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Stefan Problems with a Concentrated Capacity.

ENRICO MAGENES

Sunto. – *Vengono brevemente studiati i problemi di Stefan su «capacità concentrate», seguendo l'approccio recentemente introdotto di G. Savaré e A. Visintin.*

1. – Introduction. The basic transmission problem.

Following A. N. Tichonov [21], a partial differential equation problem modeling a physical phenomenon is a problem with a «concentrated capacity», if the order of the boundary conditions is higher or equal to the order of the partial differential equation.

In this paper we address a class of problems for the heat conduction in a n-dimensional body Ω_1, during the time interval $[0, T]$. Given a subset Γ of $\partial\Omega_1$ (Γ is the «concentrated capacity»), the classical second order parabolic heat equation in $\Omega_1 \times]0, T[$ is coupled with an initial condition at $t = 0$ and boundary conditions on $\partial\Omega_1 \times]0, T[$, that are reduced to another second order parabolic equation on $\Gamma \times]0, T[$.

In fact this is a «limit problem» for the mathematical model of the heat conduction in two disjoint bodies Ω_1 and Ω_2, whose boundaries share the concentrated capacity Γ, where the usual «transmission conditions» for the temperature and for the thermal flux are satisfied. Changes of phases in one or both the bodies Ω_1 and Ω_2, modeled by the two-phase Stefan problem, have been studied first by L. Rubinstein [16, 17] and by many Authors [1, 5, 8-13, 18-20].

In order to simplify the present exposition, here we will first consider the case where the concentrated capacity Γ coincides with $\partial\Omega_1$. Let Ω_1 be a bounded and regular (say C^2) open set of $\mathbb{R}^n$, $n \geq 2$; let Γ denote the boundary of Ω_1 and let $\nu(x)$ be the outward unit normal to Γ at the point $x \in \Gamma$. Let Ω_2 be another bounded regular open set of $\mathbb{R}^n$, sorrounding Ω_1, such that $\partial\Omega_2 = \Gamma \cup \Gamma_2$ and $\Gamma \cap \Gamma_2 = \emptyset$; let ν_2 be the outward unit normal to Γ_2.

For a fixed time interval $]0, T[$, we introduce the sets

$$Q_i := \Omega_i \times]0, T[\,, \quad i = 1, 2\,; \qquad \Sigma := \Gamma \times]0, T[\,, \quad \Sigma_2 := \Gamma_2 \times]0, T[\,.$$

For $i = 1, 2$ let ϱ_i be a strictly positive continuous function defined on $\overline{\Omega}_i$ (which stands for the heat capacity per unit volume of the two bodies) and let K_i be a $n \times n$ symmetric matrix (which stands for the thermal conductivity), whose elements belong to $C^0(\overline{\Omega}_i)$ and which satisfies the uniform ellipticity condition

$$\exists m, M > 0: \quad m|\tau|^2 \leqslant K_i(x)\,\tau\cdot\tau \leqslant M|\tau|^2, \qquad \forall \tau \in \mathbb{R}^n, \ \forall x \in \overline{\Omega}_i. \tag{1}$$

Let u_i denote the enthalpy (or energy density) and θ_i the temperature in Ω_i, $i = 1, 2$. They are related by the constitutive equations

$$\theta_i = \beta_i(u_i), \qquad \text{or equivalently} \qquad u_i \in \alpha_i(\theta_i), \tag{2}$$

where $\beta_i: \mathbb{R} \to \mathbb{R}$ is a monotone function satisfying the conditions

$$\begin{cases} (\beta_i(\xi) - \beta_i(\eta))(\xi - \eta) \geqslant c_1 |\beta_i(\xi) - \beta_i(\eta)|^2, & \forall \xi, \eta \in \mathbb{R}, \\ \beta_i(0) = 0, \quad \beta_i(\xi) \geqslant c_2 \xi - c_3, & \forall \xi \in \mathbb{R}, \\ c_1, c_2, c_3 \ \text{positive numbers}, \end{cases} \tag{3}$$

and α_i is the maximal monotone graph inverse of β_i. Finally we denote by $u_{0,i}$ the initial enthalpies and by f_i the source terms.

According to the usual weak formulation of the two phase Stefan problems (see [23], section I.4), we shall consider the following transmission problem:

PROBLEM (T.P.). – *Find u_i and θ_i, $i = 1, 2$, which satisifes the state equation* (2), *the differential equations*

$$\varrho_i \frac{\partial u_i}{\partial t} - \operatorname{div}(K_i \nabla \theta_i) = f_i, \qquad \text{in } Q_i, \qquad i = 1, 2, \tag{4}$$

the transmission conditions on Σ

$$\theta_1 = \theta_2, \qquad \nabla\theta_1 \cdot K_1 \nu = \nabla\theta_2 \cdot K_2 \nu, \qquad \text{on } \Sigma, \tag{5}$$

the initial conditions

$$u_i(\cdot, 0) = u_{i,0}, \qquad \text{in } \Omega_i, \tag{6}$$

and a lateral boundary condition on Σ_2, say, in order to fix the ideas, the homogeneous Neumann condition

$$\nabla\theta_2 \cdot K_2 \nu_2 = 0, \qquad \text{on } \Sigma_2. \tag{7}$$

Problem (T.P.) is written here in a formal way, but it could be made precise considering the equations (4) in the sense of distributions on Q_i and the other conditions in the sense of «trace theorems» in suitable Sobolev spaces.

The most convenient weak formulation of problem (T.P.) is suggested by the general theory of monotone operators in Hilbert spaces as developed by H. Brezis in [2, 3] (different formulations can be given by other types of techniques, as for the usual Stefan problem: see Section I.4 of [23] and the book [22], ch. II and IV).

DEFINITION 1.1. – *Under the assumptions*

$$f_i \in L^2(Q_i), \qquad u_{0,i} \in L^2(\Omega_i), \qquad i = 1, 2, \tag{8}$$

we say that $\{(u_i, \theta_i)\}_{i=1,2}$ *is a weak solution of problem* (*T.P.*) *if*

$$u_i \in L^2(Q_i), \qquad \theta_i \in L^2(0, T; H^1(\Omega_i)), \qquad \text{with } \theta_i = \beta_i(u_i) \text{ a.e. in } Q_i, \tag{9}$$

$$\theta_1 = \theta_2 \qquad \text{on } \Sigma, \tag{10}$$

and

$$\sum_{i=1}^{2} \iint_{Q_i} \left\{ -\varrho_i u_i \frac{\partial v_i}{\partial t} + K_i \nabla\theta_i \cdot \nabla v_i \right\} dx\, dt = \sum_{i=1}^{2} \left\{ \int_{\Omega_i} \varrho_i u_{i,0} v_i(x, 0)\, dx + \iint_{Q_i} f_i v_i \, dx\, dt \right\},$$

for every couple of test functions $v_i \in H^1(0, T; H^1(\Omega_i))$, $i = 1, 2$, *with*

$$v_1 = v_2 \qquad \text{on } \Sigma, \qquad v_1(\cdot, T) = v_2(\cdot, T) = 0. \tag{12}$$

Here the conditions (10) *and* (12) *are in the sense of the trace theorems respectively in the spaces* $L^2(0, T; H^1(\Omega_i))$ *and* $H^1(0, T; H^1(\Omega_i))$.

Then it is possible to prove the

THEOREM 1.1. – *There exists a unique weak solution of problem* (*T.P.*); *in addition*

$$u_i :]0, T[\to L^2(\Omega_i) \qquad \text{is uniformly bounded and weakly continuous}. \tag{13}$$

2. – «Blow up» of the normal conductivity.

As we said before, Stefan problems with «concentrated capacity» appear as «limit problems» of the transmission problem (T.P.). Here, following G. Savaré - A. Visintin [19], we will consider two important situations. In order to simplify the exposition, let us introduce further assumptions and notations.

For every $x \in \mathbb{R}^n$, let $d_\Gamma(x)$ be the distance of x from Γ; we shall assume that

$$d_\Gamma(x) \text{ is a function of class } C^2 \text{ in } \overline{\Omega}_2. \tag{14}$$

This assumption is equivalent to suppose Γ of class C^2 and Ω_2 contained in a suitable neighborhood of Γ, depending on its curvatures (see e.g.[4]). In particular,

(14) implies that for every $x \in \Omega_2$ there exists a unique projection x_Γ on Γ, satisfying

$$|x - x_\Gamma| = d_\Gamma(x), \tag{15}$$

so that we can define a unit vector field

$$\nu(x) := \frac{x - x_\Gamma}{d_\Gamma(x)}, \qquad \forall x \in \overline{\Omega}_2 \tag{16}$$

which is normal to (each manifold parallel to) Γ at every points of $\Omega_2 \cup \Gamma$.

The tangent and the normal spaces to Γ at the point $x \in \Gamma$ are defined, as usual in differential geometry, by

$$T_x := \{\tau \in \mathbb{R}^n : \nu(x) \cdot \tau = 0\}, \qquad N_x := \{\tau \in \mathbb{R}^n : \tau = \lambda \nu(x) \text{ for some } \lambda \in \mathbb{R}\}, \tag{17}$$

and the orthogonal projection onto T_x is

$$\mathcal{P}_x \tau := [I - \nu(x)\, \nu^T(x)]\, \tau . \tag{18}$$

The principal curvatures of Γ at x are the eigenvalues, besides 0, of the differential matrix of $\nu(x)$ (see e.g. [6])

$$S(x) := - D\nu(x). \tag{19}$$

If v^* is a regular extension to Ω_2 of a regular function $v: \Gamma \to \mathbb{R}$, then the tangential gradient of v at $x \in \Gamma$ is well defined by

$$\nabla_\Gamma(x) := \mathcal{P}_x(\nabla v^*(x)), \tag{20}$$

and it is independent of the extension v^* of v.

We also define the divergence on Γ as follows: for every tangential vector field $\tau(x) \in \mathcal{T}_x$ we set

$$\operatorname{div}_\Gamma \tau := \operatorname{div} \tau^* - \frac{\partial(\tau^* \cdot \nu)}{\partial \nu}, \tag{21}$$

where τ^* is a regular extension of τ to Ω_2 (as before, $\operatorname{div}_\Gamma \tau$ does not depend on the particular extension τ^*). In this framework, the usual Laplace-Beltrami operator, induced by the Euclidean metric on Γ has the simple form

$$\Delta_\Gamma v = \operatorname{div}_\Gamma(\nabla_\Gamma v).$$

Let us also recall that the well known Hilbert space $H^1(\Gamma)$ (cf. e.g. [7]) can be defined as the completion of $C^1(\Gamma)$ with respect to the norm induced by the scalar product

$$(u, v)_{H^1(\Gamma)} := \int_\Gamma [u(x)\, v(x) + \nabla_\Gamma u(x) \cdot \nabla_\Gamma v(x)]\, d\sigma(x),$$

where σ is the usual $n-1$ geometric measure on Γ.

Finally, we will assume that there exists a strictly positive regular function $l\colon \Gamma \to \mathbb{R}$ such that

$$\Omega_2 = \{x \in \mathbb{R}^N \setminus \overline{\Omega}_1 \colon d_\Gamma(x) < l(x_\Gamma)\} \tag{22}$$

and

$$\det(I - \lambda S(x)) > 0\,, \qquad \forall x \in \Gamma, \quad 0 \leq \lambda \leq l(x)\,. \tag{23}$$

We consider now the situation which originally motivated the introduction of Stefan problems with a «concentrated capacity», assuming $\varrho_1 = \varrho_2 = 1$ for simplicity. For every $\varepsilon > 0$, we perturb Problem (T.P.) by replacing the matrix $K_2(x)$ by

$$K_2^\varepsilon(x) := K_2(x) + \frac{1}{\varepsilon}\nu(x)\,\nu^T(x)\,, \tag{24}$$

which expresses the «blow up» of the normal conductivity.

We want to study the «limit problem», as ε goes to 0, of the family of these perturbed transmission problems. To this aim it is natural to introduce the subspace $H_\nu^1(\Omega_2)$ of $H^1(\Omega_2)$ consisting of the functions which are constant along the normal directions to Γ, i.e.

$$H_\nu^1(\Omega_2) := \{v \in H^1(\Omega_2)\colon \nu \cdot \nabla v \equiv 0\}\,. \tag{25}$$

We also denote by $L_\nu^2(\Omega_2)$ the closure of $H_\nu^1(\Omega_2)$ in $L^2(\Omega_2)$ and by Π_ν the orthogonal projection of $L^2(\Omega_2)$ on $L_\nu^2(\Omega_2)$.

THEOREM 2.1. – *Let $\{(u_i^\varepsilon, \theta_i^\varepsilon)\}_{i=1,2}$ be the weak solution of the perturbed (T.P.); let $\{(u_i, \theta_i)\}_{i=1,2}$ be the solution of the «limit problem», which is formally obtained by replacing $L^2(\Omega_2)$ and $H^1(\Omega_2)$ by $L_\nu^2(\Omega_2)$ and $H_\nu^1(\Omega_2)$ respectively in Definition* 1.1. *Then as $\varepsilon \to 0$ we have*

$$\theta_i^\varepsilon \to \theta_i \text{ strongly in } L^2(Q_i) \text{ and weakly in } L^2(0, T; H^1(\Omega_i))\,; \tag{26}$$

$$\begin{cases} u_1^\varepsilon(\cdot, t) \rightharpoonup u_1(\cdot, t) & \text{weakly in } L^2(\Omega_1), \ \ \forall t \in]0, T]\,, \\ \Pi_\nu u_2^\varepsilon(\cdot, t) \rightharpoonup u_2(\cdot, t) & \text{weakly in } L^2(\Omega_2), \ \ \forall t \in]0, T]\,, \end{cases} \tag{27}$$

the latter convergences being also strong if $u_{0,2} \in L_\nu^2(\Omega_2)$.

Finally we can give the interpretation of this «limit problem» as a Stefan problem in the concentrated capacity Γ:

THEOREM 2.2. – *Let $\{(u_i, \theta_i)\}_{i=1,2}$ be the solution of the «limit problem» defined by Theorem* 2.1. *Let $(\tilde{u}_2, \tilde{\theta}_2)$ denote the traces on Σ of (u_2, θ_2). Then $\{(u_1, \theta_1), (\tilde{u}_2, \tilde{\theta}_2)\}$ is the unique solution (in the same weak sense as in Defini-*

tion 1.1) of the following Stefan problem in the concentrated capacity Γ:

$$\text{(28)}\qquad \begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1, \\ \dfrac{\partial u_1}{\partial t} - \operatorname{div}(K_1 \nabla \theta_1) = f_1 & \text{in } Q_1, \\ u_1(x, 0) = u_{0,1}(x) & \text{in } \Omega_1, \end{cases}$$

$$\text{(29)}\qquad \theta_1 = \tilde{\theta}_2 \qquad \text{on } \Sigma,$$

$$\text{(30)}\qquad \begin{cases} \tilde{\theta}_2 = \beta_2(\tilde{u}_2) & \text{on } \Sigma, \\ \dfrac{\partial \tilde{u}_2}{\partial t} - \operatorname{div}_\Gamma(\widehat{K}_2 \nabla_\Gamma \tilde{\theta}_2) = f_2 - \nabla\theta_1 \cdot K_1 \nu & \text{on } \Sigma, \\ \tilde{u}_2(x, 0) = \widehat{u}_{0,2}(x) & \text{on } \Gamma, \end{cases}$$

where $\hat{f}_2$, $\widehat{u}_{0,2}$, $\widehat{K}_2$ can be explicitely computed from the corresponding values of f_2, $u_{0,2}$, K_2, by using the matrix $S(x)$ (and therefore they depend on the curvatures of Γ).

REMARK 2.1. – We refer to [19] for the explicit computation of $\hat{f}$, $\widehat{u}_{0,2}$, $\widehat{K}_2$. As an example, let us consider the case of $n = 3$, $K_2(x) \equiv I$. Let us denote by H_m and H_g respectively the mean and the Gaussian curvature of Γ at the point x, and let us introduce the standard parametrization of the segment s_x starting from $x \in \Gamma$ and pointing towards Ω_2 along the normal direction $\nu(x)$,

$$x_\lambda := x + \lambda\nu(x), \qquad 0 \leq \lambda \leq l(x),$$

together to the deformation measure μ_x on it

$$d\mu_x(\lambda) := [1 - 2H_m(x)\,\lambda + H_g(x)\,\lambda^2]\, d\lambda\,.$$

Then we have

$$\hat{f}_2(x, t) := \int_0^{l(x)} f_2(x_\lambda, t)\, d\mu_x(\lambda), \qquad \widehat{u}_{0,2}(x) := \int_0^{l(x)} u_{0,2}(x_\lambda)\, d\mu_x(\lambda),$$

$$\widehat{K}_2(x) := \int_0^{l(x)} (I - \lambda S(x))^{-2}\, d\mu_x(\lambda)\,.$$

The proofs of Theorems 2.1 and 2.2 are given in [19] by using in a suitable way the variational convergence in the sense of Mosco [14, 15].

REMARK 2.2. – The system (28), (29), (30) can be studied independently of the asymptotic approach given by Theorems 2.1 and 2.2. In this case $\hat{f}_2$, $\widehat{u}_{0,2}$, and $\widehat{K}_2$ are «a priori» given data and the operator $\operatorname{div}_\Gamma(\widehat{K}_2 \nabla_\Gamma)$ can be replaced by a more general elliptic operator on Γ. Thus problem (28), (29), (30) can be directly studied by using the same methods as for (T.P.) (cf. Theorem 1.1). Another approach, followed in [8, 9, 10], is to reduce the system (28), (29), (30) to a unique evolution equation in the unknown $\tilde{u}_2$. In fact, taking into account equations (28), (29), we can consider θ_1 as a function of $\tilde{\theta}_2$; consequently also the co-normal derivative of θ_1 on Σ can be viewed as depending on $\tilde{\theta}_2$ through a non local operator $\mathcal{C}$ of «Dirichlet-Neumann» type:

$$\mathcal{C}\colon \tilde{\theta}_2 \mapsto \mathcal{C}(\tilde{\theta}_2) := \nabla\theta_1 \cdot K_1 \nu\,, \qquad \text{on } \Sigma\,. \tag{31}$$

In this way it is possible to reduce the system (28, 29, 30) to a single equation on Σ in the unknown $\tilde{u}_2$:

$$\begin{cases} \tilde{\theta}_2 = \beta_2(\tilde{u}_2)\,, \\ \widehat{\varrho}_2 \dfrac{\partial \tilde{u}_2}{\partial t} - \operatorname{div}_\Gamma(\widehat{K}_2 \nabla_\Gamma \tilde{\theta}_2) = f_2 - \mathcal{C}(\tilde{\theta}_2)\,, \end{cases} \qquad \text{on } \Sigma\,, \tag{32}$$

with the initial Cauchy condition

$$\tilde{u}_2(x, 0) = \widehat{u}_{0,2}(x) \qquad \text{on } \Gamma\,. \tag{33}$$

Nevertheless, this approach seems to be more complicated, because the study of the nonlinear and nonlocal operator $\mathcal{C}$ requires some non standard estimates in the Hilbert spaces $H^{s,r}(\Sigma)$ of negative and fractional order s, r (for these spaces see [7]).

3. – «Blow up» of the «global» conductivity.

The asymptotic approach of [19], described in the previous sections, can also be applied to study the Stefan problem in a concentrated capacity arising when the global conductivity blows up and Ω_2 shrinks to Γ.

Still following [19], let us consider a family of contractions in the direction of the vector field $-\nu(x)$

$$G^\varepsilon(x) := \varepsilon x + (1-\varepsilon)\, x_\Gamma\,, \qquad 0 < \varepsilon \leq 1\,,$$

and let us introduce the shrinked sets

$$\Omega_2^\varepsilon := G^\varepsilon(\Omega_2)\,, \qquad s_x^\varepsilon := G^\varepsilon(s_x)\,.$$

If in the transmission problem (T.P.) we replace the set Ω_2 by Ω_2^ε and the data $\varrho_2, K_2, f_2, u_{0,2}$ by a family of functions $\varrho_2^\varepsilon, K_2^\varepsilon, f_2^\varepsilon, u_{0,2}^\varepsilon$ defined in Ω_2^ε and satisfying similar assumptions as $\varrho_2, K_2, f_2, u_{0,2}$, we obtain a correponding family of transmission problems depending on the parameter ε. In order to study the «limit problem» as ε goes to 0, let us assume that there exists

$$\varrho_\Gamma \in C^0(\Gamma), \text{ strictly positive}, \; f_\Gamma \in L^2(\Sigma), \; u_{0,\Gamma} \in L^2(\Gamma),$$

and a symmetric matrix K_Γ, whose elements belong to $C^0(\Gamma)$ and which satisfy (1) for every point $x \in \Gamma$, such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{Q_2^\varepsilon} |f_2^\varepsilon(x,t) - f_\Gamma(x_\Gamma, t)|^2 \, dx\, dt = 0, \qquad \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{\Omega_2^\varepsilon} |u_{0,2}^\varepsilon(x) - u_{0,\Gamma}(x_\Gamma)|^2 \, dx = 0,$$

$$\lim_{\varepsilon \to 0} \sup_{x \in \Omega_2^\varepsilon} [\, |\varepsilon K_2^\varepsilon(x) - K_\Gamma(x_\Gamma)| + |\varepsilon \varrho_2^\varepsilon(x) - \varrho_\Gamma(x_\Gamma)|\,] = 0 .$$

If $\{(u_i^\varepsilon, \theta_i^\varepsilon)\}_{i=1,2}$ is the family of solutions of these transmission problems, then for a.e. $x \in \Gamma$ we define $\overline{u}_2^\varepsilon$ and $\overline{\theta}_2^\varepsilon$ as the mean value on s_x^ε of u_2^ε, θ_2^ε:

$$\overline{u}_2^\varepsilon(x) := \fint_{s_x^\varepsilon} u_2^\varepsilon(y)\, ds(y), \qquad \overline{\theta}_2^\varepsilon(x) := \fint_{s_x^\varepsilon} \theta_2^\varepsilon(y)\, ds(y).$$

We can characterize the limit, as $\varepsilon \to 0$, of $(u_1^\varepsilon, \theta_1^\varepsilon)$ in Q_1 and of $(\overline{u}_2^\varepsilon, \overline{\theta}_2^\varepsilon)$ on Σ. More precisely we have the following result:

THEOREM 3.1. – *Let* $\varepsilon \to 0$; *then*

$\theta_1^\varepsilon \to \theta_1$ *strongly in* $L^2(0,T; H^1(\Omega_1))$, $\quad \overline{\theta}_2^\varepsilon \to \overline{\theta}_2$ *strongly in* $L^2(0,T; H^1(\Gamma))$

and, for every $t \in]0,T]$,

$u_1^\varepsilon(\cdot,t) \rightharpoonup u_1(\cdot,t)$ *weakly in* $L^2(\Omega_1)$, $\quad \overline{u}_2^\varepsilon(\cdot,t) \rightharpoonup \overline{u}_2(\cdot,t)$ *weakly in* $L^2(\Gamma)$;

moreover $\{(u_1, \theta_1), (\overline{u}_2, \overline{\theta}_2)\}$ *is the unique weak solution of the following Stefan problem*

$$\text{(34)} \qquad \begin{cases} \theta_1 = \beta_1(u_1) & \text{in } Q_1, \\ \varrho_1 \dfrac{\partial u_1}{\partial t} - \operatorname{div}(K_1 \nabla \theta_1) = f_1 & \text{in } Q_1, \\ u_1(x,0) = u_{0,1}(x) & \text{in } \Omega_1, \end{cases}$$

$$\text{(35)} \qquad \theta_1 = \overline{\theta}_2 \quad \text{on } \Sigma,$$

$$\text{(36)} \qquad \begin{cases} \overline{\theta}_2 = \beta_2(\overline{u}_2) & \text{on } \Sigma, \\ \varrho_\Gamma \dfrac{\partial \overline{u}_2}{\partial t} - l^{-1} \operatorname{div}_\Gamma(l K_\Gamma \nabla_\Gamma \overline{\theta}_2) = f_\Gamma - l^{-1} \nabla \theta_1 \cdot K_1 \nu & \text{on } \Sigma, \\ \overline{u}_2(x,0) = u_{0,\Gamma}(x) & \text{on } \Gamma, \end{cases}$$

where l *is the thikness of* Ω_2 *defined by* (22).

REMARK 3.1. – In the simplest case of constant coefficients of order of magnitude ε^{-1}, i.e. $\varrho_2^\varepsilon := \varrho_2/\varepsilon$, $K_2^\varepsilon := K_2/\varepsilon$, we obtain

$$\varrho_\Gamma(x) = \varrho_2, \qquad K_\Gamma(x) = K_2, \qquad \forall x \in \Gamma .$$

4. – Final comments.

4.1. Under stronger regularity assumptions on the data, it is possible to prove stronger regularity properties for the solutions of the problems considered in the previous sections. E.g. let us refer to the problem (28), (29), (30); if we assume that

$$(37)\begin{cases} u_{0,1} \in L^2(\Omega_1), & \beta_1(u_{0,1}) \in H^1(\Omega_1), \\ \widehat{u}_{0,2} \in L^2(\Gamma), & \beta_2(\widehat{u}_{0,2}) \in H^1(\Gamma), \end{cases} \qquad \text{with } \beta_1(u_{0,1}) = \beta_2(\widehat{u}_{0,2}) \text{ in } \Gamma ,$$

then we have

$$(38)\qquad \begin{cases} \theta_1 \in H^1(0, T; L^2(\Omega_1)) \cap L^\infty(0, T; H^1(\Omega_1)), \\ \widehat{\theta}_2 \in H^1(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^1(\Gamma)) . \end{cases}$$

As in the standard Stefan problem (cf. e.g. [22, IV, 10]) another important question to be addressed is the *continuity* of the temperature: for the problem (28), (29), (30) we refer to [11] and the references therein.

4.2. As we said in Section 1, the concentrated capacity Γ could be only a part of the boundary $\partial\Omega_1$ of Ω_1. More precisely, let us assume that Ω_1 and Ω_2 are two adjoining open bounded and regular sets of $\mathbb{R}^n$, such that $\partial\Omega_1 \cap \partial\Omega_2$ is the closure $\overline{\Gamma}$ of a regular $(n-1)$-submanifold Γ with boundary $\Gamma' := \overline{\Gamma} \setminus \Gamma$ and let us denote by $\Gamma_i := \partial\Omega_i \setminus \overline{\Gamma}$, $i = 1, 2$, the remaining part of the two boundaries and by $\nu_i(x)$, $\nu(x)$ the outward unit vector normal to Γ_i, Γ at the point x respectively. Finally, let us introduce the sets

$$Q_i := \Omega_i \times]0, T[, \qquad \Sigma_i := \Gamma_i \times]0, T[, \quad i = 1, 2; \qquad \Sigma := \Gamma \times]0, T[.$$

Then we can consider, instead of Problem (T.P.) of section 1, the new transmission problem: find u_i and θ_i which satisfy again (2), (4), (5), (6) and the homogeneous Neumann conditions on both Σ_1 and Σ_2

$$\nabla\theta_i \cdot K_i \nu_i = 0 \quad \text{ in } \Sigma_i, \qquad i = 1, 2 .$$

The same ideas, techniques and results of sections 2, 3, and 4.1 apply also to this new case (see [19, 12, 13]). We emphasize that the approach suggested by Remark 2.2 presents in this case new difficulties, because it is harder to study the properties of the operator $\mathcal{C}(\widetilde{\theta}_2)$ (see [12, 13]). Also the question of the *continuity* of the temperature is not yet completely solved (see [12], sec. 4).

4.3. The asymptotic approach of the Stefan problem with a concentrated capacity, described in sections 2, 3, and 4.2, has been developped in [19] in a very interesting and elegant abstract form, which applies to several other problems, as the porous media equations, homogeneization of nonlinear diffusion equations, self contact domains, reinforcements problems in the Calculus of Variations, problems where the concentrated capacity lies on manifolds of codimentions higher than 1; we refer to [19] and the references therein.

4.4. Regularity properties of the free boundaries, «mushy» regions, non denegeracy of the solution, classical solutions, have been studied for the standard two-phase Stefan problem (see the references in [22]). It would be interesting to study all these topics also for the Stefan problems discussed in the sections 2, 3, and 4.2 (note in particular that here we have two free boundaries, one in Q_1 and the other one on Σ).

It would also be interesting to consider the generalizations of the usual Stefan model and the related questions described in [23, sections II, III, and IV] and [22, Chap. VI-IX] in connection with the concentrated capacity problems.

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