\{B_i^1\}$, where $B_i^1 \in (\mathcal{B})_1$, so $W = \bigcup\{-B_i^1\}$ and $W \in \mathcal{T}(-(\mathcal{B})_1)$.

If $W \in \mathcal{T}(-(\mathcal{B})_1)$, then $W = \bigcup\{-B_i^1\}$, where $B_i^1 \in (\mathcal{B})_1$ for all $i \in I$. Then there are $B_i^0 \subseteq X$ such that $(B_i^0, B_i^1) \in \mathcal{B}$, for all $i \in I$, and $\bigcup\{(B_i^0, B_i^1)\} = (\bigcup\{B_i^0\}, \bigcap\{B_i^1\}) \in \mathcal{T}(\mathcal{B})$. So we have $W = \bigcup\{-B_i^1\} = -\bigcap\{B_i^1\} \in \mathcal{T}(\mathcal{B})^\Delta$.

If we try to obtain a partial topology $\mathcal{T}$ from its left and right topologies $\mathcal{T}^\nabla$ and $\mathcal{T}^\Delta$, we find that $[\mathcal{T}^\nabla, \mathcal{T}^\Delta]$ is generally finer than $\mathcal{T}$.

THEOREM 4.5. – *For any partial topology* $\mathcal{T}$, $\mathcal{T} \subseteq [\mathcal{T}^\nabla, \mathcal{T}^\Delta]$.

If $(A, B) \in \mathcal{T}$, then $A \in \mathcal{T}^\nabla$ and $-B \in \mathcal{T}^\Delta$, so $(A, B) \in [\mathcal{T}^\nabla, \mathcal{T}^\Delta]$.

In some cases we have $\mathcal{T} \subset [\mathcal{T}^\nabla, \mathcal{T}^\Delta]$. An example is given by $\mathcal{T} = \{(A, \emptyset): A \subseteq X\} \cup \{(\emptyset, A): A \subseteq X\}$, that can easily seen to be a partial topology on X different from the set of all partial set on X, the discrete partial topology on X. Then both $\mathcal{T}^\nabla$ and $\mathcal{T}^\Delta$ are the discrete classical topology on X and $[\mathcal{T}^\nabla, \mathcal{T}^\Delta] = \mathcal{D}(X)$, the discrete partial topology. At the end of this paragraph we shall give a sufficient condition for $[\mathcal{T}^\nabla, \mathcal{T}^\Delta] = \mathcal{T}$, but this requires a study of compactness in partial spaces.

We can define compactness in partial topology in the same way as in classical topology. Let $(X, \mathcal{T})$ be a partial space, we say that a class $\{(G_i^0, G_i^1)\}$ of partial sets is an *open cover* of a partial set (A, B) if $(G_i^0, G_i^1) \in \mathcal{T}$ for all $i \in I$ and $(A, B) \subseteq \bigcup\{(G_i^0, G_i^1)\}$. A partial set (A, B) is said to be a *compact partial set* if every open cover of (A, B) includes a finite subcover. The whole space is a *compact space* if $(X, \emptyset)$ is a compact partial set.

THEOREM 4.6. – *If A is compact in $(X, \mathcal{T}_0)$, $-B$ is compact in $(X, \mathcal{T}_1)$ and $A \cap B = \emptyset$, then (A, B) is compact in $(X, [\mathcal{T}_0, \mathcal{T}_1])$.*

Let $(A, B) \subseteq \bigcup\{(G_i^0, G_i^1)\}$, where $(G_i^0, G_i^1) \in [\mathcal{T}_0, \mathcal{T}_1]$ for all $i \in I$, then $G_i^0 \in \mathcal{T}_0$ and $-G_i^1 \in \mathcal{T}_1$ for all $i \in I$. Then $A \subseteq \bigcup\{G_i^0\}$ and $\bigcap\{G_i^1\} \subseteq B$ that implies $-B \subseteq \bigcup\{-G_i^1\}$. As A is compact in $(X, \mathcal{T}_0)$ and $-B$ is compact in $(X, \mathcal{T}_1)$, we have

$$A \subseteq G_{i_1}^0 \cup, \ldots, \cup G_{i_n}^0 \quad \text{and} \quad -B \subseteq -G_{j_1}^1 \cup, \ldots, \cup -G_{j_k}^1$$

for some $\{i_1, \ldots, i_n\} \subseteq I$, and some $\{j_1, \ldots, j_k\} \subseteq I$. From the second inclusion we obtain $G_{j_1}^1 \cap, \ldots, \cap G_{j_k}^1 \subseteq B$, so we have

$$((G_{i_1}^0, G_{i_1}^1) \cup, \ldots, \cup (G_{i_n}^0, G_{i_n}^1)) \cup ((G_{j_1}^0, G_{j_1}^1) \cup, \ldots, \cup (G_{j_k}^0, G_{j_k}^1)) \supseteq (A, B).$$

COROLLARY 4.7. – *If* $(X, \mathcal{T}_0)$ *and* $(X, \mathcal{T}_1)$ *are classic compact spaces then* $(X, [\mathcal{T}_0, \mathcal{T}_1])$ *is a partial compact space.*

X is compact in $(X, \mathcal{T}_0)$ and $-\emptyset$ is compact in $(X, \mathcal{T}_1)$, so by the theorem $(X, \emptyset)$ is compact in $(X, [\mathcal{T}_0, \mathcal{T}_1])$.

THEOREM 4.8. – *For any partial topology* $\mathcal{T}$ *on* X, $(X, \mathcal{T})$ *is compact iff* $(X, \mathcal{T}^\nabla)$ *is compact.*

Let $X = \bigcup\{A_i^0\}$, where $A_i^0 \in \mathcal{T}^\nabla$ for all $i \in I$. There are $A_i^1 \subseteq X$ such that $(A_i^0, A_i^1) \in \mathcal{T}$, for all $i \in I$, and

$$\bigcup\{(A_i^0, A_i^1)\} = \left(\bigcup\{A_i^0\}, \bigcap\{A_i^1\}\right) = \left(X, \bigcap\{A_i^1\}\right).$$

As $\left(X, \bigcap\{A_i^1\}\right)$ is a partial set, $\bigcap\{A_i^1\}$ must be $\emptyset$ so $\{(A_i^0, A_i^1)\}$ is an open covering of $(X, \emptyset)$ and by compactness there are $\{i_1, \ldots, i_n\} \subseteq I$ such that $(A_{i_1}^0, A_{i_1}^1) \cup, \ldots, \cup (A_{i_n}^0, A_{i_n}^1) = (X, \emptyset)$, so $A_{i_1}^0 \cup, \ldots, \cup A_{i_n}^0 = X$.

Let $\bigcup\{(A_i^0, A_i^1)\} = (X, \emptyset)$, where $(A_i^0, A_i^1) \in \mathcal{T}$ for all $i \in I$, so $X = \bigcup\{A_i^0\}$. As $(X, \mathcal{T}^\nabla)$ is compact, we have $X = A_{i_1}^0 \cup, \ldots, \cup A_{i_n}^0$ for some $i_1, \ldots, i_n \in I$, so $(A_{i_1}^0, A_{i_1}^1) \cup, \ldots, \cup (A_{i_n}^0, A_{i_n}^1) = (X, Z)$, where $Z = A_{i_1}^1 \cap, \ldots, \cap A_{i_n}^1$. As (X, Z) is a partial set, Z must be $\emptyset$, so $(X, \mathcal{T})$ is compact.

THEOREM 4.9. – *For any partial topology* $\mathcal{T}$ *on* X, n *is compact in* $(X, \mathcal{T})$ *iff* $(X, \mathcal{T}^\Delta)$ *is compact.*

Let $X = \bigcup\{A_i^1\}$, where $A_i^1 \in \mathcal{T}^\Delta$ for all $i \in I$, then $\emptyset = \bigcap\{-A_i^1\}$. For every i there is a subset A_i^0 of X such that $(A_i^0, -A_i^1) \in \mathcal{T}$, so $(A_i^0, -A_i^1) \cap n = (\emptyset, -A_i^1) \in \mathcal{T}$ because n belongs to every partial topology. Then $\bigcup\{(\emptyset, -A_i^1)\} = (\emptyset, \bigcap\{-A_i^1\}) = (\emptyset, \emptyset)$. As n is compact in $(X, \mathcal{T})$ by hypothesis, there are $\{i_1, \ldots, i_n\} \subseteq I$ such that

$$(\emptyset, \emptyset) = (\emptyset, -A_{i_1}^1) \cup, \ldots, \cup (\emptyset, -A_{i_n}^1),$$

then $\emptyset = -(A_{i_1}^1 \cup, \ldots, \cup A_{i_n}^1)$ and $X = (A_{i_1}^1 \cap, \ldots, \cap A_{i_n}^1)$.

Let $(\emptyset, \emptyset) \subseteq \bigcup\{(A_i^0, A_i^1)\}$, where $(A_i^0, A_i^1) \in \mathcal{T}$ for all $i \in I$. Then $\emptyset \subseteq \bigcup\{A_i^0\}$ and $\emptyset = \bigcap\{A_i^1\}$. As all A_i^1 are closed sets of $(X, \mathcal{T}^\Delta)$ and $(X, \mathcal{T}^\Delta)$ is compact, there are $i_1, \ldots, i_n \in I$ such that $A_{i_1}^1 \cap, \ldots, \cap A_{i_n}^1 = \emptyset$. So $(\emptyset, \emptyset) \subseteq (A_{i_1}^0, A_{i_1}^1) \cup, \ldots, \cup (A_{i_n}^0, A_{i_n}^1)$ and $(\emptyset, \emptyset)$ is compact in $(X, \mathcal{T})$.

Now we can exhibit an important point of difference with respect to classical spaces: we cannot any longer prove that closed subsets of compact spaces are compact. Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be classical topologies on an infinite set X such that $(X, \mathcal{T}_0)$ is compact and $(X, \mathcal{T}_1)$ is not compact. (We can suppose, to fix our ideas, that $\mathcal{T}_0$ is the cofinite and $\mathcal{T}_1$ is the discrete topology on X.) Then

$(X, [\mathcal{C}_0, \mathcal{C}_1])$ is compact by Theorem 4.8, but the closed set n is not compact in it. If n were compact then $(X, [\mathcal{C}_0, \mathcal{C}_1]^{\Delta})$ would be a compact space by Theorem 4.9, but $[\mathcal{C}_0, \mathcal{C}_1]^{\Delta} = \mathcal{C}_1$.

As we have seen above, given any partial topology $\mathcal{C}$ we can generate a finer partial topology $[\mathcal{C}^{\nabla}, \mathcal{C}_{\Delta}]$ from its left and right classical topologies. What are the fixed points of this process? When $[\mathcal{C}^{\nabla}, \mathcal{C}_{\Delta}] = \mathcal{C}$? We give a partial solution when $\mathcal{C}$ is generated by a field of partial sets. Firstly we introduce the concept of *full* field of partial sets. If $\mathcal{F}$ is a field of partial sets on X, we say that a subset A of X is a *component* of $\mathcal{F}$ if A is a left or a right component of $\mathcal{F}$, i.e. $A \in (\mathcal{F})_0 \cup (\mathcal{F})_1$. As $\mathcal{F}$ is a field $(\mathcal{F})_0 = (\mathcal{F})_1$, so A is a component of $\mathcal{F}$ iff $A \in (\mathcal{F})_0$ (or $A \in (\mathcal{F})_1$). We say that $\mathcal{F}$ is full if $(A, B) \in \mathcal{F}$ whenever A and B are components of $\mathcal{F}$ such that $A \cap B = \emptyset$.

THEOREM 4.10. – *A field of partial sets $\mathcal{F}$ on X is full iff* (3),

$$\forall xy (x \wedge y = n \rightarrow \exists z (z \vee n = x \,\&\, \neg z \vee n = y)),$$

holds in $\mathcal{F}$.

Firstly we assume that $\mathcal{F}$ is full and prove (3). If $x = (A, B)$, $y = (A', B')$ and $(A, B) \cap (A', B') = (\emptyset, \emptyset)$, then $A \cap A' = \emptyset$ and $B \cup B' = \emptyset$ so $B = B' = \emptyset$. As $\mathcal{F}$ is full and A, A' are components of $\mathcal{F}$ satisfying $A \cap A' = \emptyset$, we have $(A, A') \in \mathcal{F}$. If we set $z = (A, A')$ then (3) holds in $\mathcal{F}$, because $x = (A, \emptyset)$ and $y = (A', \emptyset)$. In the other direction, we suppose that (3) holds and that A and B are components of $\mathcal{F}$, with $A \cap B = \emptyset$. If A is a component of $\mathcal{F}$ then there is $A' \subseteq X$ such that $(A, A') \in \mathcal{F}$, so $(A, A') \cup n = (A, \emptyset) \in \mathcal{F}$. For the same reason we have $(B, \emptyset) \in \mathcal{F}$. As $(A, \emptyset) \cap (B, \emptyset) = n$, by (3) there is a z in $\mathcal{F}$ such that $z \cup n = (A, \emptyset)$ and $\neg z \cup n = (B, \emptyset)$. Let $z = (C, D)$, then $(C, D) \cup (\emptyset, \emptyset) = (A, \emptyset)$ so $C = A$, and $(D, C) \cup (\emptyset, \emptyset) = (B, \emptyset)$ so $D = B$. This proves that $z = (A, B)$, so $(A, B) \in \mathcal{F}$.

THEOREM 4.11. – *Let $\mathcal{F}$ be a field of partial sets on X. If $\mathcal{F}$ is full and n is compact in $(X, \mathcal{C}(\mathcal{F}))$ then $\mathcal{C}(\mathcal{F}) = [\mathcal{C}(\mathcal{F})^{\nabla}, \mathcal{C}(\mathcal{F})^{\Delta}]$.*

By Theorem 4.5, $\mathcal{C}(\mathcal{F}) \subseteq [\mathcal{C}(\mathcal{F})^{\nabla}, \mathcal{C}(\mathcal{F})^{\Delta}]$. Next we prove $[\mathcal{C}(\mathcal{F})^{\nabla}, \mathcal{C}(\mathcal{F})^{\Delta}] \subseteq \mathcal{C}(\mathcal{F})$. If $(A, B) \in [\mathcal{C}(\mathcal{F})^{\nabla}, \mathcal{C}(\mathcal{F})^{\Delta}]$, then $A \in \mathcal{C}(\mathcal{F})^{\nabla}$ and $-B \in \mathcal{C}(\mathcal{F})^{\Delta}$, with $A \cap B = \emptyset$. So by Theorem 4.4, $A \in \mathcal{C}((\mathcal{F})_0)$ and $-B \in \mathcal{C}(-(\mathcal{F})_1)$. So $A = \bigcup\{F_i^0 : i \in I\}$ where $F_i^0 \in (\mathcal{F})_0$ for all $i \in I$. If $B = \emptyset$ then $(A, B) = \bigcup\{(F_i^0, \emptyset) : i \in I\}$. As F_i^0 and $\emptyset$ are components of $\mathcal{F}$ and $\mathcal{F}$ is full, we have $(F_i^0, \emptyset) \in \mathcal{F}$ for all $i \in I$, so $(A, B) \in \mathcal{C}(\mathcal{F})$. So from now on we assume $B \neq \emptyset$. As

$-B \in \mathcal{T}(-(\mathcal{F})_1)$ we have $-B = \bigcup\{-F_j^1 : j \in J\}$, where $F_j^1 \in (\mathcal{F})_1$ for all $j \in J$, and

(6) $$B = \bigcap\{F_j^1 : j \in J\}.$$

We can suppose that $\{F_j^1 : j \in J\}$ is closed with respect to finite intersections. (Otherwise we extend it to the class

$$\mathcal{W} = \{F_{j_1}^1 \cap, \ldots, \cap F_{j_n}^1 : j_1, \ldots, j_n \in J,\ n \in \omega\}.$$

$\mathcal{W}$ is still a class of elements of $(\mathcal{F})_1$, because $\mathcal{F}$ is closed under finite unions and intersections, and we still have $\bigcap \mathcal{W} = B$.) We prove that

(7) for all F_i^0 there is a F_j^1 such that $F_1^0 \cap F_j^1 = \emptyset$.

Let's suppose, on the contrary, that

(8) there is a F_i^0 such that, for all $j \in J$, $F_i^0 \cap F_j^1 \neq \emptyset$.

We know that F_j^1 is closed in $(X, \mathcal{T}(\mathcal{F})^\Delta)$, for all $j \in J$. The same holds for F_i^0: firstly we observe that $\mathcal{T}(\mathcal{F})^\Delta = \mathcal{T}(-(\mathcal{F})_1)$, by Theorem 4.4, then we note that $F_i^0 \in (\mathcal{F})_0$ and $(\mathcal{F})_0 = (\mathcal{F})_1$, by 1) of Theorem 3.3, so $-F_i^0 \in -(\mathcal{F})_1$, then F_i^0 is closed in $\mathcal{T}((\mathcal{F})_1)$. So $\mathcal{Z} = \{F_i^0\} \cup \{F_j^1 : j \in J\}$ is a class of closed in $(X, \mathcal{T}(\mathcal{F})^\Delta)$. We prove that $\mathcal{Z}$ has the fip. Let $\mathcal{Z}'$ be a finite subset of $\mathcal{Z}$, we distinguish two cases. Case 1: $F_i^0 \in \mathcal{Z}'$. Then

$$\bigcap \mathcal{Z}' = F_i^0 \cap F_{j_1}^1 \cap, \ldots, \cap F_{j_n}^1 = F_i^0 \cap F_j^1,$$

for some $j \in J$, because we have supposed $\{F_j^1 : j \in J\}$ closed under finite intersections, and $F_i^0 \cap F_j^1 \neq \emptyset$ by (8). Case 2), $F_i^0 \notin \mathcal{Z}'$. Then

$$\bigcap \mathcal{Z}' = F_{j_1}^1 \cap, \ldots, \cap F_{j_n}^1 \supseteq B$$

and then $\bigcap \mathcal{Z}' \neq \emptyset$, because we have supposed $B \neq \emptyset$. From our hypothesis we know that n is compact in $(X, \mathcal{T}(\mathcal{F}))$ so, by Theorem 4.9, $(X, \mathcal{T}(\mathcal{F})^\Delta)$ is a compact space in which $\bigcap \mathcal{Z} \neq \emptyset$, because $\mathcal{Z}$ is a class of closed sets enjoying the fip. So we have

$$\emptyset \neq \bigcap \mathcal{Z} = F_i^0 \cap \bigcap\{F_j^1\} = F_i^0 \cap B$$

by (6). This is absurd, however, because $F_i^0 \subseteq A$ and by hypothesis $A \cap B = \emptyset$. As we have shown that (8) implies a contradiction, we have proved (7).

Now we can define, for all $i \in I$ a $f(i) \in J$ such that $F_i^0 \cap F_{f(i)}^1 = \emptyset$, by (7). As $\mathcal{F}$ is a full partial field, we have $(F_i^0, F_{f(i)}^1) \in \mathcal{F}$. Then

$$\bigcup\{(F_i^0, F_{f(i)}^1)\} = \left(\bigcup\{F_i^0\}, \bigcap\{F_{f(i)}^1\}\right) = \left(A, \bigcap\{F_{f(i)}^1\}\right) \in \mathcal{T}(\mathcal{F}).$$

On the other side we have $(\emptyset, F_j^1) \in \mathcal{F}$, for all $j \in J$, because $\emptyset$ and F_j^1 are components of $\mathcal{F}$ and $\mathcal{F}$ is full, so

$$\bigcup\{(\emptyset, F_j^1)\} = \left(\emptyset, \bigcap\{F_j^1\}\right) = (\emptyset, B) \in \mathcal{T}(\mathcal{F}).$$

As $\bigcap\{F_{f(i)}^1\} \subseteq B$, we have $(A, \bigcap\{F_{f(i)}^1\}) \cup (\emptyset, B) = (A, B)$ and then (A, B) belongs to $\mathcal{T}(\mathcal{F})$.

THEOREM 4.12. – *Let $\mathcal{F}$ be a field of partial sets on X. If $(X, \mathcal{T}(\mathcal{F}))$ is compact and $\mathcal{T}(\mathcal{F}) = [\mathcal{T}(\mathcal{F})^\nabla, \mathcal{T}(\mathcal{F})^\Delta]$, then $\mathcal{F}$ is full.*

If A and B are components of $\mathcal{F}$, then $A, B \in (\mathcal{F})_0 = (\mathcal{F})_1$. By Theorem 4.4, $\mathcal{T}(\mathcal{F})^\nabla = \mathcal{T}((\mathcal{F})_0)$ so $A \in \mathcal{T}(\mathcal{F})^\nabla$. By Theorem 4.4, $\mathcal{T}(\mathcal{F})^\Delta = \mathcal{T}(-(\mathcal{F})_1)$ so $-B \in \mathcal{T}(\mathcal{F})^\Delta$. Then $(A, B) \in [\mathcal{T}(\mathcal{F})^\nabla, \mathcal{T}(\mathcal{F})^\Delta] = \mathcal{T}(\mathcal{F})$. So there is a basic open cover $\bigcup\{(F_i^0, F_i^1)\} = (A, B)$, where $(F_i^0, F_i^1) \in \mathcal{F}$, and by compactness (A, B) is a finite union of elements of $\mathcal{F}$. As $\mathcal{F}$ is a field of partial sets, it is closed under finite unions, so $(A, B) \in \mathcal{F}$.

So «$\mathcal{F}$ is full» is a sufficient condition for $\mathcal{T}(\mathcal{F}) = [\mathcal{T}(\mathcal{F})^\nabla, \mathcal{T}(\mathcal{F})^\Delta]$, if n is compact in the space, and a necessary condition, if the whole space is compact.

5. – Partial Stone spaces.

Now we are in a good position to prove a topological characterization theorem for DMF-algebras. For any DMF-algebra $\mathcal{A} \in |\mathcal{A}|_\sim$ we define a topological space $(X, \mathcal{T}(\mathcal{A}^*))$, where $X = FP(\nabla_\mathcal{A})$ and $\mathcal{A}^*$ (see (4)) is the field of partial sets on X representing $\mathcal{A}$. $\mathcal{T}(\mathcal{A}^*)$ is indeed a topology on X because every field of partial sets on X is a base for a topology on X, by Theorem 4.1. We say that $(X, \mathcal{T}(\mathcal{A}^*))$ is the *partial Stone space* associated to $\mathcal{A}$. We shall show that every DMF-algebra $\mathcal{A}$ can be characterized as the set of all compact clopens of its partial Stone space.

We shall derive the topological properties of $\mathcal{T}(\mathcal{A}^*)$ from topological properties of the classical topologies $\mathcal{T}(\mathcal{A}^*)^\nabla$ and $\mathcal{T}(\mathcal{A}^*)^\Delta$ through the results of the preceding paragraph. Firstly we show that $\mathcal{T}(\mathcal{A}^*)^\nabla$ and $\mathcal{T}(\mathcal{A}^*)^\Delta$ are well known classical topologies, but to gain some insight in them we must recall some basic facts about the classical Stone topology associated to a bounded distributive lattice $\mathcal{B}$. We follow the neat exposition in [2, 2.1], but the original work is obviously due to [5].

Let's denote with $I(\mathcal{B})$ the complete lattice of ideals in $\mathcal{B}$, we define $h: I(\mathcal{B}) \to \mathcal{R}(FP(\mathcal{B}))$ setting $h(Z) = \{Y \in FP(\mathcal{B}): Z \cap Y \neq \emptyset\}$, for all ideal Z of

$\mathcal{B}$. One can show (see, for instance, [2, 2.1]) that

i) $h(B) = FP(\mathcal{B})$, $h(\{0\}) = \emptyset$,

ii) $Z_0 \subseteq Z_1$ iff $h(Z_0) \subseteq h(Z_1)$,

iii) $h(Z_0 \cap Z_1) = h(Z_0) \cap h(Z_1)$,

iv) $h(\bigvee \{Z_i : i \in I\}) = \bigcup \{h(Z_i) : i \in I\}$,

where $Z_i \in I(\mathcal{B})$ for all $i \in I$. Finally we define $\theta: \mathcal{B} \to \mathcal{R}(FP(\mathcal{B}))$ setting $\theta = h \circ i$, where $i: \mathcal{B} \to I(\mathcal{B})$ is the lattice monomorphism $i(b) = (b]$ (the principal ideal generated by b) So we have, for all $b \in B$,

$$\theta(b) = \{Y \in FP(\mathcal{B}): (b] \cap Y \neq \emptyset\} = \{Y \in FP(\mathcal{B}): b \in Y\}.$$

Always following [2] one can show, from i)-iv) above and from well known results about filters separating $x \neq y$ in distributive lattices, the following theorem.

THEOREM 5.1.

1) $\mathcal{C}_0^{\mathcal{B}} = \{h(Z): Z \in I(\mathcal{B})\}$ *is a topology on* $X = FP(\mathcal{B})$, *the* classical Stone topology *associated to* $\mathcal{B}$,

2) $(X, \mathcal{C}_0^{\mathcal{B}})$ *is a compact* T_0 *space*,

3) $\theta[B] = \{\theta(b): b \in B\}$ *is the set of all compact open sets, is a ring of sets isomorphic to* $\mathcal{B}$ *and a base for* $\mathcal{C}_0^{\mathcal{B}}$, *so* $\mathcal{C}_0^{\mathcal{B}} = \mathcal{C}(\theta[B])$.

Now we return to DMF-algebras and consider, for any DMF-algebra $\mathcal{A}$, the classical Stone topology associated to the bounded distributive lattice $\nabla_{\mathcal{A}}$. The following theorem shows that this classical Stone topology is the left topology of $\mathcal{C}(\mathcal{A}^*)$.

THEOREM 5.2. – $\mathcal{C}(\mathcal{A}^*)^{\nabla} = \mathcal{C}(\theta[\nabla_{\mathcal{A}}]) = \mathcal{T}_0^{\nabla_{\mathcal{A}}}$.

We have $\mathcal{C}(\mathcal{A}^*)^{\nabla} = \mathcal{C}((\mathcal{A}^*)_0)$ by Theorem 4.4 and $(\mathcal{A}^*)_0 = \theta[\nabla_{\mathcal{A}}]$ by 3) of Theorem 3.3, so $\mathcal{C}(\mathcal{A}^*)^{\nabla} = \mathcal{C}_0^{\nabla_{\mathcal{A}}}$ by 3) of Theorem 5.1.

If we denote with $F(\mathcal{B})$ the complete lattice of all filters of a bounded distributive lattice $\mathcal{B}$, we can define $k: F(\mathcal{B}) \to \mathcal{R}(FP(\mathcal{B}))$ setting $k(Z) = \{Y \in FP(\mathcal{B}): Z \nsubseteq Y\}$, for all filter Z in $\mathcal{B}$. One can show, as above, that

i′) $k(B) = FP(\mathcal{B})$, $k(\{1\}) = \emptyset$,

ii′) $Z_0 \subseteq Z_1$ iff $k(Z_0) \subseteq k(Z_1)$,

iii′) $k(Z_0 \cap Z_1) = k(Z_0) \cap k(Z_1)$,

iv′) $k(\bigvee \{Z_i : i \in I\}) = \bigcup \{k(Z_i) : i \in I\}$,

where $Z_i \in F(\mathcal{B})$ for all $i \in I$. Then we define $i: \mathcal{B} \to F(\mathcal{B})$ setting $i(b) = [b)$, the principal filter generated by b, and finally we define $\eta: \mathcal{B} \to \mathcal{R}(FP(\mathcal{B}))$ setting

$\eta = k \circ i$. So we have, for all $b \in B$,

(9) $$\eta(b) = \{Y \in FP(\mathcal{B}): [b) \not\subseteq Y\} = \{Y \in FP(\mathcal{B}): b \notin Y\} = -\theta(b).$$

As above we can derive from i′)-iv′) above the following theorem.

THEOREM 5.3.

1) $\mathcal{T}_1^{\mathcal{B}} = \{k(Z): Z \in F(\mathcal{B})\}$ *is a classical topology on* $X = FP(\mathcal{B})$, *the* dual Stone topology *associated to* $\mathcal{B}$,

2) $(X, \mathcal{T}_1^{\mathcal{B}})$ *is a compact* T_0 *space,*

3) $\eta[B] = \{\eta(b): b \in B\}$ *is the set of all compact open sets, a ring of sets antisomorphic to* $\mathcal{B}$ *and a base for* $\mathcal{T}_1^{\mathcal{B}}$, *so* $\mathcal{T}_1^{\mathcal{B}} = \mathcal{T}(\eta[B])$.

Going back to DMF-algebras we consider, for any DMF-algebra $\mathcal{A}$, the dual Stone topology associated to the bounded distributive lattice $\nabla_{\mathcal{A}}$. The following theorem shows that this dual Stone topology is the right topology of $\mathcal{T}(\mathcal{A}^*)$.

THEOREM 5.4. – $\mathcal{T}(\mathcal{A}^*)^{\Delta} = \mathcal{T}(\eta[\nabla_{\mathcal{A}}]) = \mathcal{T}_1^{\nabla_{\mathcal{A}}}$.

We have $\mathcal{T}(\mathcal{A}^*)^{\Delta} = \mathcal{T}(-(\mathcal{A}^*)_1)$ by Theorem 4.4 and $-(\mathcal{A}^*)_1 = -(\mathcal{A}^*)_0 = \{-\theta(x): x \in \nabla_{\mathcal{A}}\} = \{\eta(x): x \in \nabla_{\mathcal{A}}\}$ by 3) of Theorem 3.3 and (9), so $\mathcal{T}(\mathcal{A}^*)^{\Delta} = \mathcal{T}_1^{\nabla_{\mathcal{A}}}$ by 3) of Theorem 5.3.

Now we can prove the topological characterization theorem.

THEOREM 5.5. – *For any DMF-algebra* $\mathcal{A}$, *the partial Stone space* $(X, \mathcal{T}(\mathcal{A}^*))$ *is a compact partial space where* $\mathcal{A}^*$ *is the set of all compact clopen sets.*

By Theorem 5.2 and 5.4, we know that $\mathcal{T}(\mathcal{A}^*)^{\nabla}$ and $\mathcal{T}(\mathcal{A}^*)^{\Delta}$ are respectively the classical and the dual Stone topology, so $(X, \mathcal{T}(\mathcal{A}^*)^{\nabla})$ and $(X, \mathcal{T}(\mathcal{A}^*)^{\Delta})$ are respectively the classical and the dual Stone space and they are compact spaces by 2) of Theorems 5.1 and 5.3. By Corollary 4.7, $(X, [\mathcal{T}(\mathcal{A}^*)^{\nabla}, \mathcal{T}(\mathcal{A}^*)^{\Delta}])$ is a compact partial space and by Theorem 4.5, $\mathcal{T}(\mathcal{A}^*) \subseteq [\mathcal{T}(\mathcal{A}^*)^{\nabla}, \mathcal{T}(\mathcal{A}^*)^{\Delta}]$ so $(X, \mathcal{T}(\mathcal{A}^*))$ is a compact space.

We show that every element of $\mathcal{A}^*$ is compact clopen. As $\mathcal{A}^*$ is a field of partial sets, every element of $\mathcal{A}^*$ is clopen. By (3.4) the elements of $\mathcal{A}^*$ are of kind $(\theta(a \vee n), \theta(\neg a \vee n))$, where a varies in A. As $a \vee n$ belongs to $\nabla_{\mathcal{A}}$, $\theta(a \vee n)$ is compact in the classical Stone space $(X, \mathcal{T}(\mathcal{A}^*)^{\nabla})$ by 3) of Theorem 5.1. As $\neg a \vee n$ belongs to $\nabla_{\mathcal{A}}$, $-\theta(\neg a \vee n)$ is compact in the dual Stone space $(X, \mathcal{T}(\mathcal{A}^*)^{\Delta})$, because $-\theta(\neg a \vee n) = \eta(\neg a \vee n)$ by (9) and $\eta(\neg a \vee n)$ is compact by 3) of Theorem 5.3. So $(\theta(a \vee n), \theta(\neg a \vee n))$ is compact in

$(X, [\mathcal{C}(\mathcal{A}^*)^\nabla, \mathcal{C}(\mathcal{A}^*)^\Delta])$ by Theorem 4.6. But $\mathcal{C}(\mathcal{A}^*) \subseteq [\mathcal{C}(\mathcal{A}^*)^\nabla, \mathcal{C}(\mathcal{A}^*)^\Delta]$ by Theorem 4.5, so $(\theta(a \vee n), \theta(\neg a \vee n))$ is also compact in $(X, \mathcal{C}(\mathcal{A}^*))$

We show that if (A, B) is a compact clopen set in $(X, \mathcal{C}(\mathcal{A}^*))$, then $(A, B) \in \mathcal{A}^*$. As (A, B) is an open set, we have $(A, B) = \bigcup\{(C_i^0, C_i^1)\}$, where $(C_i^0, C_i^1) \in \mathcal{A}^*$ for all $i \in I$. By compactness there is a finite subset $K \subseteq I$ such that $(A, B) = \bigcup\{(C_i^0, C_i^1): i \in K\}$. But $\mathcal{A}^*$ is field of partial sets, so is closed under finite unions, then $(A, B) \in \mathcal{A}^*$.

This result can be used to obtain a topological characterization of those DMF-algebras which arise with π-constructions i.e. DMF-algebras in which (2.3) holds, by Theorem 2.6.

THEOREM 5.6. – *Lat* $\mathcal{A}$ *be a DMF-algebra, then* (2.3) *holds in* $\mathcal{A}$ *iff* $\mathcal{C}(\mathcal{A}^*) = [\mathcal{C}(\mathcal{A}^*)^\nabla, \mathcal{C}(\mathcal{A}^*)^\Delta]$.

We know that $(X, \mathcal{C}(\mathcal{A}^*))$ is a compact partial space, by Theorem 5.5, and $\mathcal{A}^*$ is a field of partial sets by the representation theorem for DMF-algebras. So by Theorem 4.12, if $\mathcal{C}(\mathcal{A}^*) = [\mathcal{C}(\mathcal{A}^*)^\nabla, \mathcal{C}(\mathcal{A}^*)^\Delta]$ then $\mathcal{A}^*$ is full. By Theorem 4.10, (2.3) holds in $\mathcal{A}^*$ and then in $\mathcal{A}$.

Let's suppose that (2.3) holds in $\mathcal{A}$, then it holds also in $\mathcal{A}^*$ and then $\mathcal{A}^*$ is full, by Theorem 4.10. Now we observe that n is compact in $(X, \mathcal{C}(\mathcal{A}^*))$, because $\mathcal{A}^*$ is made of compact clopens, by Theorem 5.5, and $n \in \mathcal{A}^*$, because $\mathcal{A}^*$ is a field of partial sets. So we can apply Theorem 4.11 and conclude that $\mathcal{C}(\mathcal{A}^*) = [\mathcal{C}(\mathcal{A}^*)^\nabla, \mathcal{C}(\mathcal{A}^*)^\Delta]$.

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