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Some Remarks on Almost-Positivity of $\psi do's$.

CESARE PARENTI - ALBERTO PARMEGGIANI

Sunto. – *Per una classe di operatori pseudodifferenziali a caratteristiche multiple vengono date condizioni necessarie e sufficienti per la validità di stime dal basso «ottimali».*

1. – Introduction.

The problem of understanding almost-positivity (i.e. lower bounds) of differential and, more generally, of pseudodifferential operators ($\psi do's$), has been started long time ago by the pioneering work of Gårding [4], who established his famous inequality for elliptic operators. Namely, let $P = P^*$ be a self-adjoint classical ψdo of order m on some open set $X \subset \boldsymbol{R}^n$. Then, the following two properties are equivalent (1):

$$p_m(x, \xi) > 0, \qquad \forall (x, \xi) \in T^*X \backslash 0 . \tag{1}$$

For any $\mu < m/2$ *and any compact* $K \subset X$ *there exist* $c_{\mu, K}, C_{\mu, K} > 0$ *such that*

$$(Pu, u) \geq c_{\mu, K} \|u\|^2_{m/2} - C_{\mu, K} \|u\|^2_{\mu}, \qquad \forall u \in C_0^\infty(K) . \tag{2}$$

The difficult problem is next to understand what happens when one relaxes the ellipticity condition (1). Hörmander (see [7]) proved the equivalence of the following:

$$p_m(x, \xi) \geq 0 , \qquad \forall (x, \xi) \in T^*X \backslash 0 . \tag{3}$$

For any compact $K \subset X$ *there exists* $C_K > 0$ *such that*

$$(Pu, u) \geq - C_K \|u\|^2_{(m-1)/2}, \qquad \forall u \in C_0^\infty(K) \tag{4}$$

(the so-called *Sharp Gårding Inequality*).

Notice that the above inequalities depend only on the principal symbol of P. It was Melin [9], who studied how almost-positivity is influenced by the lower order

(1) Unexplained notation used throughout are standard, and can be found in Hörmander's books [8], Vol. I and III.

terms of the total symbol of P. Precisely, he proved the equivalence of the following:

For any $\varepsilon > 0$, *for any* $\mu < (m-1)/2$ *and any compact* $K \subset X$ *there exists* $C_{\varepsilon,\mu,K} > 0$ *such that*

$$(5) \qquad (Pu, u) \geqslant -\varepsilon \|u\|^2_{(m-1)/2} - C_{\varepsilon,\mu,K} \|u\|^2_{\mu}, \qquad \forall u \in C_0^\infty(K)$$

(the so-called *Melin inequality*);

$$(6) \qquad \begin{cases} p_m(x, \xi) \geqslant 0, & \forall (x, \xi) \in T^*X \backslash 0, \\ p_m(x, \xi) = 0 \Rightarrow p^s_{m-1}(x, \xi) + \mathrm{Tr}^+(F_{x,\xi}) \geqslant 0, \end{cases}$$

where

$$p^s_{m-1}(x, \xi) = p_{m-1}(x, \xi) + \frac{i}{2} \sum_{j=1}^n \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}(x, \xi)$$

is the *subprincipal symbol* of P, and $\mathrm{Tr}^+(F_{x,\xi})$ is the *positive trace of the fundamental matrix* $F_{x,\xi}$ defined by

$$Q_{x,\xi}(v) = \langle Q_{x,\xi} v, v \rangle = \sigma(v, F_{x,\xi} v), \qquad v \in T_{(x,\xi)} T^*X,$$

$Q_{x,\xi}$ being the Hessian of $p_m/2$ at (x, ξ) and with $\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j$ the canonical symplectic form on T^*X. In explicit form, $\mathrm{Tr}^+(F_{x,\xi}) = \sum_{\mu > 0} \mu$ with $i\mu$ in the spectrum of $F_{x,\xi}$.

It is crucial to observe that in conditions (6) above, no assumption on the geometry of the *characteristic set*

$$\Sigma := \{(x, \xi) \in T^*X \backslash 0 \,;\; p_m(x, \xi) = 0\}$$

is made.

In fact, supposing that:

(a) *Σ is a smooth sub-manifold of $T^*X \backslash 0$,*

(b) *σ has constant rank on the connected components of Σ (i.e. $\Sigma \ni \varrho \mapsto \dim(T_\varrho \Sigma \cap T_\varrho \Sigma^\sigma)$ is locally constant, $T_\varrho \Sigma^\sigma$ being the symplectic orthogonal of $T_\varrho \Sigma$),*

(c) $p_m(x, \xi)$ *vanishes* exactly *to second order on Σ,*

Hörmander [7] proved the following result: (6) above is equivalent to

For any compact $K \subset X$ *there exists* $C_K > 0$ *such that*

$$(7) \qquad (Pu, u) \geqslant -C_K \|u\|^2_{m/2-1}, \qquad \forall u \in C_0^\infty(K).$$

We recall that inequalities (5) and (7) play a central role in a number of problems, such as the well-posedness of the Cauchy problem for weakly hyperbolic operators (see Hörmander [7]), the Weyl-asymptotics for degenerate elliptic operators on compact manifolds (see Menikoff-Sjöstrand [10], Mohamed [11]), just to mention a few of them.

From a somewhat different viewpoint, Fefferman and Phong [3] proved the following sharp result:

If $m = 2$ and the total *symbol $p(x, \xi)$ of P is non-negative on T^*X, then for any compact $K \subset X$ there exists $C_K > 0$ such that*

$$\operatorname{Re}(Pu, u) \geq -C_K \|u\|_0^2, \qquad \forall u \in C_0^\infty(K).$$

Our point of view here will be focussed upon *invariant conditions*. More precisely, we are concerned with finding a suitable generalization of inequality (7) above, when the principal symbol vanishes to order higher than second on Σ. To this purpose, the main point will consist in finding the *correct* generalization of conditions (6) above.

Some «experiments» in this direction can be found in [12], [13], [14], where examples with p_m vanishing to fourth order are treated. Soon after, we found the «kind-of-forgotten» Mohamed's paper [11], where, among other results, he proves a generalization of Melin's inequality (5) (see below for more details).

In this paper we shall deal with the geometric machinery required by the statement of our generalization of (7) (see Theorems 4.1 and 4.10 below). The geometric setting will be developed by largely using the methods introduced by Boutet, Grigis and Helffer in [2]. The proof of Theorems 4.1 and 4.10, which uses Mohamed's results, will appear elsewhere.

2. – Operators with multiple characteristics and related invariants.

Let X be an open subset of $\boldsymbol{R}^n$ (or, more generally, a C^∞ n-dimensional manifold without boundary) and let $\Sigma \subset T^*X \backslash 0$ be a C^∞ conic submanifold. With $m \in \boldsymbol{R}$ and $k \in \boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$, we denote by $N^{m,k}(X, \Sigma)$ (see [16], [2]) *the set of all classical symbols of order* m, $p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$, such that for any $j \geq 0$ one has

$$|p_{m-j}(x, \xi)| \lesssim |\xi|^{m-j} \operatorname{dist}_\Sigma(x, \xi)^{(k-2j)_+}, \tag{8}$$

where $t_+ := \max\{t, 0\}$, $\operatorname{dist}_\Sigma(x, \xi)$ denotes the distance of $(x, \xi/|\xi|)$ to Σ, and the relation $f \lesssim g$ means that for any conic set $\Gamma \subset T^*X \backslash 0$ with compact base, there exists a constant $C_\Gamma > 0$ for which

$$f(x, \xi) \leq C_\Gamma g(x, \xi), \qquad \forall (x, \xi) \in \Gamma.$$

By $\mathrm{OPN}^{m,k}(X, \Sigma)$ *we denote the corresponding class of (properly-supported) $\psi do's$.*

We will say that p (or the corresponding operator $P = \mathrm{Op}(p)$) is *transversally elliptic* (with respect to Σ) iff the principal symbol p_m *vanishes exactly to kth-order on* Σ, i.e.

$$|p_m(x,\xi)| \geqslant |\xi|^m \mathrm{dist}_\Sigma(x,\xi)^k. \tag{9}$$

It is useful to recall the following algebra properties:

$$\begin{cases} A \in \mathrm{OPN}^{m,k}(X,\Sigma),\ B \in \mathrm{OPN}^{m',k'}(X,\Sigma) \Rightarrow AB \in \mathrm{OPN}^{m+m',k+k'}(X,\Sigma), \\ A \in \mathrm{OPN}^{m,k}(X,\Sigma) \Rightarrow A^* \in \mathrm{OPN}^{m,k}(X,\Sigma), \\ A \in \mathrm{OPN}^{m,k}(X,\Sigma) \Rightarrow A \in \mathrm{OPN}^{m+l,k+2l}(X,\Sigma), \qquad \forall l \in \boldsymbol{Z}_+ . \end{cases} \tag{10}$$

Moreover, transversal ellipticity is obviously preserved by composition and by taking adjoints.

We will also need the invariance of the above classes under canonical change of variables (see [1] for a proof of this nontrivial fact).

Let $X, Y \subset \boldsymbol{R}^n$ *be open sets and let*

$$\chi\colon T^*X\backslash 0 \to T^*Y\backslash 0$$

be a smooth homogeneous (of degree one in the fibers) canonical transformation. Let $\Lambda_\chi \subset (T^*Y\backslash 0)\times(T^*X\backslash 0)$ *(resp.* $\Lambda_{\chi^{-1}} \subset (T^*X\backslash 0)\times(T^*Y\backslash 0)$*) be the canonical relation associated with* χ *(resp.* χ^{-1}*) and finally denote by*

$$F \in I^0(Y\times X, \Lambda_\chi) \qquad (\text{resp.}\ \ F^{-1} \in I^0(X\times Y, \Lambda_{\chi^{-1}}))$$

an elliptic Fourier integral operator of order 0, associated with Λ_χ *(resp.* $\Lambda_{\chi^{-1}}$*) (see* [6]*), with* $FF^{-1} \equiv I$, $F^{-1}F \equiv I$. *Then*

$$P \in \mathrm{OPN}^{m,k}(X,\Sigma) \Rightarrow \tilde{P} := FPF^{-1} \in \mathrm{OPN}^{m,k}(Y,\chi(\Sigma)). \tag{11}$$

It turns out that there are invariants naturally attached to operators in the class considered above. Their definition relies on a crucial result of Helffer [5]. First of all, we need to recall what the *Weyl-symbol* of a classical *ψdo* is. Let $p(x,\xi) \sim \sum_{j\geqslant 0} p_{m-j}(x,\xi)$ be a classical symbol. Then we define the *Weyl-symbol* p_w as

$$p_w(x,\xi) = e^{\langle D_x, D_\xi\rangle/2i} p(x,\xi) \sim \sum_{j\geqslant 0}\left(\sum_{l+r=j} \frac{1}{r!}\left(\frac{1}{2i}\langle D_x, D_\xi\rangle\right)^r p_{m-l}(x,\xi)\right). \tag{12}$$

Notice that $p_w \in N^{m,k}(X,\Sigma)$ iff p does.

The aforementioned result may then be stated as follows.

THEOREM 2.1. – *Let* $p \in N^{m,k}(X, \Sigma)$ *and let* $\chi: T^*X \backslash 0 \to T^*Y\backslash 0$ *be as above. Put*

$$\mathrm{Op}(\tilde{p}) = F\,\mathrm{Op}(p)\,F^{-1} \in \mathrm{OPN}^{m,k}(Y, \chi(\Sigma)).$$

Then

$$\tilde{p}_w \circ \chi - p_w \in \mathrm{N}^{m,k+1}(X, \Sigma). \tag{13}$$

We are now in a position to define the main invariant attached to an operator $\mathrm{Op}(p) \in \mathrm{OPN}^{m,k}(X, \Sigma)$. Write

$$p_w(x, \xi) \sim \sum_{j \geq 0} q_{m-j}(x, \xi).$$

For any $\varrho \in \Sigma$, and any $v \in T_\varrho T^*X$, let V be a smooth section of TT^*X, defined in a neighborhood of ϱ, with $V(\varrho) = v$. Define

$$p_\varrho^{(k)}(v) = \sum_{0 \leq j \leq k/2} \frac{1}{(k-2j)!}(V^{k-2j} q_{m-j})(\varrho). \tag{14}$$

It is easy to see that the above definition is independent of the extension V of v, and that the map

$$\begin{cases} p^{(k)}: TT^*X|_\Sigma \to \boldsymbol{C}, \\ (\varrho, v) \mapsto p^{(k)}(\varrho, v) := p_\varrho^{(k)}(v), \end{cases} \tag{15}$$

is smooth.

The next proposition lists a number of important properties of $p^{(k)}$.

PROPOSITION 2.2.

1) *The map* $p^{(k)}$ *is a polynomial map of degree* $\leq k$ *in the fibers of* $TT^*X|_\Sigma$.

2) *Denote by* $\Lambda_\varrho(k) \subset T_\varrho T^*X$ *the lineality of the polynomial* $p_\varrho^{(k)}(\cdot)$. *Then*

$$T_\varrho \Sigma \subset \Lambda_\varrho(k), \qquad \forall \varrho \in \Sigma.$$

Equality holds for every $\varrho \in \Sigma$ *iff* p *is transversally elliptic.*

3) *If* $\chi: T^*X\backslash 0 \to T^*Y\backslash 0$ *is a symplectomorphism as above, and* $\mathrm{Op}(\tilde{p}) = F\,\mathrm{Op}(p)\,F^{-1}$, *then*

$$\tilde{p}_{\chi(\varrho)}^{(k)}(d\chi(\varrho)\,v) = p_\varrho^{(k)}(v), \qquad \forall \varrho \in \Sigma, \quad \forall v \in T_\varrho T^*X. \tag{16}$$

The proof uses Theorem 2.1 and it is straightforward.

Having defined the polynomials $p_\varrho^{(k)}(\cdot)$, $\varrho \in \Sigma$, we «quantize» them as follows.

For any fixed $\varrho \in \Sigma$, let

$$\zeta\colon T^*\boldsymbol{R}^n \simeq \boldsymbol{R}^n_y \times (\boldsymbol{R}^n)'_\eta \to T_\varrho T^*X$$

be a linear symplectomorphism. Pulling back gives

$$p_\varrho^{(k)}(\zeta(y,\eta)) =: p_\zeta(y,\eta), \tag{17}$$

and we can define

$$\mathrm{Op}^w(p_\zeta)(y, D_y)\colon \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}(\boldsymbol{R}^n)$$

as

$$\mathrm{Op}^w(p_\zeta)\, f(x) = (2\pi)^{-n} \int\!\!\int e^{i\langle x-y,\eta\rangle} p_\zeta\left(\frac{x+y}{2}, \eta\right) f(y)\, dy\, d\eta, \quad f \in \mathcal{S}(\boldsymbol{R}^n). \tag{18}$$

The crucial observation is now that if $\zeta'\colon T^*\boldsymbol{R}^n \to T_\varrho T^*X$ is another linear symplectic map, then

$$p_{\zeta'} = p_\zeta \circ (\zeta^{-1} \circ \zeta'), \tag{19}$$

whence, as it is well-known (see [8], Vol. III, Thm. 18.5.9), there exists a unitary operator $U\colon L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$ (uniquely determined up to a complex constant factor of modulus 1), which is also an automorphism of $\mathcal{S}(\boldsymbol{R}^n)$ and $\mathcal{S}'(\boldsymbol{R}^n)$, such that

$$\mathrm{Op}^w(p_{\zeta'}) = U^{-1}\,\mathrm{Op}^w(p_\zeta)\, U. \tag{20}$$

As a consequence, *with every $P = \mathrm{Op}(p) \in \mathrm{OPN}^{m,k}(X, \Sigma)$, we can associate, for any fixed $\varrho \in \Sigma$, a family P_ϱ of differential operators of order $\leq k$ acting in $\mathcal{S}(\boldsymbol{R}^n)$ (and $\mathcal{S}'(\boldsymbol{R}^n)$) as*

$$P_{\varrho,\zeta}(y, D_y) = \mathrm{Op}^w(p_\zeta)(y, D_y), \tag{21}$$

where $\zeta\colon T^*\boldsymbol{R}^n \to T_\varrho T^*X$ is any linear symplectic map. By (20), *$P_{\varrho,\zeta}$ are all unitarily equivalent* (and therefore their «spectral properties» are independent of ζ. This point will be made clear later on).

We group together in the next proposition some useful properties of the family P_ϱ, $\varrho \in \Sigma$.

PROPOSITION 2.3.

1) *Let $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$ and $\tilde{P} = FPF^{-1} \in \mathrm{OPN}^{m,k}(Y, \chi(\Sigma))$ be obtained as above through a homogeneous canonical transformation $\chi\colon T^*X \backslash 0 \to T^*Y \backslash 0$. Then*

$$\tilde{P}_{\chi(\varrho),\, d\chi(\varrho)\circ\zeta} = P_{\varrho,\zeta} \tag{22}$$

for any $\varrho \in \Sigma$ and any linear symplectic map $\zeta\colon T^*\boldsymbol{R}^n \to T_\varrho T^*X$.

2) *If* $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$, *then*

$$(P_{\varrho,\zeta})^* = (P^*)_{\varrho,\zeta} \tag{23}$$

for any $\varrho \in \Sigma$ *and* ζ *as above.*

3) *If* $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$, $Q \in \mathrm{OPN}^{m',k'}(X, \Sigma)$, *then*

$$(PQ)_{\varrho,\zeta} = P_{\varrho,\zeta} Q_{\varrho,\zeta} \tag{24}$$

for any $\varrho \in \Sigma$ *and* ζ *as above.*

4) *Suppose* $P = \mathrm{Op}(p) \in \mathrm{OPN}^{m,k}(X, \Sigma)$, $X \subset \boldsymbol{R}^n$, *with* $p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi)$, *and, for a fixed* $\varrho \in \Sigma$, *let* ζ *be the canonical indentification of* $\boldsymbol{R}^n_x \times (\boldsymbol{R}^n)'_\xi$ *with* $T_\varrho T^* X$. *Then*

$$P_{\varrho,\zeta}(x, D_x) = \sum_{|\alpha| + |\beta| + 2j = k} \frac{1}{\alpha!\beta!} (\partial_x^\alpha \partial_\xi^\beta p_{m-j})(\varrho)\, x^\alpha D_x^\beta. \tag{25}$$

PROOF.

1) (22) is a trivial consequence of (16).

2) To prove (23), we recall that if $P = \mathrm{Op}(p)$ then $P^* = \mathrm{Op}(p^*)$, $p^* \sim \sum_{\alpha \geq 0} \partial_\xi^\alpha D_x^\alpha \bar{p}/\alpha!$, and hence (see [8])

$$(p^*)_w = \bar{p}_w.$$

Thus

$$p_\varrho^{*(k)}(v) = \overline{p_\varrho^{(k)}(v)}, \qquad \forall \varrho \in \Sigma, \quad \forall v \in T_\varrho T^* X.$$

As a consequence,

$$(\mathrm{Op}^w(p_\varrho^{(k)} \circ \zeta))^* = \mathrm{Op}^w(p_\varrho^{*(k)} \circ \zeta),$$

for any ϱ, ζ, which is (23).

3) We recall (see [8]) that if $P = \mathrm{Op}(p)$, $Q = \mathrm{Op}(q)$, $PQ = \mathrm{Op}(r)$, then

$$r_w(x, \xi) = e^{i\sigma(D_x, D_\xi; D_y, D_\eta)/2}(p_w(x, \xi)\, q_w(y, \eta))\big|_{\substack{x=y\\ \xi=\eta}} =: (p_w \# q_w)(x, \xi),$$

with $\sigma(D_x, D_\xi; D_y, D_\eta) = \langle D_y, D_\xi \rangle - \langle D_x, D_\eta \rangle$. Thus

$$r_\varrho^{(k+k')} \circ \zeta = (p_\varrho^{(k)} \circ \zeta) \# (q_\varrho^{(k')} \circ \zeta),$$

and (24) follows by Weyl-quantization.

4) (25) is an immediate consequence of the following identity:

$$(p_\varrho^{(k)} \circ \zeta)(x, \xi) = e^{\langle D_x, D_\xi \rangle / 2i} \left(\sum_{|\alpha| + |\beta| + 2j = k} \frac{1}{\alpha!\beta!} (\partial_x^\alpha \partial_\xi^\beta p_{m-j})(\varrho)\, x^\alpha \xi^\beta \right). \qquad \blacksquare$$

Using (25), we can explicitly compute P_ϱ. Some meaningful examples are given below.

Case $k=1$.

$$P_\varrho(x, D_x) = \sum_{j=1}^{n}\left(\frac{\partial p_m}{\partial x_j}(\varrho)\, x_j + \frac{\partial p_m}{\partial \xi_j}(\varrho)\, D_{x_j}\right).$$

Case $k=2$.

$$p_\varrho^{(2)}(x, \xi) = \frac{1}{2}\left\langle \text{Hess}\; p_m(\varrho)\begin{bmatrix} x \\ \xi \end{bmatrix}, \begin{bmatrix} x \\ \xi \end{bmatrix}\right\rangle + p_{m-1}^{s}(\varrho),$$

where $p_{m-1}^{s} = p_{m-1} + i\langle \partial_x, \partial_\xi \rangle p_m/2$. Equivalently,

$$p_\varrho^{(2)}(x, \xi) = \sigma\left(\begin{bmatrix} x \\ \xi \end{bmatrix}, F(\varrho)\begin{bmatrix} x \\ \xi \end{bmatrix}\right) + p_{m-1}^{s}(\varrho),$$

where

$$F(\varrho) = \frac{1}{2} J\; \text{Hess}\; p_m(\varrho), \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

is the *fundamental matrix* of $p_m/2$ at ϱ.

Observe that transverse ellipticity means

$$\text{Ker}\; F(\varrho) = \text{Ker Hess}\, p_m(\varrho) = T_\varrho \Sigma, \qquad \forall \varrho \in \Sigma.$$

A SOURCE OF EXAMPLES. – Let $P_j \in \text{OPN}^{m,k}(X, \Sigma), j = 1, 2, \ldots, N, N \geqslant 1$. For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \boldsymbol{Z}_+^N$, define

$$P^\alpha = P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_N^{\alpha_N}$$

(then $P^\alpha \in \text{OPN}^{m|\alpha|, k|\alpha|}(X, \Sigma)$). For α as above, with $|\alpha| \leqslant \mu$, $\mu \in \boldsymbol{N}$, let A_α be a classical *ψdo* of order 0, with principal symbol a_α. Define

$$R = \sum_{|\alpha| \leqslant \mu} A_\alpha P^\alpha \in \text{OPN}^{m\mu, k\mu}(X, \Sigma).$$

By Proposition 2.3,

$$R_\varrho = \sum_{|\alpha| \leqslant \mu} a_\alpha(\varrho)\, P_\varrho^\alpha, \qquad \varrho \in \Sigma,$$

where $P_\varrho^\alpha = P_{1,\varrho}^{\alpha_1} P_{2,\varrho}^{\alpha_2} \ldots P_{N,\varrho}^{\alpha_N}$. In case all the P_j's are transversally elliptic, a sufficient condition in order for R to be transversally elliptic may be given as follows.

For $\gamma \geqslant 0$ define

$$\Lambda_\gamma = \{z \in \boldsymbol{C};\ \text{Re}\, z \geqslant 0,\ |\text{Im}\, z| \leqslant \gamma\ \text{Re}\, z\}.$$

Suppose that for some $\gamma \geq 0$, *we have*

$$p_{m,j}(x, \xi) \in \Lambda_\gamma, \qquad j = 1, \ldots, N, \quad \forall (x, \xi) \in T^*X \backslash 0 . \tag{26}$$

Moreover, suppose that *for any conic set with compact base* $\Gamma \subset T^*X\backslash 0$, *there exists* $C_\Gamma > 0$ *and* $\beta \in \boldsymbol{Z}_+^N$, *with* $|\beta| = \mu$, *for which*

$$\Big| \sum_{|\alpha| = \mu} a_\alpha(x, \xi)\, \tau^\alpha \Big| \geq C_\Gamma \prod_{j=1}^{N} |\tau_j|^{\beta_j}, \tag{27}$$

for every $(x, \xi) \in \Gamma$ *and for every* $\tau = (\tau_1, \ldots, \tau_N)$, $\tau_j \in \Lambda_\gamma$, $j = 1, 2, \ldots, N$. We leave it to the reader to check that conditions (26), (27) and the transversal ellipticity of the P_j's indeed imply the transversal ellipticity of R. We note in passing that the cases studied in [12], [13], [14] fall in this latter setup.

3. – Setting of the problem and necessary conditions.

The natural generalization of inequality (7) of the Introduction to the present framework may be stated as follows.

Suppose $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$. When is it true that

For any compact $K \subset X$ *there exists* $C_K > 0$ *such that*

$$(Pu, u) \geq -C_K \|u\|^2_{m/2 - (k+2)/4}, \qquad \forall u \in C_0^\infty(K)? \tag{28}$$

The reason why the Sobolev exponent $m/2 - (k+2)/4$ is chosen in (28), is that we look for a lower bound which depends only on the first $k/2$ terms of the total symbol of P, precisely the terms which, due to conditions (8), vanish on Σ. Notice that when $k = 2$, (28) reduces to (7) (see also [12], [13], [14], when $k = 4$).

One could also ask for a generalization of Melin's inequality (5) of the Introduction. Namely, supposing $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$, when is it true that

For any $\varepsilon > 0$, *any* $\mu < m/2 - k/4$ *and any compact* $K \subset X$ *there exists* $C_{\varepsilon, \mu, K} > 0$ *such that*

$$(Pu, u) \geq -\varepsilon \|u\|^2_{m/2 - k/4} - C_{\varepsilon,\mu,K} \|u\|^2_\mu, \qquad \forall u \in C_0^\infty(K)? \tag{29}$$

Obviously inequality (29) is a consequence of (28), the converse being in general false.

The following proof of a necessary condition for (29) to hold relies on completely standard arguments that we recall for the sake of completeness.

THEOREM 3.1. – *Let* $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$ *satisfy inequality* (29). *Denote by* $P_\varrho(x, D_x)$, $x \in \boldsymbol{R}^n$, $\varrho \in \Sigma$, *the operator attached as above to* P. *Then*

$$(P_\varrho f, f) \geq 0, \qquad \forall f \in \mathcal{S}(\boldsymbol{R}^n). \tag{30}$$

PROOF. – Take $\varrho = (x^0, \xi^0) \in \Sigma$, $|\xi^0| = 1$, and fix any compact neighborhood K of x^0. Let $K' \subset X$ be a compact such that $\mathrm{supp}(Pu) \subset K'$ if $u \in C_0^\infty(K)$. Take $\chi \in C_0^\infty(X)$ with $\chi \equiv 1$ near $K \cup K'$, so that $Pu = \chi P(\chi u) = \mathrm{Op}(p)\, u$, for $u \in C_0^\infty(K)$ and $p \in N^{m,k}(X, \Sigma)$, with p coinciding with the total symbol of P over K. Let now $v \in C_0^\infty(\boldsymbol{R}^n)$ and $t \geq 1$. Put

$$u_t(x) = e^{it^2\langle x, \xi^0\rangle} v(t(x - x^0))\,. \tag{31}$$

For t large, $u_t \in C_0^\infty(K)$ and one computes

$$\begin{cases} \widehat{u}_t(\xi) = t^{-n} e^{i\langle x^0, t^2\xi^0 - \xi\rangle} \widehat{v}(\xi/t - t\xi^0)\,, \\ \|u\|_s^2 = t^{4s-n}(\|v\|_0^2 + o(1)) \qquad \text{as } t \to +\infty\,, \end{cases} \tag{32}$$

for any $s \in \boldsymbol{R}$.

On the other hand,

$$Pu_t(x) = e^{it^2\langle x, \xi^0\rangle} \phi_t(t(x - x^0))$$

with

$$\phi_t(x) = (2\pi)^{-n} \int e^{i\langle x, \eta\rangle} p(x^0 + x/t, t\eta + t^2\xi^0)\, \widehat{v}(\eta)\, d\eta\,.$$

Taylor expanding then yields

$$p(x^0 + x/t, t\eta + t^2\xi^0) =$$

$$= t^{4(m/2 - k/4)} \sum_{|\alpha| + |\beta| + 2j = k} \frac{1}{\alpha!\beta!} (\partial_x^\alpha \partial_\xi^\beta p_{m-j})(\varrho)\, x^\alpha \eta^\beta + O(t^{4(m/2 - (k+1)/4)})\,.$$

Hence,

$$(Pu_t, u_t) = t^{4(m/2 - k/4) - n}(P_\varrho(y, D_y)\, v, v) + O(t^{4(m/2 - (k+1)/4) - n}) \geq$$

$$\geq -\varepsilon t^{4(m/2 - k/4) - n}(\|v\|_0^2 + o(1)) - C_{\varepsilon, \mu, K} t^{4\mu - n}(\|v\|_0^2 + o(1))\,,$$

as $t \to +\infty$. Dividing by $t^{4(m/2 - k/4) - n}$, and letting $t \to +\infty$, $\varepsilon \to 0+$, gives

$$(P_\varrho v, v) \geq 0, \qquad \forall v \in C_0^\infty(\boldsymbol{R}^n)\,. \qquad \blacksquare$$

Some remarks are in order.

1) It follows from the proof that (29) implies

$$p_m(x, \xi) \geq 0 \qquad \text{on } T^*X \backslash 0\,.$$

Thus, k is an even integer (as we shall suppose from now on).

2) It is also clear that the above proof is purely pointwise-microlocal. One could define $P_\varrho(x, D_x)$ directly by (25) and get the same consequence (30).

To prove sufficiency, it seems that one is forced to require the invariant setting developed earlier.

In [11], Mohamed proves the following sharp lower bound.

THEOREM 3.2. – *Let* $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$, *and suppose*:

1) *P is transversally elliptic.*

2) *For any* $\varrho \in \Sigma$, $(P_\varrho f, f) \geq 0$, $\forall f \in \mathcal{S}(\boldsymbol{R}^n)$.

3) *For any* $\varrho \in \Sigma$, $P_\varrho \colon \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}(\boldsymbol{R}^n)$ *is injective.*

4) $m > k/2$.

Then, for any compact $K \subset X$ *there exists* $C_K > 0$ *such that*

$$(Pu, u) \geq -C_K \|u\|_0^2, \qquad \forall u \in C_0^\infty(K). \tag{33}$$

It is now an easy matter to obtain Melin inequality (29) from Mohamed's result. Precisely, we have the

THEOREM 3.3. – *Let* $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$, *and suppose*:

1) *P is transversally elliptic.*

2) *For any* $\varrho \in \Sigma$, $(P_\varrho f, f) \geq 0$, $\forall f \in \mathcal{S}(\boldsymbol{R}^n)$.

Then inequality (29) *holds.*

PROOF. – We can suppose $m > k/2$, for otherwise we write

$$(Pu, u) = (\Lambda^{-\sigma} P \Lambda^{-\sigma} \Lambda^\sigma u, \Lambda^\sigma u),$$

with $\sigma < m/2 - k/4$, and observe that

$$\Lambda^{-\sigma} P \Lambda^{-\sigma} \in \mathrm{OPN}^{m-2\sigma, k}(X, \Sigma),$$

it is transversally elliptic, and for any $\varrho = (x, \xi) \in \Sigma$,

$$(\Lambda^{-\sigma} P \Lambda^{-\sigma})_\varrho = |\xi|^{-2\sigma} P_\varrho .$$

To prove (29), define

$$P_\varepsilon = P + \varepsilon \Lambda^{m-k/2}, \qquad \varepsilon > 0 .$$

It is immediate to check that conditions (1) to (4) in Theorem 3.3 are satisfied by P_ε. Hence, from (33), we obtain that for any $\varepsilon > 0$, any compact $K \subset X$, there exists $C_{\varepsilon, K} > 0$, such that

$$(P_\varepsilon u, u) = (Pu, u) + \varepsilon(\Lambda^{m-k/2} u, u) \geq -C_{\varepsilon, K} \|u\|_0^2, \qquad \forall u \in C_0^\infty(K),$$

which yields (29) with $\mu = 0$. ■

We stress upon the fact that in the double-characteristic case (i.e. $k = 2$), Theorem 3.3 does not recover Melin's result in its full strength, for in the latter Σ

need not be smooth and, *most importantly*, P is not required to be transversally elliptic. We do not know to what extent condition (1) in Theorem 3.3 can be relaxed.

We now turn to Hörmander's counterpart, that is inequality (28).

As a matter of fact, it is impossible to get (28) from (29) by perturbation arguments.

Our approach will be the content of the next section.

4. – Geometrical assumptions and precised invariants.

From now on we suppose that $\Sigma \subset T^*X\backslash 0$ satisfies the following assumption (H) (assumptions (H1) to (H3) below):

(H1) Σ *has fixed codimension.*

(H2) *The symplectic form* σ *has constant rank on* Σ, *i.e.* $\dim(T_\varrho\Sigma \cap T_\varrho\Sigma^\sigma) = \text{constant}$, *for any* $\varrho \in \Sigma$.

(H3) *The canonical* 1-*form* $\sum_{j=1}^{n} \xi_j \, dx_j$ *does not vanish identically on* $T_\varrho\Sigma$, $\varrho \in \Sigma$.

When Σ is *involutive*, i.e. $T_\varrho\Sigma^\sigma \subset T_\varrho\Sigma$, for any $\varrho \in \Sigma$, we have the following result.

THEOREM 4.1. – *Let* $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$, *and suppose*:

1) P *is transversally elliptic.*

2) *For any* $\varrho \in \Sigma$, $(P_\varrho f, f) \geq 0$, $\forall f \in \mathcal{S}(\boldsymbol{R}^n)$.

3) Σ *satisfies* (H) *and is involutive.*

Then inequality (28) *holds.*

We now turn to the more difficult case in which Σ is *non-involutive*, that is when

$$T_\varrho\Sigma^\sigma \not\subset T_\varrho\Sigma, \qquad \forall \varrho \in \Sigma.$$

By virtue of condition (H2), we have only two possibilities:

either

$$T_\varrho\Sigma \cap T_\varrho\Sigma^\sigma = (0), \qquad \forall \varrho \in \Sigma.$$

(the *symplectic case*), or

$$\begin{cases} T_\varrho\Sigma^\sigma / (T_\varrho\Sigma \cap T_\varrho\Sigma^\sigma) \neq (0), \\ T_\varrho\Sigma \cap T_\varrho\Sigma^\sigma \neq (0), \end{cases} \qquad \forall \varrho \in \Sigma,$$

(the *non-involutive and non-symplectic case*).

We will only deal with the second case, leaving to the reader the required adjustments needed when Σ is symplectic.

Let $2\nu = \dim(T_\varrho \Sigma^\sigma/(T_\varrho \Sigma \cap T_\varrho \Sigma^\sigma))$, and $l = \dim(T_\varrho \Sigma \cap T_\varrho \Sigma^\sigma)$.

It is important to notice that the integers ν and l above are *independent of* $\varrho \in \Sigma$, by virtue of condition (H). In fact,

$$\begin{cases} 2\nu + l = \text{codim}\, \Sigma\,, \\ 2(n - (\nu + l)) = \text{rk}\, \sigma|_\Sigma. \end{cases} \tag{34}$$

In the sequel a special role will be played by a particular class of local symplectic coordinates near Σ, whose existence is guaranteed by Thm. 21.2.4 of [8], Vol. III.

Precisely: *Given any* $\varrho_0 \in \Sigma$, *there exist*

(i) *a conic neighborhood* $\Gamma \subset T^*X \backslash 0$ *of* ϱ_0,

(ii) *an open conic set* $\tilde{\Gamma} \subset T^*\boldsymbol{R}^\nu \times T^*\boldsymbol{R}^l \times (T^*\boldsymbol{R}^{n-(\nu+l)} \backslash 0)$,

(iii) *a smooth symplectomorphism* (*homogeneous of degree 1 in the fibers*) $\chi\colon \Gamma \to \tilde{\Gamma}$, *for which*

$$\chi(\Gamma \cap \Sigma) = \{(y, \eta) = (y', \eta', y'', \eta'', y''', \eta''') \in \tilde{\Gamma}; \quad y' = \eta' = 0,\ \eta'' = 0\}. \tag{35}$$

Such a map will be called *a canonical flattening of* Σ (*near* ϱ_0).

Remark that in condition (iii) above, «symplectic» means $\chi^*(\tilde{\sigma}) = \sigma$, where

$$\tilde{\sigma} = \sum_{j=1}^{\nu} d\eta'_j \wedge dy'_j + \sum_{j=1}^{l} d\eta''_j \wedge dy''_j + \sum_{j=1}^{n-(\nu+l)} d\eta'''_j \wedge dy'''_j.$$

Denote by $N\Sigma = TT^*X/T\Sigma$ the *normal bundle* to Σ ($\dim N_\varrho \Sigma = 2\nu + l$, $\varrho \in \Sigma$). Any canonical flattening $\chi\colon \Gamma \to \tilde{\Gamma}$ of Σ induces a local trivialization of the vector-bundle $N\Sigma$.

Precisely, if we identify, by means of (35), $\chi(\Gamma \cap \Sigma)$ with an open conic set of $\boldsymbol{R}^l \times (T^*\boldsymbol{R}^{n-(\nu+l)} \backslash 0)$, we can define the map

$$\begin{cases} \zeta_\chi\colon \chi(\Gamma \cap \Sigma) \times \boldsymbol{R}^{2\nu+l} \to N\Sigma|_{\Gamma \cap \Sigma}\,, \\ \zeta_\chi(\chi(\varrho); (z', \zeta', \zeta'')) = \left(\varrho, \left[d\chi^{-1}(\chi(\varrho)) \begin{bmatrix} z' \\ \zeta' \\ 0 \\ \zeta'' \\ 0 \\ 0 \end{bmatrix} \right]\right), \end{cases} \tag{36}$$

$\varrho \in \Gamma \cap \Sigma$, $(z', \zeta') \in \boldsymbol{R}^{2\nu}$, $\zeta'' \in \boldsymbol{R}^l$, where $[v]$ denotes the *residue class* in $N_\varrho \Sigma$ of $v \in T_\varrho T^*X$.

The following lemma shows the nature of the structure-group associated with the trivializations ζ_χ.

LEMMA 4.2. – *Let* $\chi\colon \Gamma \to \tilde{\Gamma}$, $\chi'\colon \Gamma' \to \tilde{\Gamma}'$ *be two canonical flattenings of* Σ *with* $\Gamma \cap \Gamma' \cap \Sigma \neq \emptyset$. *Then the map*

$$\zeta_{\chi'}^{-1} \circ \zeta_\chi\colon \chi(\Gamma \cap \Gamma' \cap \Sigma) \times \boldsymbol{R}^{2\nu + l} \to \chi'(\Gamma \cap \Gamma' \cap \Sigma) \times \boldsymbol{R}^{2\nu + l}$$

takes the form

$$(37) \qquad (\zeta_{\chi'}^{-1} \circ \zeta_\chi)\left(\chi(\varrho); \begin{bmatrix} \begin{bmatrix} z' \\ \zeta' \end{bmatrix} \\ \zeta'' \end{bmatrix}\right) = \left(\chi'(\varrho); \begin{bmatrix} \alpha(\varrho) & \beta(\varrho) \\ 0 & \gamma(\varrho) \end{bmatrix} \begin{bmatrix} \begin{bmatrix} z' \\ \zeta' \end{bmatrix} \\ \zeta'' \end{bmatrix}\right),$$

where

$$(38) \qquad \begin{cases} \alpha \in C^\infty(\Gamma \cap \Gamma' \cap \Sigma, \mathrm{Sp}(2\nu, \boldsymbol{R})), \\ \gamma \in C^\infty(\Gamma \cap \Gamma' \cap \Sigma, \mathrm{GL}(l, \boldsymbol{R})), \\ \beta \in C^\infty(\Gamma \cap \Gamma' \cap \Sigma, \mathrm{Mat}(2\nu \times l, \boldsymbol{R})). \end{cases}$$

The proof relies on standard symplectic linear algebra (see [8], Vol. III, paragraph 21.2) and will be given in Appendix 1.

From now on, we will consider $N\Sigma$ *as a vector-bundle over* Σ *with respect to the structure given by* (36), (37).

Notice that to any canonical flattening $\chi\colon \Gamma \to \tilde{\Gamma}$ of Σ, there corresponds, for every $\varrho \in \Gamma \cap \Sigma$, a symplectic basis of $T_\varrho T^*X$, given by the linear symplectic map

$$(39) \qquad \begin{cases} \theta_\varrho\colon T^*\boldsymbol{R}^n \simeq T^*\boldsymbol{R}^\nu \times T^*\boldsymbol{R}^l \times T^*\boldsymbol{R}^{n-(\nu+l)} \to T_\varrho T^*X, \\ \theta_\varrho(z', \zeta', z'', \zeta'', z''', \zeta''') = d\chi^{-1}(\chi(\varrho)) \begin{bmatrix} z' \\ \zeta' \\ z'' \\ \zeta'' \\ z''' \\ \zeta''' \end{bmatrix}. \end{cases}$$

Supposing that $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$ (Σ as above), we can write down the operator $P_{\varrho, \theta_\varrho}$, $\varrho \in \Gamma \cap \Sigma$ (see (21)). Precisely, consider the polynomial map $p^{(k)}$ defined in (15). It induces naturally, by virtue of Proposition 2.2, a smooth mapping, still denoted by $p^{(k)}$,

$$p^{(k)}\colon N\Sigma \to \boldsymbol{C},$$

which is a polynomial of degree $\leq k$ in the fibers.

Now consider

$$(40) \qquad (p^{(k)} \circ \zeta_\chi)(\chi(\varrho); z', \zeta', \zeta''),$$

as a polynomial in $((z', \zeta'), \zeta'') \in \boldsymbol{R}^{2\nu} \times \boldsymbol{R}^{l}$, and observe that it is *elliptic* if P is *transversally elliptic*.

We claim that for every $\varrho \in \Gamma \cap \Sigma$,

$$P_{\varrho, \theta_\varrho}(z', D_{z'}, z'', D_{z''}, z''', D_{z'''}) = \mathrm{Op}^w(p^{(k)} \circ \zeta_\chi)(\chi(\varrho); z', D_{z'}, D_{z''}) \otimes \mathrm{Id}_{z'''}. \tag{41}$$

The proof is obvious.

From (41) it follows that $P_{\varrho, \theta_\varrho}$ may be thought of as an unbounded operator in $L^2(\boldsymbol{R}^{\nu + l})$. The problem with this is that even if P is transversally elliptic, $P_{\varrho, \theta_\varrho}$ does not have discrete spectrum in $L^2(\boldsymbol{R}^{\nu + l})$. However, if we Weyl-quantize the polynomial (40) with respect to the variables (z', ζ') *only*, thus thinking of ζ'' as a parameter together with ϱ, the resulting «k-th order oscillator» has in fact discrete spectrum in $L^2(\boldsymbol{R}^{\nu})$, when P is *transversally elliptic*. The related «eigenvalues» are henceforth functions of the parameters ϱ, ζ''. Since our generalization of Hörmander's inequality is completely based upon spectral properties of P_ϱ, the crux of the matter consists of giving them *invariance*.

We now make a digression to develop some abstract setting required to achieve the aforementioned invariance.

DEFINITION 4.3. – *Let E, F be smooth manifolds, and let*

$$r \colon E \to F$$

be a surjective submersion. We say that the triple (E, F, r) is a smooth symplectic fibration of rank 2ν $(\nu \geq 1)$ *if: For some atlas $\{(\eta, V_\eta)\}$ of F, with $V_\eta \subset \boldsymbol{R}^N$, $N = \dim F$, $\eta \colon V_\eta \to \eta(V_\eta) \subset F$ smooth diffeomorphisms, there exist smooth diffeomorphisms*

$$\psi_\eta \colon V_\eta \times \boldsymbol{R}^{2\nu} \to r^{-1}(\eta(V_\eta))$$

with

$$r\left(\psi_\eta\left(x, \begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right)\right) = \eta(x), \qquad \forall x \in V_\eta, \quad \forall \begin{bmatrix} z' \\ \zeta' \end{bmatrix} \in \boldsymbol{R}^{2\nu},$$

such that

$$\begin{cases} \psi_{\eta'}^{-1} \circ \psi_\eta \colon (V_\eta \cap V_{\eta'}) \times \boldsymbol{R}^{2\nu} \to (V_\eta \cap V_{\eta'}) \times \boldsymbol{R}^{2\nu}, \\ (\psi_{\eta'}^{-1} \circ \psi_\eta)\left(x, \begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) = \left(\eta'^{-1}(\eta(x)),\, a_{\eta'\eta}(x) \begin{bmatrix} z' \\ \zeta' \end{bmatrix} + b_{\eta'\eta}(x)\right), \end{cases} \tag{42}$$

where

$$\begin{cases} a_{\eta'\eta} \in C^\infty(V_\eta \cap V_{\eta'}, \mathrm{Sp}(2\nu, \boldsymbol{R})), & a_{\eta\eta}(x) \equiv I_{2\nu}, \quad \forall x \in V_\eta, \\ b_{\eta'\eta} \in C^\infty(V_\eta \cap V_{\eta'}, \boldsymbol{R}^{2\nu}), & b_{\eta\eta}(x) \equiv 0, \quad \forall x \in V_\eta, \end{cases} \tag{43}$$

and with the following cocycle conditions satisfied:

$$\begin{cases} a_{\eta''\eta}(x) = a_{\eta''\eta'}(x)\, a_{\eta'\eta}(x)\,, \\ b_{\eta''\eta}(x) = a_{\eta''\eta'}(x)\, b_{\eta'\eta}(x) + b_{\eta''\eta'}(x), \end{cases} \qquad \forall x \in V_\eta \cap V_{\eta'} \cap V_{\eta''}. \tag{44}$$

DEFINITION 4.4. – *Suppose* (E, F, r) *is a symplectic fibration of rank* 2ν, *and let* $q\colon E \to \boldsymbol{C}$ *be a smooth function. We say that* $q \in \boldsymbol{S}^k_{\mathrm{reg}}(E)$, $k \in \boldsymbol{R}$, *if for any trivialization* $\psi_\eta\colon V_\eta \times \boldsymbol{R}^{2\nu} \to r^{-1}(\eta(V_\eta))$ *as above, we have*

$$(q \circ \psi_\eta)\left(x, \begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) =: q_\eta(x; z', \zeta') \in C^\infty(V_\eta, \boldsymbol{S}^k_{\mathrm{reg}}(\boldsymbol{R}^{2\nu}))\,. \tag{45}$$

Recall that $\boldsymbol{S}^k(\boldsymbol{R}^n)$ denotes the space of all smooth functions f such that for every $\alpha \in \boldsymbol{Z}^n_+$:

$$\sup_{y \in \boldsymbol{R}^n} (1 + |y|)^{|\alpha| - k}\, |\partial^\alpha_y f(y)| < +\infty\,.$$

By $\boldsymbol{S}^k_{\mathrm{reg}}(\boldsymbol{R}^n)$ we denote the elements $f \in \boldsymbol{S}^k(\boldsymbol{R}^n)$ which admit an asymptotic expansion $f(y) \sim \sum_{j \geq 0} f_{k-j}(y)$, where f_{k-j} is (positively) homogeneous of degree $k-j$, $j \geq 0$.

DEFINITION 4.5. – *We say that* $q \in \boldsymbol{S}^k_{\mathrm{reg}}(E)$ *is* elliptic *iff for any* ψ_η *as above*

$$q_\eta(x; z', \zeta') \sim \sum_{j \geq 0} q_{k-j,\,\eta}(x; z', \zeta')\,,$$

with

$$q_{k,\,\eta}(x; z', \zeta') \neq 0\,, \qquad \forall (x; z', \zeta') \in V_\eta \times (\boldsymbol{R}^{2\nu} \setminus \{0\})\,. \tag{46}$$

REMARK 4.6. – *Definitions* 4.4-4.5 *do not depend on the particular* $\{\psi_\eta\}$ *chosen, as it will be shown in Appendix* 2.

It makes now sense to Weyl-quantize each $q_\eta(x; \cdot)$ by setting

$$\mathrm{Op}^w(q_\eta)(x; z', D_{z'})\colon \mathcal{S}(\boldsymbol{R}^\nu) \to \mathcal{S}(\boldsymbol{R}^\nu), \qquad x \in V_\eta. \tag{47}$$

By (42) and [8], Vol. III, Thm. 18.5.9, we have the following relation between $\mathrm{Op}^w(q_\eta)$ and $\mathrm{Op}^w(q_{\eta'})$ on $V_\eta \cap V_{\eta'}$:

$$\mathrm{Op}^w(q_{\eta'})((\eta'^{-1} \circ \eta)(x); \cdot) = U(x)^{-1}\, \mathrm{Op}^w(q_\eta)(x; \cdot)\, U(x), \qquad \forall x \in V_\eta \cap V_{\eta'}, \tag{48}$$

where $x \mapsto U(x)$ is a smooth family of unitary operators in $L^2(\boldsymbol{R}^\nu)$, which are also automorphisms of $\mathcal{S}(\boldsymbol{R}^\nu)$ and $\mathcal{S}'(\boldsymbol{R}^\nu)$.

Suppose now that $q \in S^k_{\rm reg}(E)$ has the following properties:

$$q \text{ is elliptic and } k > 0\,, \tag{49}$$

for every η as above

$$\begin{cases} (\mathrm{Op}^w(q_\eta)(x;\cdot)\,\phi,\,\psi) = (\phi,\,\mathrm{Op}^w(q_\eta)(x;\cdot)\,\psi)\,, \\ (\mathrm{Op}^w(q_\eta)(x;\cdot)\,\phi,\,\phi) \geqslant 0\,, \end{cases} \quad \forall x \in V_\eta, \quad \forall \phi,\,\psi \in \mathcal{S}(\boldsymbol{R}^\nu)\,. \tag{50}$$

Then it is well-known (see, e.g., [15]) that every such $\mathrm{Op}^w(q_\eta)(x;\cdot)$ admits a unique self-adjoint realization as an unbounded operator in $L^2(\boldsymbol{R}^\nu)$, whose spectrum

$$\mathrm{Spec}\ \mathrm{Op}^w(q_\eta)(x;\cdot) \subset [0,\,+\infty)$$

is discrete, made of eigenvalues (with finite multiplicity) tending to $+\infty$.

As a consequence, we can define

$$\lambda_\eta(x) = \min\,(\mathrm{Spec}\ \mathrm{Op}^w(q_\eta)(x;\cdot)), \qquad x \in V_\eta. \tag{51}$$

Observe that $\lambda_\eta\colon V_\eta \to [0,\,+\infty)$ is a-priori only a continuous function. Furthermore, by (48) we have

$$\lambda_{\eta'}(\eta'^{-1}(\eta(x))) = \lambda_\eta(x)\,, \qquad \forall x \in V_\eta \cap V_{\eta'}. \tag{52}$$

(52) yields the existence of a well-defined continuous function

$$\lambda\colon F \to [0,\,+\infty) \qquad (\lambda(\eta(x)) = \lambda_\eta(x), \quad x \in V_\eta)\,, \tag{53}$$

which will be called *the ground energy of q*.

DEFINITION 4.7. – *We say that $q \in S^k_{\rm reg}(E)$, satisfying* (49), (50), *is* tame-degenerate *iff*

$$\lambda(\eta(x^0)) = 0 \;\Rightarrow\; \dim \mathrm{Ker}\big(\mathrm{Op}^w(q_\eta)(x;\cdot) - \lambda(\eta(x))\big) = \text{constant}\,, \tag{54}$$

for any x in a suitable neighborhood of x^0 contained in V_η.

It is important to observe that (54) is a condition on the ground energy of q and it yields (see Appendix 3) that λ is actually *smooth* in a neighborhood of its zero-set.

We now show how, in case Σ is non-involutive and non-symplectic, the family P_ϱ of Section 2 fits in the abstract setting just developed.

We put $E = N\Sigma$ and $F = (T\Sigma \cap T\Sigma^\sigma)'$, the dual bundle of the vector-bundle on Σ (of rank l) $T\Sigma \cap T\Sigma^\sigma$. *As regards the map $r\colon E \to F$, we define it by*

$$r(\varrho,[v]) = (\varrho,\,\sigma_\varrho(v,\cdot))\,, \qquad \varrho \in \Sigma, \quad v \in T_\varrho T^*X\,. \tag{55}$$

It is straightforward to check that r is a surjective morphism between the vector-bundles E and F (whose kernel can be canonically identified with the vector-bundle $T\Sigma^\sigma/(T\Sigma \cap T\Sigma^\sigma)$. We next define the atlas $\{(\eta,\,V_\eta)\}$ of F.

Any canonical flattening $\chi\colon \Gamma \to \tilde{\Gamma}$ of Σ induces a trivialization of F by the map

$$(56)\qquad \begin{cases} \eta_\chi\colon \chi(\Gamma \cap \Sigma) \times \boldsymbol{R}^l \to F|_{\Gamma\cap\Sigma}, \\ \eta_\chi(\chi(\varrho), \zeta'') = r\left(\varrho, \left[d\chi^{-1}(\chi(\varrho))\begin{bmatrix} 0\\0\\0\\ \zeta''\\0\\0\end{bmatrix}\right]\right). \end{cases}$$

Hence, we choose

$$\eta = \eta_\chi \qquad \text{and} \qquad V_\eta = \chi(\Gamma \cap \Sigma) \times \boldsymbol{R}^l,$$

with χ ranging in the set of the canonical flattenings of Σ. The corresponding ψ_η are defined as follows:

$$(57)\qquad \begin{cases} \psi_\eta\colon (\chi(\Gamma\cap\Sigma)\times \boldsymbol{R}^l)\times \boldsymbol{R}^{2\nu} \to r^{-1}(\eta_\chi(\chi(\Gamma\cap\Sigma)\times\boldsymbol{R}^l)), \\ \psi_\eta\left((\chi(\varrho), \zeta''), \begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) := \zeta_\chi(\chi(\varrho); z', \zeta', \zeta''), \end{cases}$$

where ζ_χ is defined in (36).

From (37) we get

$$(58)\qquad (\psi_{\eta'}^{-1}\circ\psi_\eta)\left((\chi(\varrho), \zeta''), \begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) = \left((\chi'(\varrho), \gamma(\varrho)\,\zeta''),\ \alpha(\varrho)\begin{bmatrix} z' \\ \zeta' \end{bmatrix} + \beta(\varrho)\,\zeta''\right),$$

whence, upon calling $x = (\chi(\varrho), \zeta'')$,

$$(59)\qquad \begin{cases} (\chi'(\varrho), \gamma(\varrho)\,\zeta'') = \eta_{\chi'}^{-1}(\eta_\chi(x)), \\ a_{\eta'\eta}(x) = \alpha(\varrho), \qquad b_{\eta'\eta}(x) = \beta(\varrho)\,\zeta''. \end{cases}$$

We leave it to the reader to check that conditions (43), (44) are satisfied.

Suppose now that $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$, Σ as above, satisfies the following conditions:

$$(60)\qquad \begin{cases} 1)\ P = P^*. \\ 2)\ P \textit{ is transversally elliptic}. \\ 3)\ \textit{For any } \varrho\in\Sigma, \quad (P_\varrho f, f) \geq 0, \quad \forall f \in \mathcal{S}(\boldsymbol{R}^n). \end{cases}$$

The preceding discussion allows us to define the *ground energy of* P as follows.

We take as q in Definition 4.4 the polynomial map $p^{(k)}: N\Sigma \to \boldsymbol{R}$, and check that $p^{(k)} \in \boldsymbol{S}^k_{\text{reg}}(E)$ satisfies (49) and (50). Observe that

$$p_\eta^{(k)}\left((\chi(\varrho),\, \zeta'');\begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) = (p^{(k)}\circ\psi_\eta)\left((\chi(\varrho),\, \zeta'');\begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right) =$$

$$= (p^{(k)}\circ\zeta_\chi)(\chi(\varrho);\, z',\, \zeta',\, \zeta'') = \sum_{|\alpha|+|\beta|+|\gamma|\leq k} c_{\alpha\beta\gamma}(\varrho)\, z'^{\alpha}\zeta'^{\beta}\zeta''^{\gamma},$$

for some smooth coefficients $c_{\alpha\beta\gamma}(\varrho)$, $\varrho \in \Gamma \cap \Sigma$.

It is then obvious that $p_\eta^{(k)} \in C^\infty(\chi(\Gamma\cap\Sigma)\times \boldsymbol{R}^l,\, \boldsymbol{S}^k_{\text{reg}}(\boldsymbol{R}^{2\nu}))$, with *principal symbol*

$$p_\eta^{(k)}\left((\chi(\varrho),\, \zeta''=0),\begin{bmatrix} z' \\ \zeta' \end{bmatrix}\right),$$

which does not vanish when $(z',\zeta') \in \boldsymbol{R}^{2\nu}\setminus\{0\}$, *because of the transversal ellipticity of* P.

From (41) we get the fundamental relation

$$\text{(61)}\qquad P_{\varrho,\,\theta_\varrho}(z',\, D_{z'},\, z'',\, D_{z''},\, z''',\, D_{z'''})\,\phi(z',\, z'',\, z''') =$$

$$(\mathrm{Op}^w(p^{(k)}\circ\zeta_\chi)(\chi(\varrho);\, z',\, D_{z'},\, D_{z''})\otimes \mathrm{Id}_{z'''})\,\phi(z',\, z'',\, z''') =$$

$$(2\pi)^{-l}\int e^{i\langle z'',\,\zeta''\rangle}\,\mathrm{Op}^w(p_\eta^{(k)})((\chi(\varrho),\, \zeta'');\, z',\, D_{z'})\,\widehat{\phi}(z',\, \zeta'',\, z''')\, d\zeta'',$$

for every $\phi \in \mathcal{S}(\boldsymbol{R}^{\nu+l+(n-\nu-l)})$, where

$$\widehat{\phi}(z',\, \zeta'',\, z''') = \int e^{-i\langle z'',\,\zeta''\rangle}\phi(z',\, z'',\, z''')\, dz''.$$

Condition (50) is now an immediate consequence of (61) and $P_\varrho = P_\varrho^*$.

Hence, it makes sense to give the following definition.

DEFINITION 4.8. – *If* $P \in \mathrm{OPN}^{m,\,k}(X,\Sigma)$ *satisfies conditions* (60), *we define the* ground energy λ of P *to be the ground energy* λ *of* $p^{(k)}$, *and say that* P *is* tame-degenerate *exactly when* $p^{(k)}$ *is.*

REMARK 4.9. – *Note that* λ *is a function on* $(T\Sigma \cap T\Sigma^\sigma)'$ *with values in* $[0, +\infty)$.

Remark that in all the preceding discussion we have supposed Σ to be non-involutive (i.e. $\nu \geqslant 1$) and non-symplectic (i.e. $l \geqslant 1$). When Σ is symplectic, i.e. when

$$T_\varrho \Sigma \cap T_\varrho \Sigma^\sigma = (0), \qquad \forall \varrho \in \Sigma ,$$

all the arguments above greatly simplify. In fact, $N\Sigma$ is canonically identified with the bundle $T\Sigma^\sigma$ (of rank 2ν), while $(T\Sigma \cap T\Sigma^\sigma)'$ reduces to a rank 0 vector-bundle over Σ, thus canonically identified with Σ itself. As before, we can define the ground energy λ of $P \in \mathrm{OPN}^{m,k}(X, \Sigma)$ satisfying (60). In this case, λ is naturally a function on Σ with values in $[0, +\infty)$, and, accordingly, the fact that P is tame-degenerate makes sense.

We are finally in a position to state the non-involutive counterpart of Theorem 4.1.

THEOREM 4.10. – *Let* $P = P^* \in \mathrm{OPN}^{m,k}(X, \Sigma)$, *and suppose*:

1) P *is transversally elliptic.*
2) *For any* $\varrho \in \Sigma$, $(P_\varrho f, f) \geqslant 0$, $\forall f \in \mathcal{S}(\boldsymbol{R}^n)$.
3) Σ *satisfies* (H) *and is non-involutive.*
4) P *is tame-degenerate.*

Then inequality (28) *holds.*

The proofs of Theorems 4.1 and 4.10 are too long to be given here. They will appear elsewhere.

EXAMPLES AND REMARKS.

1) The case $k = 2$. Let $P = P^* \in \mathrm{OPN}^{m,2}(X, \Sigma)$, Σ satisfying condition (H), be transversally elliptic. Recall from the Introduction that by $F(\varrho)$, $\varrho \in \Sigma$, we denote the *fundamental matrix* of $p_m/2$, i.e.

$$\sigma(v, F(\varrho)\, v) = \frac{1}{2} \langle \mathrm{Hess}\, p_m(\varrho)\, v, v \rangle, \qquad v \in T_\varrho T^* X ,$$

and by $p^s_{m-1}(\varrho) = p_{m-1}(\varrho) + i\langle \partial_x, \partial_\xi \rangle\, p_m(\varrho)/2$ the *subprincipal symbol* of P at $\varrho \in \Sigma$.

Supposing that $p_m(x, \xi) \geqslant 0$ on $T^*X\backslash 0$, let us check the equivalence of the following properties:

(*a*) $$(P_\varrho f, f) \geqslant 0, \qquad \forall \varrho \in \Sigma, \ \ \forall f \in \mathcal{S}(\boldsymbol{R}^n),$$

(*b*) $$p^s_{m-1}(\varrho) + \mathrm{Tr}^+ F(\varrho) \geqslant 0, \qquad \forall \varrho \in \Sigma ,$$

where $\mathrm{Tr}^+ F(\varrho) := \sum_{\mu > 0} \mu(\varrho)$, with $i\mu(\varrho) \in \mathrm{Spec}\, F(\varrho)$. By [8], Vol. III, Thm. 21.5.3, there exists a linear symplectic map $\zeta\colon T^*\boldsymbol{R}^n \to T_\varrho T^*X$ such that

(62) $$\sigma\left(\zeta\begin{bmatrix}x\\ \xi\end{bmatrix}, F(\varrho)\,\zeta\begin{bmatrix}x\\ \xi\end{bmatrix}\right) = \sum_{j=1}^{\nu}\mu_j(\varrho)(x_j'^2+\xi_j'^2)+\sum_{j=1}^{l}\xi_j''^2,$$

where $2\nu = \dim(T_\varrho\Sigma^\sigma/(T_\varrho\Sigma\cap T_\varrho\Sigma^\sigma))$ and $l=\dim(T_\varrho\Sigma\cap T_\varrho\Sigma^\sigma)$.

If $p_\varrho^{(2)}$ denotes the polynomial map associated with P, then

(63) $$\mathrm{Op}^w(p_\varrho^{(2)}\circ\zeta)(x, D_x) = \sum_{j=1}^{\nu}\mu_j(\varrho)(x_j'^2+D_{x_j'}^2)+\sum_{j=1}^{l}D_{x_j''}^2+p_{m-1}^s(\varrho).$$

It follows that condition (a) is equivalent to

(64) $$(\mathrm{Op}^w(p_\varrho^{(2)}\circ\zeta)(x, D_x)\,f, f)\geqslant 0\,,\qquad \forall f\in\mathcal{S}(\boldsymbol{R}^n)\,.$$

A standard density argument shows that in turn (64) amounts to

(65) $$\left(\sum_{j=1}^{\nu}\mu_j(\varrho)(x_j'^2+D_{x_j'}^2)\,\phi,\phi\right)+\sum_{j=1}^{l}\xi_j''^2+p_{m-1}^s(\varrho)\geqslant 0\,,$$

for any $\phi(x')\in\mathcal{S}(\boldsymbol{R}^\nu)$, $\|\phi\|_0^2=1$, and every $\xi''\in\boldsymbol{R}^l$.

Let

(66) $$h_k(t) := \pi^{1/4}(2^k k!)^{-1/2}\left(\frac{d}{dt}-t\right)^k e^{-t^2/2},\qquad k=0,1,\ldots$$

be the k-th Hermite function, and define

(67) $$\phi_\beta(x')=\prod_{j=1}^{\nu}h_{\beta_j}(x_j'),\qquad \beta=(\beta_1,\ldots,\beta_\nu)\in\boldsymbol{Z}_+^\nu\,.$$

It is well-known that

(68) $$\sum_{j=1}^{\nu}\mu_j(\varrho)(x_j'^2+D_{x_j'}^2)\,\phi_\beta(x')=\left(\sum_{j=1}^{\nu}\mu_j(\varrho)(2\beta_j+1)\right)\phi_\beta(x'),$$

whence, because of the density in $\mathcal{S}$ of the Hermite functions, (65) is equivalent to

(69) $$\sum_{j=1}^{\nu}\mu_j(\varrho)(2\beta_j+1)+\sum_{j=1}^{l}\xi_j''^2+p_{m-1}^s(\varrho)\geqslant 0,\qquad \forall\beta\in\boldsymbol{Z}_+^\nu,\quad \forall\xi''\in\boldsymbol{R}^l,$$

which is in turn equivalent to condition (b).

When Σ is non-involutive, we can explicitly compute the ground energy of P. Namely,

(70) $$\lambda(\varrho, v)=\mathrm{Tr}^+F(\varrho)+\sigma(v, F(\varrho)\,v)+p_{m-1}^s(\varrho)\,,$$

$v\in\mathrm{Ker}\,(F(\varrho)^2)/\mathrm{Ker}\,F(\varrho)\simeq(T_\varrho\Sigma\cap T_\varrho\Sigma^\sigma)'$.

In fact, (69) gives all the eigenvalues of

$$\mathrm{Op}^w(p_\varrho^{(2)} \circ \zeta)(x', D_{x'}; \xi'').$$

Since for every (ϱ, v) the multiplicity of the eigenvalue $\lambda(\varrho, v)$ is 1, P is *automatically tame-degenerate.*

Theorems 4.1 and 4.10 therefore recover Hörmander inequality.

2) For k larger than 2, the explicit knowledge of $\lambda(\varrho, v)$ is clearly out of reach in general. In the next example, which is a variation of the ones treated in [12], [13], [14], the function λ may be explicitly computed.

Consider, for $j = 1, 2, \ldots, N$, N operators $P_j = P_j^* \in \mathrm{OPN}^{m,2}(X, \Sigma)$ (Σ satisfying condition (H)) such that:

(A) *All the P_j's are transversally elliptic with*

$$p_{m,j}(x, \xi) \geq 0, \qquad \forall (x, \xi) \in T^*X \backslash 0 .$$

(B) $[F_j(\varrho), F_k(\varrho)] = F_j(\varrho)\, F_k(\varrho) - F_k(\varrho)\, F_j(\varrho) = 0$, *for any* $\varrho \in \Sigma$, *and any* $j, k = 1, 2, \ldots, N$, F_j *being the fundamental matrix of* $p_{m,j}/2$.

As a consequence, the operators $P_{j,\varrho}$, $j = 1, 2, \ldots, N$, commute with each other. In fact, it follows from the aforementioned Thm. 21.5.3 of Hörmander ([8], Vol. III), that for any $\varrho \in \Sigma$ we can find a symplectic map $\zeta\colon T^*\boldsymbol{R}^n \to T_\varrho T^*X$ such that

$$\sigma\left(\zeta\begin{bmatrix} x \\ \xi \end{bmatrix}, F_j(\varrho)\,\zeta\begin{bmatrix} x \\ \xi \end{bmatrix}\right) = \sum_{h=1}^{\nu} \mu_{jh}(\varrho)(x_h'^2 + \xi_h'^2) + \sum_{h=1}^{l} \alpha_{jh}(\varrho)\, \xi_h''^2, \tag{71}$$

$j = 1, 2, \ldots, N$, where $\mu_{jh}(\varrho)$, $\alpha_{jh}(\varrho) > 0$, for any j, h.

Hence, for $j = 1, \ldots, N$,

$$P_{j,\varrho} = \sum_{h=1}^{\nu} \mu_{jh}(\varrho)(x_h'^2 + D_{x_h'}^2) + \sum_{h=1}^{l} \alpha_{jh}(\varrho)\, D_{x_h''}^2 + p_{m-1,j}^s(\varrho), \tag{72}$$

which yields immediately the commutativity.

Note that, because of conditions (A) and (B),

$$\mathrm{Ker}\, F_j(\varrho) = \mathrm{Ker}\, F_k(\varrho) = T_\varrho \Sigma \ \text{ and } \ \mathrm{Ker}\,(F_j(\varrho)^2) = \mathrm{Ker}\,(F_k(\varrho)^2), \quad \forall \varrho \in \Sigma, \quad \forall j, k .$$

Consider now the operator

$$R := \sum_{|\alpha| \leq \mu} A_\alpha P^\alpha, \qquad P^\alpha = P_1^{\alpha_1} \ldots P_N^{\alpha_N}, \tag{73}$$

where $A_\alpha = A_\alpha^*$ are classical ψdo's of order 0, with principal symbol a_α.

Suppose that *for any conic set $\Gamma \subset T^*X \backslash 0$ with compact base there exist*

$C_\Gamma > 0$ *and* $\gamma \in \boldsymbol{Z}_+^N$, $|\gamma| = \mu$, *for which*

$$\sum_{|\alpha|=\mu} a_\alpha(x, \xi)\, \tau^\alpha \geq C_\Gamma \prod_{j=1}^N \tau_j^{\gamma_j}, \tag{74}$$

for every $(x, \xi) \in \Gamma$ *and* $\tau \in [0, +\infty)^N$.

It is then obvious that $R \in \mathrm{OPN}^{m\mu, 2\mu}(X, \Sigma)$ is transversally elliptic. Furthermore,

$$R_\varrho = \sum_{|\alpha| \leq \mu} a_\alpha(\varrho)\, P_\varrho^\alpha, \qquad P_\varrho^\alpha = P_{1,\varrho}^{\alpha_1} \dots P_{N,\varrho}^{\alpha_N}, \qquad \varrho \in \Sigma, \tag{75}$$

is self-adjoint, because the $P_{j,\varrho}$'s are self-adjoint and commute. Remark that to obtain the lower bound for $\mathrm{Re}(Ru, u)$, we have to look at

$$\frac{1}{2}(R + R^*)_\varrho = R_\varrho.$$

Let us define, for $j = 1, 2, \dots, N$,

$$\tau_j(\varrho, v; \beta) = \sum_{h=1}^{\nu} \mu_{jh}(\varrho)(2\beta_h + 1) + \sigma(v, F_j(\varrho)\, v) + p_{m-1,j}^s(\varrho), \tag{76}$$

$\varrho \in \Sigma$, $v \in \mathrm{Ker}\,(F_j(\varrho)^2)/\mathrm{Ker}\, F_j(\varrho)$, $\beta \in \boldsymbol{Z}_+^\nu$.

We claim that

$$(R_\varrho f, f) \geq 0, \qquad \forall f \in \mathcal{S}(\boldsymbol{R}^n)$$

iff

$$Q(\varrho, v; \beta) := \sum_{|\alpha| \leq \mu} a_\alpha(\varrho) \prod_{j=1}^N \tau_j(\varrho, v; \beta)^{\alpha_j} \geq 0, \tag{77}$$

for every v as above, and every $\beta \in \boldsymbol{Z}_+^\nu$.

The proof uses Hermite functions as in the case $k = 2$ above. As a consequence, the ground energy (in the non-involutive case, of course) of R (more precisely of $(R + R^*)/2$) is

$$\lambda(\varrho, v) = \min_{\beta \in \boldsymbol{Z}_+^\nu} Q(\varrho, v; \beta). \tag{78}$$

Furthermore, the dimension of the eigenspace corresponding to $\lambda(\varrho, v)$ is

$$\# J(\varrho, v), \tag{79}$$

where

$$J(\varrho, v) := \{\beta \in \boldsymbol{Z}_+^\nu;\ Q(\varrho, v; \beta) = \lambda(\varrho, v)\}, \tag{80}$$

so that the tame-degeneracy of R amounts to requiring

$$\# J(\varrho, v) = \text{constant}$$

in some neighborhood of every (ϱ_0, v_0) for which $\lambda(\varrho_0, v_0) = 0$.

3) The foregoing example, though very special, already shows how difficult it can be to check whether tame-degeneracy holds (see, for instance, [12]).

However, we are not able to avoid in our proof the tame-degeneracy condition.

Appendix 1.

Let us consider the following geometric setting.

Suppose S is a real vector space of dimension $2n$ with a symplectic form σ, and let $V \subset S$ be a subspace satisfying

$$\begin{cases} \dim (V \cap V^\sigma) = l \geqslant 1\,, \\ \dim (V^\sigma/(V \cap V^\sigma)) = 2\nu \geqslant 2 \end{cases} \tag{81}$$

(so that $2\nu + l = \dim (S/V)$ and $2(n - (\nu + l)) = \operatorname{rk} \sigma|_V$). Put

$$\begin{cases} r\colon S/V \to (V \cap V^\sigma)'\,, \\ [v] \mapsto \sigma(v, \cdot)\,, \end{cases} \tag{82}$$

$[\cdot]$ denoting the residue class in S/V. Consider on $V^\sigma/(V \cap V^\sigma)$ the symplectic form

$$\widehat{\sigma}([u]', [v]') := \sigma(u, v)\,, \tag{83}$$

$[\cdot]'$ denoting the residue class in $V^\sigma/(V \cap V^\sigma)$. We have the canonical map

$$\begin{cases} i\colon V^\sigma/(V \cap V^\sigma) \to S/V\,, \\ [u]' \mapsto [u]\,. \end{cases} \tag{84}$$

It is trivial to check that we have the exact sequence

$$0 \to \frac{V^\sigma}{V \cap V^\sigma} \xrightarrow{i} \frac{S}{V} \xrightarrow{r} (V \cap V^\sigma)' \to 0\,. \tag{85}$$

Given two linear maps

$$\begin{cases} \mu\colon T^* \boldsymbol{R}^\nu \to V^\sigma/(V \cap V^\sigma), & \mu \text{ symplectic}\,, \\ L\colon \boldsymbol{R}^l \to V \cap V^\sigma, & L \text{ isomorphism}\,, \end{cases} \tag{86}$$

let

$$\zeta_{\mu, L}\colon T^* \boldsymbol{R}^\nu \times T^* \boldsymbol{R}^l \times T^* \boldsymbol{R}^{n - (\nu + l)} \to S \tag{87}$$

be a linear symplectic map satisfying

(88) $$\begin{cases}[\zeta_{\mu,L}(y',\eta';0,0;0,0)]=i(\mu(y',\eta')),\\ \zeta_{\mu,L}(0,0;y'',0;0,0)=Ly''.\end{cases}$$

The existence of $\zeta_{\mu,L}$ is guaranteed by the linear Darboux Theorem. Observe that

(89) $$\zeta_{\mu,L}(y',\eta';y'',\eta'';y''',\eta''')\in V\Leftrightarrow y'=\eta'=0,\ \eta''=0,$$

whence we have an induced isomorphism

(90) $$\begin{cases}\hat{\zeta}_{\mu,L}\colon T^*\boldsymbol{R}^\nu\times(\boldsymbol{R}^l)'\to S/V,\\ \hat{\zeta}_{\mu,L}(y',\eta';\eta''):=[\zeta_{\mu,L}(y',\eta';0,\eta'';0,0)].\end{cases}$$

Let now μ', L' be linear maps having the same properties of μ, L, respectively, and let $\zeta_{\mu',L'}$ be a corresponding linear symplectic map satisfying the corresponding (88). We have the following proposition.

PROPOSITION 4.11. – *There exists a unique linear map $\alpha:(\boldsymbol{R}^l)'\to T^*\boldsymbol{R}^\nu$ such that*

(91) $$\hat{\zeta}_{\mu',L'}(x',\xi';\xi'')=\hat{\zeta}_{\mu,L}((\mu^{-1}\circ\mu')(x',\xi')+\alpha\xi'';{}^t(L'^{-1}\circ L)\,\xi'').$$

PROOF. – Since

$$\hat{\zeta}_{\mu',L'}(x',\xi';0)=i(\mu'(x',\xi')),$$

we immediately have

(92) $$\hat{\zeta}_{\mu',L'}(x',\xi';0)=\hat{\zeta}_{\mu,L}((\mu^{-1}\circ\mu')(x',\xi');0).$$

For any $w\in V\cap V^\sigma$, we have

$$w=\zeta_{\mu,L}(0,0;y'',0;0,0)=\zeta_{\mu',L'}(0,0;Ty'',0;0,0),$$

for a unique $y''\in\boldsymbol{R}^l$, with $T:=L'^{-1}\circ L$. As

$$\langle r(\hat{\zeta}_{\mu',L'}(0,0;\xi'')),w\rangle=\sigma(\zeta_{\mu',L'}(0,0;0,\xi'';0,0),\zeta_{\mu',L'}(0,0;Ty'',0;0,0))=$$

$$\langle\xi'',Ty''\rangle=\langle{}^tT\xi'',y''\rangle=$$

$$\sigma(\zeta_{\mu,L}(0,0;0,{}^tT\xi'';0,0),\zeta_{\mu,L}(0,0;y'',0;0,0))=\langle r(\hat{\zeta}_{\mu,L}(0,0;{}^tT\xi'')),w\rangle,$$

we conclude that

(93) $$\hat{\zeta}_{\mu',L'}(0,0;\xi'')-\hat{\zeta}_{\mu,L}(0,0;{}^tT\xi'')\in\operatorname{Ker}r=\operatorname{Im}i.$$

It follows that

$$(\widehat{\zeta}^{-1}_{\mu, L} \circ \widehat{\zeta}_{\mu', L'})(0, 0; \xi'') = (0, 0; {}^t T\xi'') + (y_0', \eta_0'; \eta_0''), \tag{94}$$

for some $(y_0', \eta_0'; \eta_0'')$, for which

$$\widehat{\zeta}_{\mu, L}(y_0', \eta_0'; \eta_0'') = [\zeta_{\mu, L}(y_0', \eta_0'; 0, \eta_0''; 0, 0)] \in \operatorname{Im} i\,.$$

On the other hand, one has

$$\zeta_{\mu, L}(y', \eta'; y'', \eta''; y''', \eta''') \in V^\sigma \Leftrightarrow \eta'' = 0, \quad y''' = \eta''' = 0\,,$$

so that $[\widehat{\zeta}_{\mu, L}(y_0', \eta_0'; \eta_0'')] \in \operatorname{Im} i$ iff there exist $\overline{y}', \overline{\eta}'$ for which

$$\zeta_{\mu, L}(y_0', \eta_0'; 0, \eta_0''; 0, 0) - \zeta_{\mu, L}(\overline{y}', \overline{\eta}'; 0, 0; 0, 0) \in V\,, \tag{95}$$

i.e.

$$y_0' = \overline{y}', \qquad \eta_0' = \overline{\eta}', \qquad \eta_0'' = 0\,.$$

Hence

$$(\widehat{\zeta}^{-1}_{\mu, L} \circ \widehat{\zeta}_{\mu', L'})(0, 0; \xi'') - (0, 0; {}^t T\xi'') = (\alpha_1(\xi''), \alpha_2(\xi''); 0)\,, \tag{96}$$

for a well-defined linear map

$$(\boldsymbol{R}^l)' \ni \xi'' \mapsto (\alpha_1(\xi''), \alpha_2(\xi'')) =: \alpha\xi'' \in T^*\boldsymbol{R}^\nu. \tag{97}$$

Finally, (96) and (92) give (91). ■

Lemma 4.2 is now a trivial consequence of the above Proposition.

Appendix 2.

Showing that Definitions 4.4 and 4.5 do not depend on the particular trivializations ψ_η, amounts to proving the following result.

Let $\Omega_y \subset \boldsymbol{R}^n$ be an open set, and suppose we are given

$$f(y; \xi) \in C^\infty(\Omega_y, \boldsymbol{S}^k_{\mathrm{reg}}(\boldsymbol{R}^N_\xi))\,,$$

$$f(y; \xi) \sim \sum_{j \geq 0} f_{k-j}(y; \xi)\,.$$

Let $\phi\colon \Omega_x' \subset \boldsymbol{R}^n \to \Omega_y$ be a smooth diffeomorphism, and $A(x) \in C^\infty(\Omega_x', GL(N, \boldsymbol{R}))$, $b(x) \in C^\infty(\Omega_x', \boldsymbol{R}^N)$ be given. Define

$$\tilde{f}(x; \zeta) = f(\phi(x); A(x)\zeta + b(x))\,. \tag{98}$$

We claim that

$$\text{(99)}\qquad \begin{cases} \tilde{f}(x;\zeta) \in C^\infty(\Omega'_x, \boldsymbol{S}^k_{\mathrm{reg}}(\boldsymbol{R}^N_\zeta)) \\ \tilde{f}(x;\zeta) \sim \sum_{r\geq 0} \tilde{f}_{k-r}(x;\zeta), \end{cases}$$

with

$$\tilde{f}_{k-r}(x;\zeta) = \sum_{|\alpha|+j=r} \frac{b(x)^\alpha}{\alpha!}(\partial_\xi^\alpha f_{k-j})(\phi(x); A(x)\zeta), \qquad r \geq 0 .$$

In particular,

$$\text{(100)}\qquad \tilde{f}_k(x;\zeta) = f_k(\phi(x); A(x)\,\zeta),$$

hence $\tilde{f}$ is elliptic iff f is.

The proof is almost obvious. In fact, by remarking that

$$\text{(101)}\qquad \begin{cases} \tilde{f}(x;\zeta) \in C^\infty(\Omega'_x, \boldsymbol{S}^k(\boldsymbol{R}^N_\zeta)), \\ (\partial_\xi^\alpha f)(\phi(x); A(x)\,\zeta) \in C^\infty(\Omega'_x, \boldsymbol{S}^{k-|\alpha|}(\boldsymbol{R}^N_\zeta)), \qquad \forall \alpha \in \boldsymbol{Z}^N_+, \end{cases}$$

Taylor's formula easily yields

$$\text{(102)}\qquad \tilde{f}(x;\zeta) \sim \sum_{\alpha\geq 0} \frac{b(x)^\alpha}{\alpha!}(\partial_\xi^\alpha f)(\phi(x); A(x)\,\zeta).$$

Since

$$\text{(103)}\qquad (\partial_\xi^\alpha f)(\phi(x); A(x)\,\zeta) \sim \sum_{j\geq 0} (\partial_\xi^\alpha f_{k-j})(\phi(x); A(x)\,\zeta),$$

we immediately obtain (99) and (100). ∎

Appendix 3.

The smoothness of the lowest eigenvalue λ, supposed to have constant multiplicity, is a consequence of a general, and well-known, argument that we reproduce here for the sake of completeness.

Let B and H be two (complex) Hilbert spaces such that $B \hookrightarrow H$ with *compact immersion and dense image.*

Let $x \mapsto A(x)$ be a smooth family of linear continuous operators from B to H, for x in some open set $U \subset \boldsymbol{R}^n$ (say).

For every $x \in U$, consider $A(x)$ as an unbounded operator in H, with domain

B, and suppose

(104) $$\begin{cases} A(x) = A(x)^*, \\ (A(x)\,u, u) \geqslant 0, \qquad \forall x \in U, \quad \forall u \in B\,. \end{cases}$$

Since

$$(A(x) - \lambda)^{-1} \colon H \to B \hookrightarrow H$$

is a compact operator, it follows that the spectrum of $A(x)$, Spec $A(x) \subset [0, +\infty)$, consists of eigenvalues with finite multiplicity. In particular, it makes sense to define

(105) $$\lambda(x) = \min\,(\mathrm{Spec}\,A(x))\,, \qquad x \in U\,,$$

which a-priori is a continuous function of x with values in $[0, +\infty)$. Put

(106) $$V_x = \mathrm{Ker}\,(A(x) - \lambda(x))\,, \qquad x \in U\,.$$

We claim that *if V_x has constant dimension in a neighborhood of some $x^0 \in U$, then $\lambda(x)$ is* smooth *in the same neighborhood.*

To prove the claim, fix a compact neighborhood $\omega \subset U$ of x^0, and $\varepsilon > 0$ such that, for $x \in \omega$, we have

$$|\lambda(x) - \lambda(x^0)| < \varepsilon\,, \qquad \mathrm{Spec}\,A(x) \cap \{\zeta \in \boldsymbol{C};\ |\zeta - \lambda(x^0)| \leqslant 2\varepsilon\} = \{\lambda(x)\}\,,$$

and

$$\dim V_x = \text{constant}\,.$$

Then, for $x \in \omega$, define

(107) $$\pi(x) = \frac{1}{2\pi i} \oint_{|\zeta - \lambda(x^0)| = \varepsilon} (\zeta - A(x))^{-1}\, d\zeta\,.$$

It turns out that $\pi(x)$ is a smooth self-adjoint projection onto V_x, which makes $\cup_{x \in \omega} V_x$ into a smooth Hermitian vector-bundle on ω. Since the restriction of $A(x)$ to V_x is just $\lambda(x)\,\mathrm{Id}_{V_x}$, we get the required smoothness of λ in ω.

To identify the above abstract setting to the one considered in Section 4, suppose we have $a(x; (z', \zeta')) \in C^\infty(U, \boldsymbol{S}^k_{\mathrm{reg}}(\boldsymbol{R}^{2\nu}))$, $k > 0$, and put $A(x) = \mathrm{Op}^w(a)(x; z', D_{z'})$, with $H = L^2(\boldsymbol{R}^\nu)$ and

$$B = \{u \in L^2(\boldsymbol{R}^\nu);\ (1 + |z'|^2 + |D_{z'}|^2)^{k/2} u \in L^2(\boldsymbol{R}^\nu)\}. \qquad \blacksquare$$

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