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Asymptotic Behavior and Non-Existence Theorems for Semilinear Dirichlet Problems Involving Critical Exponent on Unbounded Domains of the Heisenberg Group.

E. LANCONELLI - F. UGUZZONI

Sunto. – *In questa nota dimostriamo stime asintotiche ottimali per le soluzioni deboli non negative del problema al contorno*

$$(*)\qquad -\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)} \quad in\ \Omega\,, \quad u = 0 \ \ in\ \partial\Omega\,.$$

$\Delta_{\mathbb{H}^n}$ *è il Laplaciano di Kohn sul gruppo di Heisenberg* $\mathbb{H}^n$, Ω *è un aperto non limitato e* $Q = 2n+2$ *è la dimensione omogenea di* $\mathbb{H}^n$. *Utilizziamo successivamente le stime ottenute per dimostrare un teorema di non esistenza per* (*) *nel caso in cui* Ω *sia un semispazio di* $\mathbb{H}^n$ *con bordo parallelo al centro del gruppo.*

1. – Introduction.

Let $\Delta_{\mathbb{H}^n}$ be the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ and let $Q = 2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. The aim of this paper is to study asymptotic behavior and to establish non-existence results for nonnegative weak solutions to the semilinear boundary value problem

$$(1.1)\qquad \begin{cases} -\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)} & \text{in } \Omega\,, \\ u \in S_0^1(\Omega)\,, \end{cases}$$

where Ω is an *unbounded* open subset of $\mathbb{H}^n$ and $S_0^1(\Omega)$ is a Folland-Stein's Sobolev space (see definition below). Equations like that in (1.1) naturally arise in the study of the Yamabe problem for the Cauchy-Riemann manifolds [JL1-2]. The exponent

$$\frac{Q+2}{Q-2}$$

is a *critical exponent* for semilinear Dirichlet problems related to $\Delta_{\mathbb{H}^n}$, as well as the exponent

$$\frac{N+2}{N-2}$$

is critical for semilinear Poisson equations in $\mathbb{R}^N$, $N \geq 3$.

In order to be more precise we need to introduce additional notation and to recall some known results. The Heisenberg group $\mathbb{H}^n$, whose points will be denoted by $\xi = (z, t) = (x, y, t)$, is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ with composition law defined by

$$\xi \circ \xi' = (z + z', t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle))$$

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^n$. The Kohn Laplacian on $\mathbb{H}^n$ is the operator

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

where

$$X_j = \partial_{x_j} + 2y_j \partial_t , \qquad Y_j = \partial_{y_j} - 2x_j \partial_t$$

for all $j \in \{1, \ldots, n\}$. We set

$$\nabla_{\mathbb{H}^n} = (X_1, \ldots, X_n, Y_1, \ldots, Y_n).$$

A natural group of dilations on $\mathbb{H}^n$ is given by

$$\delta_\lambda(\xi) = (\lambda z, \lambda^2 t), \qquad \lambda > 0 . \tag{1.2}$$

The Jacobian determinant of δ_λ is λ^Q where

$$Q = 2n + 2$$

is the *homogeneous dimension* of $\mathbb{H}^n$. The operator $\Delta_{\mathbb{H}^n}$ is invariant with respect to the left translations of $\mathbb{H}^n$ and homogeneous of degree two with respect to the dilations δ_λ. More precisely, if we set

$$\tau_\xi(\xi') = \xi \circ \xi' \tag{1.3}$$

we have

$$\Delta_{\mathbb{H}^n}(u \circ \tau_\xi) = (\Delta_{\mathbb{H}^n} u) \circ \tau_\xi , \qquad \Delta_{\mathbb{H}^n}(u \circ \delta_\lambda) = \lambda^2 (\Delta_{\mathbb{H}^n} u) \circ \delta_\lambda .$$

A remarkable analogy between the Kohn Laplacian and the classical Laplace operator is that a fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is given by [F]

$$\Gamma(\xi) = \frac{c_Q}{d(\xi)^{Q-2}} , \tag{1.4}$$

where c_Q is a suitable positive constant and

$$d(\xi) = (|z|^4 + t^2)^{1/4}\,. \tag{1.5}$$

Moreover, if we define $d(\xi, \xi') = d(\xi'^{-1} \circ \xi)$, then d is a distance on $\mathbb{H}^n$ (see [Cy] for a complete proof of this statement). We shall denote by $B_d(\xi, r)$ the d-ball of center ξ and radius r. By the left translation invariance of the distance d, we have $\tau_\xi(B_d(0, r)) = B_d(\xi, r)$. Moreover, since d is homogeneous of degree 1 with respect to the dilations δ_λ, we also have $\delta_\lambda(B_d(0, r)) = B_d(0, \lambda r)$ and $|B_d(\xi, r)| = r^Q |B_d(0, 1)|$. Here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. We also recall that the Lebesgue measure is a Haar measure on $\mathbb{H}^n$.

A basic role in the functional analysis on the Heisenberg group is played by the following Sobolev-type inequality:

$$\|\varphi\|_{Q^*}^2 \leq B_Q \|\nabla_{\mathbb{H}^n} \varphi\|_2^2 \qquad \forall \varphi \in C_0^\infty(\mathbb{H}^n) \tag{1.6}$$

where

$$Q^* := \frac{2Q}{Q-2} \tag{1.7}$$

and B_Q is a positive constant whose best value has been determined by Jerison and Lee in [JL2]. Hereafter $\|\cdot\|_p$ will denote the usual L^p-norm. If Ω is an open subset of $\mathbb{H}^n$, we shall denote by $S^1(\Omega)$ the Sobolev space of the functions $u \in L^{Q^*}(\Omega)$ such that $\nabla_{\mathbb{H}^n} u \in L^2(\Omega)$. The norm in $S^1(\Omega)$ is given by

$$\|u\|_{S^1(\Omega)} = \|u\|_{Q^*} + \|\nabla_{\mathbb{H}^n} u\|_2\,. \tag{1.8}$$

We denote by $S_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to (1.8). By means of (1.6), this norm is equivalent in $S_0^1(\Omega)$ to that generated by the inner product

$$\langle u, v\rangle_{S_0^1} = \int_\Omega \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} v\rangle\,.$$

Thus $S_0^1(\Omega)$ is a Hilbert space. We emphasize that, for general unbounded domains, the space $S_0^1(\Omega)$ is not embedded in $L^2(\Omega)$.

A nonnegative weak solution of the Dirichlet problem (1.1) is a function $u \in S_0^1(\Omega)$, $u \geq 0$, such that

$$\int_\Omega \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} \varphi\rangle = \int_\Omega u^{Q^*-1} \varphi \qquad \forall \varphi \in S_0^1(\Omega)\,. \tag{1.9}$$

We explicitly remark that, for every u, $\varphi \in S_0^1(\Omega)$, $u \geq 0$, we have

$$u^{(Q+2)/(Q-2)} \varphi \in L^1(\Omega)\,.$$

Indeed $\varphi \in L^{2Q/(Q-2)}(\Omega)$, $u^{(Q+2)/(Q-2)} \in L^{2Q/(Q+2)}(\Omega)$ and

$$\frac{Q-2}{2Q} + \frac{Q+2}{2Q} = 1 .$$

We also remark that every classical solution of (1.1) satisfies the integral identity (1.9) since $X_j^* = -X_j$ and $Y_j^* = -Y_j$, for $j = 1, \ldots, n$.

When $\Omega = \mathbb{H}^n$, a positive solution to the equation in (1.1) is the following C^∞ function:

$$U(x, y, t) = U(z, t) = \frac{c_0}{((1 + |z|^2)^2 + t^2)^{(Q-2)/4}} \tag{1.10}$$

where c_0 is a suitable positive constant. Moreover, *every* nontrivial nonnegative weak solution of (1.1) with $\Omega = \mathbb{H}^n$ takes the form

$$u(\xi) = \lambda^{(Q-2)/2} U(\delta_\lambda(\eta^{-1} \circ \xi)) , \qquad \xi \in \mathbb{H}^n$$

for suitably $\lambda > 0$ and $\eta \in \mathbb{H}^n$. This deep result of Jerison and Lee [JL2] is the Kohn-Laplacian counterpart of a celebrated Theorem of Talenti [T] for the classical Laplace operator.

The following theorem shows that for any unbounded open subset Ω of $\mathbb{H}^n$, *every* nonnegative weak solution of (1.1) behaves at infinity like the function U in (1.10). This is one of the principal results of this note.

THEOREM 1.1. – *Let Ω be an arbitrary unbounded open subset of $\mathbb{H}^n$ and let u be a nonnegative weak solution of the Dirichlet problem* (1.1). *Then there exists a constant $M > 0$ such that*

$$u(\xi) \leq M U(\xi) \qquad \forall \xi \in \Omega .$$

We will use Theorem 1.1 as a crucial step in proving a nonexistence result on *halfspaces* for the Dirichlet problem (1.1). We next give our motivation for studying this problem. A nonnegative function $u \in S_0^1(\Omega)$ is a weak solution of (1.1) iff u is a critical point of the functional

$$I \colon S_0^1(\Omega) \to \mathbb{R} , \qquad I(u) = \frac{1}{2} \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 - \frac{1}{Q^*} \int_\Omega u^{Q^*},$$

where Q^* is defined in (1.7). The exponent Q^* is the critical Sobolev exponent for $\Delta_{\mathbb{H}^n}$ since, even if Ω is bounded, the continuous embedding

$$S_0^1(\Omega) \quad L^{Q^*}(\Omega)$$

is not compact. As a consequence, the Palais-Smale sequences of I are in general not compact. Therefore, standard variational techniques cannot be applied in looking for critical points of I. On the other hand, as Citti proved in [C], the loss of compactness of the Palais-Smale sequences of I only depends on the weak solutions of the so called problems at infinity:

$$-\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)}, \qquad u > 0 \ \text{ in } \ \mathbb{H}^n$$

and

$$-\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)}, \qquad u > 0 \ \text{ in } \ \Pi\,, \quad u = 0 \ \text{ in } \ \partial\Pi$$

where Π is any halfspace of $\mathbb{H}^n$. Citti's work is an extension to the Heisenberg group $\mathbb{H}^n$ of some relevant results of Benci-Cerami [BeC], P. L. Lions [L] and Brezis-Nirenberg [BN] concerning the critical semilinear Poisson equation in $\mathbb{R}^N$, $N \geqslant 3$,

$$-\Delta u = u^{(N+2)/(N-2)}.$$

For this equation Esteban and Lions proved in [EL] the following nonexistence theorem: *for every halfspace Π of $\mathbb{R}^N$ the «problem at infinity»*

$$\begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \ \Pi\,, \\ u \in W_0^1(\Pi)\,, \end{cases} \tag{1.11}$$

has no nontrivial nonnegative weak solutions.

Here $W_0^1(\Pi)$ denotes the closure of $C_0^\infty(\Pi)$ with respect to the usual Sobolev norm $u \mapsto \|\nabla u\|_2$ and Δ is the classical Laplace operator. Using this result and by means of algebraic topology techniques, Bahri and Coron [BC] were able to prove the following celebrated theorem: *let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geqslant 3$, and suppose there exists $m \in \mathbb{N}$ such that the omology group $H_m(\Omega)$ is nontrivial. Then the Dirichlet problem*

$$\begin{cases} -\Delta u = u^{(N+2)/(N-2)} & \text{in } \ \Omega\,, \\ u \in W_0^1(\Omega)\,, \end{cases}$$

has a nontrivial nonnegative weak solution.

Bahri and Coron techniques seem to be appropriate for application in the context of the Heisenberg group, provided the above nonexistence theorem of Esteban and Lions can be extended. However, at the authors' knowledge, no nonexistence result has been established for the critical semilinear Kohn-Laplace equation on halfspaces. In a recent paper, Birindelli, Capuzzo Dolcetta and Cutrì proved nonexistence theorems on cones of $\mathbb{H}^n$, but they only treat sub-critical equations [BCC].

In the following theorem we provide a first answer to the problem raised above.

THEOREM 1.2. – *Let Π be a halfspace of $\mathbb{H}^n$ whose boundary is parallel to the center of $\mathbb{H}^n$. Then the problem at infinity*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)} & \text{in } \Pi, \\ u \in S_0^1(\Pi), \end{cases} \tag{1.12}$$

has no nontrivial nonnegative weak solutions.

The center of $\mathbb{H}^n$ is the set $\{(0, t) \mid t \in \mathbb{R}\}$; a halfspace of $\mathbb{H}^n$ is merely a halfspace of $\mathbb{R}^{2n+1}$.

Problem (1.12), in spite of its similarity to problem (1.1), presents a much higher difficulty level, mainly due to the lack of good a priori estimates for $\partial_t u$. We explicitly remark that the differential operator ∂_t is homogeneous of degree two with respect to the dilations δ_λ in (1.2), thus $\partial_t u$ should be considered as a *second derivative* for a solution u of (1.12). A condition on $\partial_t u$ and in particular its square summability, would allow us to apply Theorem 2.4 of [GL2] in proving our Theorem 1.2. Starting from Theorem 1.1 we will actually show that

$$|\partial_t u| \leq MU \tag{1.13}$$

where U is the function (1.10) and M is a suitable positive constant. This estimate implies that $\partial_t u \in L^2(\Pi)$ only when $n > 1$, however it is sufficient for proving Theorem 1.2 for every $n \geq 1$.

To obtain inequality (1.13) we will use an argument based on the representation of the harmonic part of $\partial_t u$ as a fixed point for a mean value operator modeled on the geometry of Π. We should add that our method is not applicable to the halfspace

$$\Pi_t = \{(z, t) \mid t > 0\}$$

or, equivalently, to any other halfspace whose boundary intersects the center of $\mathbb{H}^n$ at a single point.

The paper is organized as follows. In section 2 we prove L^p and Hölder continuity properties for nonnegative weak solutions of (1.1). Section 3 and section 4 are devoted to the proof of Theorem 1.1 and of Theorem 1.2 respectively.

2. – L^p and Hölder properties of solutions.

Throughout this section we shall always denote by u a nonnegative weak solution of (1.1). Ω will be supposed to be an arbitrary (bounded or unbounded) open subset of $\mathbb{H}^n$. Our aim is to prove L^p and Hölder continuity properties of u. The main results are contained in Proposition 2.1, Proposition 2.6 and Proposition 2.7.

We will use boot-strap and iteration techniques inspired to those of Brezis-Kato [BK] and Moser ([GT], Chapter 8). We would like to stress that the major difficulties lay in proving that $u \in L^p$ for $Q^*/2 < p < Q^*$, which leads to novel and significant modifications of the standard schemes.

PROPOSITION 2.1. – *We have* $u \in L^p(\Omega)$ *for every* $p \in]\,Q^*/2,\, +\infty\,]$.

PROOF. – The proof will directly follow from Lemma 2.2, Lemma 2.3 and Lemma 2.4. ∎

LEMMA 2.2. – *We have* $u \in L^p(\Omega)$ *for every* $p \in [Q^*,\, +\infty[$.

PROOF. – Since $u \in L^{Q^*}(\Omega)$ there exists $M > 0$ such that

$$\int_{\{u > M\}} u^{Q^*} < \left(\frac{1}{2B_Q}\right)^{Q/2} \tag{2.1}$$

where B_Q is the constant defined in (1.6). Let $\eta \in C^\infty([0, +\infty[)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in $[0, M]$, $\eta \equiv 0$ in $[2M, +\infty[$. We set

$$V = (1 - \eta(u))\, u^{4/(Q-2)}, \qquad g = \eta(u)\, u^{(Q+2)/(Q-2)}\,.$$

Then u is a weak solution of

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = Vu + g & \text{in } \Omega\,, \\ u \in S_0^1(\Omega)\,, \end{cases} \tag{2.2}$$

where

$$V \in L^{Q/2}(\Omega)\,, \qquad g \in L^{2Q/(Q+2)}(\Omega) \cap L^\infty(\Omega)\,. \tag{2.3}$$

Moreover, (2.1) yields $\|V\|_{Q/2} < 1/2B_Q$. Hence, for every $\varphi \in S_0^1(\Omega)$, we have

$$\int_\Omega V\varphi^2 \leqslant \|V\|_{Q/2} \|\varphi\|^2_{Q^*} \leqslant B_Q \|V\|_{Q/2} \|\nabla_{\mathbb{H}^n} \varphi\|_2^2 < \frac{1}{2} \|\nabla_{\mathbb{H}^n} \varphi\|_2^2\,. \tag{2.4}$$

We now proceed by essentially adapting the proof of Lemma 4.1 of [GL2], taking into account the fact that the Sobolev space $\overset{\circ}{S}{}_1^2$ in [GL2] is different from S_0^1. For every $k \in \mathbb{N}$ we define $V_k = \min\{V, k\}$. Since (2.3) and (2.4) hold, Lax Milgram's Theorem implies the existence of exactly one weak solution u_k of

$$\begin{cases} -\Delta_{\mathbb{H}^n} u_k = V_k u_k + g & \text{in } \Omega\,, \\ u_k \in S_0^1(\Omega)\,, \end{cases} \tag{2.5}$$

such that

$$\|u_k\|_{Q^*} \leqslant \sqrt{B_Q}\, \|u_k\|_{S_0^1} \leqslant 2\sqrt{B_Q}\, \|g\|_{(S_0^1)'} \leqslant 2B_Q \|g\|_{2Q/(Q+2)}\,. \tag{2.6}$$

The same argument also yields uniqueness for the problem (2.2). Hence, by the boundness of (u_k) in $S_0^1(\Omega)$, taking a subsequence if necessary, we have

$$u_k \to u \quad \text{weakly in} \quad S_0^1(\Omega). \tag{2.7}$$

We now want to prove that, for every $p \in [Q^*, +\infty[$ there exists $c_p > 0$ such that

$$\sup_{k \in \mathbb{N}} \|u_k\|_p \leqslant c_p . \tag{2.8}$$

We set $\beta = Q/(Q-2)$. Since $\beta > 1$ and (2.8) holds for $p = Q^*$ (see (2.6)), we only need to prove (2.8) for βp, under the hypothesis that (2.8) holds for p. We fix $k \in \mathbb{N}$ and, for sake of brevity, we set $v = u_k$. We then define, for every $m \in \mathbb{N}$,

$$v_m = \min\{v^+, m\}, \qquad \varphi_m = v_m^{p-1}, \qquad f_m = v_m^{p/2}.$$

We remark that v, v_m, φ_m, $f_m \in S_0^1(\Omega)$, since $p \geqslant Q^* > 2$. Moreover v_m, φ_m, f_m are all nonnegative. Choosing φ_m as a test function in the weak formulation of (2.5) and setting $\alpha_p = (4(p-1))/p^2$, we obtain

$$\alpha_p \int_\Omega |\nabla_{\mathbb{H}^n} f_m|^2 = \int_\Omega \langle \nabla_{\mathbb{H}^n} v_m, \nabla_{\mathbb{H}^n} \varphi_m \rangle = \int_\Omega \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle =$$
$$\int_\Omega (V_k v \varphi_m + g\varphi_m) \leqslant \int_{\{v \leqslant m\}} V_k f_m^2 + k \int_{\{v > m\}} v^p + \int_\Omega g\varphi_m . \tag{2.9}$$

Since $V \in L^{Q/2}(\Omega)$, we can choose $M_p > 0$ such that

$$\left(\int_{\{V > M_p\}} V^{Q/2} \right)^{2/Q} < \frac{\alpha_p}{2B_Q} .$$

Then

$$\int_\Omega V f_m^2 \leqslant M_p \int_{\{V \leqslant M_p\}} f_m^2 + \int_{\{V > M_p\}} V f_m^2 \leqslant$$
$$M_p \|f_m\|_2^2 + \frac{\alpha_p}{2B_Q} \|f_m\|_{Q^*}^2 \leqslant M_p \|f_m\|_2^2 + \frac{\alpha_p}{2} \|\nabla_{\mathbb{H}^n} f_m\|_2^2 . \tag{2.10}$$

From (2.9) and (2.10) it follows

$$\frac{\alpha_p}{2B_Q} \|v_m\|_{\beta p}^p = \frac{\alpha_p}{2B_Q} \|f_m\|_{Q^*}^2 \leqslant \frac{\alpha_p}{2} \|\nabla_{\mathbb{H}^n} f_m\|_2^2 \leqslant$$
$$M_p \|f_m\|_2^2 + k \int_{\{v > m\}} v^p + \int_\Omega g\varphi_m \leqslant M_p \|f_m\|_2^2 + k \int_{\{v > m\}} v^p + \|g\|_p \|v_m\|_p^{p-1} .$$

From these inequalities, as m goes to infinity, we obtain

$$\frac{\alpha_p}{2B_Q}\|v^+\|_{\beta p}^p \leqslant M_p\|v^+\|_p^p + \|g\|_p\,\|v^+\|_p^{p-1} \leqslant \qquad \text{(since (2.8) holds for } p)$$

$$M_p c_p^p + c_p^{p-1}\|g\|_p \leqslant$$

$$M_p c_p^p + c_p^{p-1}\|g\|_\infty^{1-Q^*/p}\|g\|_{Q^*}^{Q^*/p} < +\infty \qquad \text{(for (2.3))}.$$

Since a similar estimate can be proved for v^-, (2.8) holds when replacing p with βp. Therefore (2.8) holds for every $p \in [Q^*, +\infty[$.

We now fix $p \in [Q^*, +\infty[$ and we define

$$C = \{f \in S_0^1(\Omega) \,|\, \|f\|_p \leqslant c_p\}.$$

Fatou's Lemma ensures that C is a closed convex subset of $S_0^1(\Omega)$. By (2.8) (u_k) is contained in C. Hence, by (2.7), $u \in C$. In particular $u \in L^p(\Omega)$. ■

The next lemma is one of the crucial steps in the proof of Theorem 1.1.

LEMMA 2.3. – *We have $u \in L^p(\Omega)$ for every $p \in]Q^*/2, Q^*]$.*

Although the proof follows the lines of the previous one, it presents many more difficulties. Indeed we need to choose *ad hoc* truncated potentials and particular test functions.

PROOF. – We fix $\varepsilon \in]0, 1/2[$ and we set

$$M_\varepsilon = \frac{1}{(1-2\varepsilon)^2}. \tag{2.11}$$

Since $u \in L^{Q^*}(\Omega)$ there exists $\delta > 0$ such that

$$\int_{\{u<2\delta\}} u^{Q^*} < \left(\frac{1}{M_\varepsilon B_Q}\right)^{Q/2}. \tag{2.12}$$

Let $\eta \in C^\infty([0, +\infty[)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ in $[0, \delta]$, $\eta \equiv 0$ in $[2\delta, +\infty[$. We set

$$V = \eta(u)\,u^{4/(Q-2)}, \qquad g = (1-\eta(u))\,u^{(Q+2)/(Q-2)}.$$

Then u is a weak solution of

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = Vu + g & \text{in } \Omega, \\ u \in S_0^1(\Omega), \end{cases}$$

and, by Lemma 2.2,

$$V \in L^{Q/2}(\Omega) \cap L^{\infty}(\Omega), \qquad g \in L^p(\Omega) \qquad \forall p \in [1, +\infty[. \tag{2.13}$$

Moreover, by (2.12),

$$\int_{\Omega} V\varphi^2 \leqslant \|V\|_{Q/2} \|\varphi\|_{Q^*}^2 < \frac{1}{M_\varepsilon} \|\nabla_{\mathbb{H}^n} \varphi\|_2^2 \qquad \forall \varphi \in S_0^1(\Omega) \tag{2.14}$$

(note that $1/M_\varepsilon < 1$). For every $k \in \mathbb{N}$ let $\eta_k \in C^{\infty}([0, +\infty[)$ be such that $0 \leqslant \eta_k \leqslant 1$, $\eta_k \equiv 0$ in $[0, \delta/(k+1)]$, $\eta_k \equiv 1$ in $[\delta/k, +\infty[$. We define $V_k = \eta_k(u)V$ so that

$$V_k \in L^1(\Omega) \cap L^{\infty}(\Omega) \tag{2.15}$$

and

$$V_k \nearrow V .$$

Exactly in the same way as in the proof of Lemma 2.2, we can see that for every $k \in \mathbb{N}$ problem (2.5) admits a weak solution u_k such that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{S_0^1(\Omega)} \leqslant c = c(Q, u, \varepsilon) \tag{2.16}$$

and $u_k \to u$ weakly in $S_0^1(\Omega)$.

We now want to prove that

$$\sup_{k \in \mathbb{N}} \|u_k\|_{Q^*(1-\varepsilon)} \leqslant c_0 = c_0(Q, u, \varepsilon) . \tag{2.17}$$

We fix $k \in \mathbb{N}$ and, for sake of brevity, we set $v = u_k$. Then we define, for every $m \in \mathbb{N}$,

$$v_m = \begin{cases} m^{2\varepsilon/(1-2\varepsilon)} (v^+)^{1/(1-2\varepsilon)} & \text{where } v \leqslant \dfrac{1}{m} , \\ v & \text{where } v > \dfrac{1}{m} , \end{cases}$$

$$\varphi_m = v_m^{1-2\varepsilon} = \begin{cases} m^{2\varepsilon} v^+ & \text{where } v \leqslant \dfrac{1}{m} , \\ v^{1-2\varepsilon} & \text{where } v > \dfrac{1}{m} , \end{cases}$$

$$f_m = v_m^{1-\varepsilon} = \begin{cases} m^{(2\varepsilon(1-\varepsilon))/(1-2\varepsilon)} (v^+)^{(1-\varepsilon)/(1-2\varepsilon)} & \text{where } v \leqslant \dfrac{1}{m} , \\ v^{1-\varepsilon} & \text{where } v > \dfrac{1}{m} . \end{cases}$$

We remark that $v, v_m, \varphi_m, f_m \in S_0^1(\Omega)$, since $0 < 1 - 2\varepsilon < 1 - \varepsilon < 1$. Moreover $v_m, \varphi_m, f_m \geqslant 0$ and $v_m \to v^+$ pointwise, as $m \to +\infty$. Choosing φ_m as a test function in the weak formulation of (2.5) and setting $c_\varepsilon = ((1-\varepsilon)/(1-2\varepsilon))^2$, we obtain

$$\int_\Omega |\nabla_{\mathbb{H}^n} f_m|^2 = (1-2\varepsilon)\, c_\varepsilon \int_\Omega \langle \nabla_{\mathbb{H}^n} v_m, \nabla_{\mathbb{H}^n} \varphi_m \rangle = (1-2\varepsilon)\, c_\varepsilon \int_{\{v>1/m\}} \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle +$$

$$c_\varepsilon \int_{\{0<v\leqslant 1/m\}} m^{2\varepsilon/(1-2\varepsilon)} v^{1/(1-2\varepsilon)-1} \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle \leqslant$$

$$c_\varepsilon \int_{\{v>1/m\}} \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle + c_\varepsilon \int_{\{0<v\leqslant 1/m\}} \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle$$

(since $\langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle \geqslant 0$ in Ω)

$$= c_\varepsilon \int_\Omega \langle \nabla_{\mathbb{H}^n} v, \nabla_{\mathbb{H}^n} \varphi_m \rangle = c_\varepsilon \int_\Omega (V_k v \varphi_m + g \varphi_m) =$$

$$c_\varepsilon \left(\int_{\{v>1/m\}} V_k f_m^2 + m^{2\varepsilon} \int_{\{0<v\leqslant 1/m\}} V_k v^2 + \int_{\{v>1/m\}} g v^{1-2\varepsilon} + m^{2\varepsilon} \int_{\{0<v\leqslant 1/m\}} g v \right) \leqslant$$

$$c_\varepsilon \left(\frac{1}{M_\varepsilon} \|\nabla_{\mathbb{H}^n} f_m\|_2^2 + \frac{1}{m^{2-2\varepsilon}} \|V_k\|_1 + \|g\|_{\frac{Q^*}{Q^*-1+2\varepsilon}} \|v\|_{Q^*}^{1-2\varepsilon} + \frac{1}{m^{1-2\varepsilon}} \|g\|_1 \right).$$

Here we have used (2.14). Reading from (2.11) that $c_\varepsilon / M_\varepsilon < 1$, from (2.13), (2.15), (2.16) and (1.6), we obtain

$$\|\nabla_{\mathbb{H}^n} f_m\|_2^2 < \frac{c_1}{m^{1-2\varepsilon}} + c_2$$

with $c_1 = c_1(Q, u, \varepsilon, k)$ and $c_2 = c_2(Q, u, \varepsilon)$. Hence

$$\|v_m\|_{Q^*(1-\varepsilon)}^{2(1-\varepsilon)} = \|f_m\|_{Q^*}^2 \leqslant B_Q \|\nabla_{\mathbb{H}^n} f_m\|_2^2 \leqslant \frac{c_1 B_Q}{m^{1-2\varepsilon}} + c_2 B_Q$$

and letting $m \to +\infty$, Fatou's Lemma yields

$$\|v^+\|_{Q^*(1-\varepsilon)} \leqslant (c_2 B_Q)^{1/(2(1-\varepsilon))} =: c_0 = c_0(Q, u, \varepsilon).$$

A similar estimate can be proved for v^-. Then, inequality (2.17) holds. We can now conclude as in the proof of Lemma 2.2 and obtain

$$u \in L^{Q^*(1-\varepsilon)}(\Omega)$$

for every fixed $\varepsilon \in]0, 1/2[$. ■

The following lemma completes the proof of Proposition 2.1.

LEMMA 2.4. – *We have* $u \in L^\infty(\Omega)$.

PROOF. – It is sufficient to prove that there exist $c_0 > 0$ and a sequence of positive numbers $(p_N)_{N \in \mathbb{N}}$ such that $p_N \to +\infty$ and

$$\sup_{N \in \mathbb{N}} \|u\|_{p_N} \leqslant c_0 . \tag{2.18}$$

We shall use the notation of Lemma 2.2. By Lemmas 2.2 and 2.3, we know that

$$g \in L^1(\Omega) \cap L^\infty(\Omega), \qquad V \in L^p(\Omega) \qquad \forall p \in \left] \frac{Q}{4}, +\infty \right[.$$

Setting $r = \sqrt{\beta} = \sqrt{Q/(Q-2)}$, from (2.9) we get

$$\alpha_p \int_\Omega |\nabla_{\mathbb{H}^n} f_m|^2 \leqslant \|V\|_{r/(r-1)} \|v_m^p\|_r + k \int_{\{v > m\}} v^p + \|g\|_{rp/(rp-p+1)} \|v_m\|_{rp}^{p-1} \leqslant$$

$$c\|v^+\|_{rp}^p + k \int_{\{v > m\}} v^p + c\|v^+\|_{rp}^{p-1} .$$

We have used here that $\sup_{p \in [1, +\infty]} \|g\|_p < c$ and $r/(r-1) > 1/(\beta - 1) = (Q-2)/2 \geqslant Q/4$. From this estimate, since

$$\frac{\alpha_p}{B_Q} \|v_m\|_{\beta p}^p = \frac{\alpha_p}{B_Q} \|f_m\|_{Q^*}^2 \leqslant \alpha_p \|\nabla_{\mathbb{H}^n} f_m\|_2^2 ,$$

letting $m \to +\infty$ we obtain

$$\frac{\alpha_p}{B_Q} \|v^+\|_{\beta p}^p \leqslant c(\|v^+\|_{rp}^p + \|v^+\|_{rp}^{p-1})$$

where c does not depend on p. On the other hand, being $p \geqslant Q^* > 2$, $\alpha_p = (4(p-1))/p^2 > 1/p$. Therefore, if we set $H_p = \max\{1, \|v^+\|_p\}$ we get

$$H_{\beta p} \leqslant (cp)^{1/p} H_{rp} \qquad \forall p \in [Q^*, +\infty[$$

where c is another positive constant not depending on p. Choosing $p = r^j Q^*$, $j \in \mathbb{N}$, and recalling that $\beta = r^2$, we have

$$H_{r^{j+2}Q^*} \leqslant (cr^j)^{1/r^j Q^*} H_{r^{j+1}Q^*} .$$

Therefore, for $N \in \mathbb{N}$, $N \geqslant 3$, setting $p_N = r^j Q^*$, we obtain

$$\|v^+\|_{p_N} \leqslant H_{r^N Q^*} \leqslant \left(\prod_{j=1}^{N-2} (cr^j)^{1/r^j Q^*} \right) H_{r^2 Q^*} \leqslant c' H_{r^2 Q^*} \leqslant c_0$$

(note that $r > 1$ ensures that $\sum_{j=1}^{+\infty} j/r^j$ is convergent). Since an analogous estimate holds for v^-, (2.18) follows. ■

We now want to prove some more properties of u.

LEMMA 2.5. – *There exists a positive constant c such that*

$$\|u\|_{L^\infty(B_d(\xi,1))} \leqslant c\|u\|_{L^{Q^*}(B_d(\xi,2))} \qquad \forall \xi \in \mathbb{H}^n .$$

Here we have set $u=0$ outside Ω.

PROOF. – Let us fix $r, R \in \mathbb{R}$ such that $1 \leqslant r < R \leqslant 2$ and choose $\alpha \in C^\infty([0, +\infty[)$ such that $0 \leqslant \alpha \leqslant 1$, $\alpha \equiv 1$ in $[0, r]$, $\alpha \equiv 0$ in $[R, +\infty[$ and $|\alpha'| \leqslant 2/(R-r)$. We set $\eta_0 = \alpha(d)$ and, for every $\xi \in \mathbb{H}^n$, define $\eta_\xi = \eta_0 \circ \tau_{\xi^{-1}}$. Thus $\eta_\xi \equiv 1$ in $B_d(\xi, r)$, $\eta_\xi \equiv 0$ in $\mathbb{H}^n \setminus B_d(\xi, R)$ and

$$\|\nabla_{\mathbb{H}^n} \eta_\xi\|_\infty = \|\nabla_{\mathbb{H}^n} \eta_0\|_\infty = \|\alpha'(d)\nabla_{\mathbb{H}^n} d\|_\infty \leqslant \|\alpha'\|_\infty \leqslant \frac{2}{R-r} . \tag{2.19}$$

We now fix $\xi \in \mathbb{H}^n$ and $p \in [Q^*, +\infty[$ and we set $\eta = \eta_\xi$, $\varphi = \eta^2 u^{p-1}$, $\psi = \eta u^{p/2}$, $B_r = B_d(\xi, r)$ and $B_R = B_d(\xi, R)$. We remark that $\varphi, \psi \in S_0^1(\Omega)$. We also set $\beta = Q^*/2$. Finally, we define

$$I = \int \eta^2 u^{p-2} |\nabla_{\mathbb{H}^n} u|^2 .$$

Choosing φ as a test function in (1.9), we get

$$\begin{aligned}\int u^{Q^*-1}\varphi = \int \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n}\varphi\rangle \geqslant \int \eta^2 \langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n}(u^{p-1})\rangle - 2\int \eta u^{p-1} |\nabla_{\mathbb{H}^n} u|\, |\nabla_{\mathbb{H}^n}\eta| \geqslant \\ (p-1)\int \eta^2 u^{p-2}\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} u\rangle - \frac{1}{2} I - 2\int |\nabla_{\mathbb{H}^n}\eta|^2 u^p = \\ \left(p - \frac{3}{2}\right) I - 2\int |\nabla_{\mathbb{H}^n}\eta|^2 u^p \geqslant \frac{1}{2} I - 2 \int |\nabla_{\mathbb{H}^n}\eta|^2 u^p .\end{aligned}$$

Hence, recalling (2.19), we obtain

$$\begin{aligned}\|u\|^p_{L^{\beta p}(B_r)} \leqslant \|\psi\|^2_{Q^*} \leqslant c\|\nabla_{\mathbb{H}^n}\psi\|_2^2 = \\ c\|\eta\nabla_{\mathbb{H}^n}(u^{p/2}) + u^{p/2}\nabla_{\mathbb{H}^n}\eta\|_2^2 \leqslant 2c\left(\frac{p^2}{4} I + \|u^{p/2}\nabla_{\mathbb{H}^n}\eta\|_2^2\right) \leqslant \\ c\left(p^2\left(\int u^{Q^*-1}\varphi + \int |\nabla_{\mathbb{H}^n}\eta|^2 u^p\right) + \|u^{\frac{p}{2}}\nabla_{\mathbb{H}^n}\eta\|_2^2\right) \leqslant \\ cp^2\left(\|u\|_\infty^{Q^*-2}\int_{B_R}\eta^2 u^p + \frac{4}{(R-r)^2}\int_{B_R} u^p\right) \leqslant \\ cp^2\left(1 + \frac{1}{(R-r)^2}\right)\int_{B_R} u^p \leqslant c\left(\frac{p}{R-r}\right)^2 \|u\|^p_{L^p(B_R)} .\end{aligned} \tag{2.20}$$

The positive constant c only depends on Q and u. For every $m \in \mathbb{Z}^+$ we now define $p_m = \beta^m Q^*$, $r_m = 1 + 1/2^m$, $B_m = B_d(\xi, r_m)$, $H_m = \|u\|_{L^{p_m}(B_m)}$. Letting $p = p_m$, $r = r_{m+1}$ and $R = r_m$ in (2.20) we get

$$H_{m+1} = \|u\|_{L^{\beta p_m}(B_{m+1})} \leq c^{1/p_m} \left(\frac{p_m}{r_m - r_{m+1}} \right)^{2/p_m} H_m =$$

$$c^{1/\beta^m Q^*} (2^{m+1} \beta^m Q^*)^{2/\beta^m Q^*} H_m = c^{c\beta^{-m}} (2\beta)^{cm\beta^{-m}} H_m .$$

Hence, for every $N \in \mathbb{N}$ we have

$$H_N \leq H_0 \prod_{m=0}^{N-1} \left(c^{c\beta^{-m}} (2\beta)^{cm\beta^{-m}} \right) \leq cH_0 .$$

Letting $N \to +\infty$ we finally obtain

$$\|u\|_{L^{Q^*}(B_d(\xi, 2))} = H_0 \geq cH_N \geq c\|u\|_{L^{p_N}(B_d(\xi, 1))} \xrightarrow{N \to +\infty} c\|u\|_{L^\infty(B_d(\xi, 1))} .$$

The positive constant c only depends on Q and u. ■

PROPOSITION 2.6. – *We have* $u(\xi) \to 0$, *as* $d(\xi) \to \infty$, ξ *a.e. in* Ω.

PROOF. – It is an immediate consequence of Lemma 2.5. ■

PROPOSITION 2.7. – *Let us suppose that* Ω *satisfies the following boundary regularity condition: there exist two positive constants* δ *and* r_0 *such that*

$$(2.21) \qquad |B_d(\xi, r) \setminus \Omega| \geq \delta |B_d(\xi, r)| \qquad \forall \xi \in \partial\Omega \qquad \forall r \in]0, r_0[.$$

Then, if we continue u *on* $\mathbb{H}^n$ *by setting* $u = 0$ *outside* Ω, *there exist* $\alpha \in]0, 1[$ *and* $M > 0$ *such that*

$$|u(\xi) - u(\xi')| \leq M(d(\xi, \xi'))^\alpha \qquad \forall \xi, \xi' \in \mathbb{H}^n$$

(*i.e., following Folland-Stein* [FS], u *belongs to the Hölder space* $\Gamma^\alpha(\mathbb{H}^n)$). *In particular* $u \in C(\overline{\Omega})$, $u = 0$ *in* $\partial\Omega$.

PROOF. – We refer to the Appendix, Theorem A.1 (we recall that $u \in L^\infty(\Omega)$ by Lemma 2.4). ■

COROLLARY 2.8. – *If* $\Omega = \Pi$ *is a halfspace then there exists* $\alpha \in]0, 1[$ *such that* $u \in \Gamma^\alpha(\mathbb{H}^n)$, *setting* $u = 0$ *outside* Π. *In particular* $u \in C(\overline{\Pi})$, $u = 0$ *in* $\partial\Pi$.

PROOF. – We only need to prove that every halfspace Π satisfies (2.21). When $\xi = 0$ the d-balls centered at ξ are symmetric with respect to ξ (see (1.5)) and the condition (2.21) is obviously satisfied (with $\delta = 1/2$). On the other hand any other case can be reduced to this one by a left translation of the group $\mathbb{H}^n$ (which is a bijective affine transformation mapping halfspaces into half-spaces). ■

PROPOSITION 2.9. – *It is* $u \in C^{\infty}(\Omega)$. *Moreover, if* $\Omega = \Pi$ *is a halfspace with boundary parallel to the t-axis, then* $u \in C^{\infty}(\overline{\Pi})$.

PROOF. – The first part of the statement follows from Proposition 2.1, by means of a standard regularization technique, based on the results of Folland and Stein [FS]. The additional hypothesis that $\partial\Pi$ is parallel to the t-axis ensures that $\partial\Pi$ does not have characteristic points. This fact together with Corollary 2.8 gives the boundary regularity (see [KN]; see also [J]). ■

REMARK 2.10. – *If* Ω *satisfies* (2.21) (*in particular if* $\Omega = \Pi$ *is a halfspace*) *then* u *is a classical solution of*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = u^{(Q+2)/(Q-2)} & \text{in } \Omega\,, \\ u \geqslant 0 & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,, \\ u(\xi) \to 0 & \text{as } d(\xi) \to \infty\,, \end{cases}$$

by means of Propositions 2.6, 2.7 *and* 2.9.

3. – Proof of Theorem 1.1.

In this section Ω will be an arbitrary unbounded open subset of $\mathbb{H}^n$ and u will denote any fixed nonnegative weak solution of the boundary value problem (1.1).

Our method, partially inspired by a technique used in [CGL], is based on a representation formula which can be found in [GL1]: if v is a C^2 function in an open subset A of $\mathbb{H}^n$ and $\overline{B_d(\xi, r)} \subseteq A$ then

$$v(\xi) = (M_r v)(\xi) - \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \int_{B_d(\xi, \varrho)} \left(\Gamma(\xi, \xi') - \frac{c_Q}{\varrho^{Q-2}} \right) \Delta_{\mathbb{H}^n} v(\xi')\, d\xi'\, d\varrho \tag{3.1}$$

where M_r is the mean value operator defined by

$$(M_r v)(\xi) = \frac{1}{\alpha_Q r^Q} \int_{B_d(\xi, r)} \psi(\xi, \xi')\, v(\xi')\, d\xi'\,. \tag{3.2}$$

Here we have set $\Gamma(\xi, \xi') = \Gamma(\xi'^{-1} \circ \xi)$ and $\psi(\xi, \xi') = \psi(\xi'^{-1} \circ \xi)$ where

$$\psi(\xi) = \frac{|z|^2}{d(\xi)^2}\,. \tag{3.3}$$

Moreover

$$\alpha_Q = \int_{B_d(0,1)} \psi(\xi)\, d\xi\,.$$

Let us introduce the function

$$w = \Gamma * f\colon\ \mathbb{H}^n \to \mathbb{R}\,, \qquad w(\xi) = \int_{\mathbb{H}^n} \Gamma(\xi, \xi')\, f(\xi')\, d\xi'\,, \tag{3.4}$$

where we have denoted

$$f = u^{Q^* - 1} \tag{3.5}$$

and u is set be zero outside Ω. From Proposition 2.1 it follows that

$$f \in L^1(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)\,. \tag{3.6}$$

Hence, by means of the results of Folland and Stein in [FS], we get

$$w \in L^p(\mathbb{H}^n)\,, \qquad \forall p \in \left]\frac{Q^*}{2}\,,\ +\infty\right[\,, \tag{3.7}$$

$$-\Delta_{\mathbb{H}^n} w = f \quad \text{weakly in } \mathbb{H}^n\,. \tag{3.8}$$

Moreover $w \geq 0$, $\Delta_{\mathbb{H}^n}(u - w) = 0$ in Ω and $(u - w)^+ := \max\{0, u - w\} \in S_0^1(\Omega)$. Then $(u - w)^+ = 0$ in Ω, i.e.

$$0 \leq u \leq w \quad \text{in } \Omega\,. \tag{3.9}$$

The following lemma is another key point in the proof of Theorem 1.1.

LEMMA 3.1. – *For every* $s \in]0, Q-2[$, *it holds*

$$w(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \qquad \text{as } d(\xi) \to \infty\,. \tag{3.10}$$

PROOF. – We set $p = Q/s$ and $V = u^{Q^* - 2}$. For sake of brevity, for every $\xi \in \mathbb{H}^n$ and $r > 0$, we also define

$$N_r(\xi) = \int_{B_d(\xi, r)} \Gamma_\xi V\,, \qquad I_r(\xi) = \int_{B_d(\xi, r)} \Gamma_\xi V w\,, \qquad M_r(\xi) = (M_r w)(\xi)\,,$$

where $\Gamma_\xi = \Gamma(\xi, \cdot)$ and $M_r w$ is the mean value operator defined in (3.2). We now fix $q \in]Q/4, Q/2[$ and we set $p' = p/(p-1)$ and $q' = q/(q-1)$. Since $p, q' \in]Q^*/2, +\infty[$ we have

$$u, w \in L^p(\mathbb{H}^n) \cap L^{q'}(\mathbb{H}^n)\,, \tag{3.11}$$

$$\Gamma \in L^p(\mathbb{H}^n \setminus B_d(0, 1)) \cap L^{q'}(\mathbb{H}^n \setminus B_d(0, 1))\,, \tag{3.12}$$

$$V \in L^q(\mathbb{H}^n) \cap L^\infty(\mathbb{H}^n)\,, \tag{3.13}$$

$$V(\xi) \to 0\,, \qquad \text{as } d(\xi) \to \infty\,, \tag{3.14}$$

by means of (3.7) and Propositions 2.1 and 2.6. Hence we can establish the following estimates

$$M_r(\xi) \leqslant \frac{c}{r^Q} \int_{B_d(\xi,\, r)} w \leqslant \frac{c}{r^Q} \|w\|_p \|1\|_{L^{p'}(B_d(\xi,\, r))} = \frac{c}{r^Q} r^{Q/p'} = \frac{c}{r^s} \tag{3.15}$$

and

$$\begin{aligned} N_r(\xi) \leqslant \|\Gamma_\xi\|_{L^1(B_d(\xi,\, 1))} \|V\|_{L^\infty(B_d(\xi,\, r))} + \|\Gamma_\xi\|_{L^{q'}(\mathbb{H}^n \setminus B_d(\xi,\, 1))} \|V\|_{L^q(B_d(\xi,\, r))} \leqslant \\ c(\|V\|_{L^\infty(B_d(\xi,\, r))} + \|V\|_{L^q(B_d(\xi,\, r))})\,. \end{aligned} \tag{3.16}$$

In particular (3.13) and (3.16) give

$$\sup_{\xi \in \mathbb{H}^n,\, r > 0} N_r(\xi) \leqslant c\,. \tag{3.17}$$

Let us now recall (3.8) and write the representation formula (3.1) for w [1]. From (3.5) and (3.9), $f = Vu \leqslant Vw$, and we get

$$\begin{aligned} w(\xi) = M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(\xi,\, \varrho)} \left(\Gamma_\xi - \frac{c_Q}{\varrho^{Q-2}} \right) f \right) d\varrho \leqslant \\ M_r(\xi) + \frac{Q}{r^Q} \int_0^r \varrho^{Q-1} \left(\int_{B_d(\xi,\, r)} \Gamma_\xi V w \right) d\varrho = M_r(\xi) + I_r(\xi) \leqslant \text{ (by (3.15)) } \frac{c}{r^s} + I_r(\xi)\,. \end{aligned} \tag{3.18}$$

Hence

$$\begin{aligned} I_r(\xi) \leqslant \int_{B_d(\xi,\, r)} \Gamma_\xi(\eta)\, V(\eta) \left(\frac{c}{r^s} + I_r(\eta) \right) d\eta \leqslant \\ \frac{c}{r^s} N_r(\xi) + \int_{B_d(\xi,\, r)} \Gamma_\xi V I_r \leqslant \text{ (by (3.17)) } \frac{c}{r^s} + \int_{B_d(\xi,\, r)} \Gamma_\xi V I_r\,. \end{aligned} \tag{3.19}$$

Moreover

$$\begin{aligned} \int_{B_d(\xi,\, r)} \Gamma_\xi V I_r = \int_{B_d(\xi,\, r)} \Gamma(\xi, \eta)\, V(\eta) \left(\int_{B_d(\eta,\, r)} \Gamma(\eta, \zeta)\, V(\zeta)\, w(\zeta)\, d\zeta \right) d\eta = \\ \int_{B_d(\xi,\, 2r)} V(\zeta)\, w(\zeta) \left(\int_{B_d(\xi,\, r) \cap B_d(\zeta,\, r)} \Gamma(\xi, \eta)\, \Gamma(\eta, \zeta)\, V(\eta)\, d\eta \right) d\zeta\,. \end{aligned} \tag{3.20}$$

[1] (3.1) is proved in [GL1] only for smooth functions, but it can be easily extended to any function as regular as w.

We now define $A = B_d(\xi, r) \cap B_d(\zeta, r)$, $A_1 = \{\eta \in A \mid d(\eta, \zeta) \leqslant d(\xi, \zeta)/2\}$ and $A_2 = A \backslash A_1$. Then $d(\xi, \eta) \geqslant d(\xi, \zeta)/2$ for every $\eta \in A_1$ and

$$\int_A \Gamma_\xi \Gamma_\zeta V =$$

$$\int_{A_1} \Gamma_\xi \Gamma_\zeta V + \int_{A_2} \Gamma_\xi \Gamma_\zeta V \leqslant c\Gamma(\xi, \zeta)\left(\int_{A_1} \Gamma_\zeta V + \int_{A_2} \Gamma_\xi V\right) \leqslant c\Gamma(\xi, \zeta)(N_r(\zeta) + N_r(\xi)) .$$

This estimate and (3.20) finally yield

$$\int_{B_d(\xi, r)} \Gamma_\xi V I_r \leqslant c \sup_{\zeta \in B_d(\xi, 2r)} N_r(\zeta)\left(\int_{B_d(\xi, 2r)} \Gamma_\xi V w\right). \tag{3.21}$$

From now on we will take $r = r(\xi) = d(\xi)/4$. From (3.16) it follows that

$$\sup_{\zeta \in B_d(\xi, 2r(\xi))} N_{r(\xi)}(\zeta) \leqslant c(\|V\|_{L^\infty(\mathbb{H}^n \backslash B_d(0, r(\xi)))} + \|V\|_{L^q(\mathbb{H}^n \backslash B_d(0, r(\xi)))}) .$$

Then, by means of (3.13) and (3.14),

$$\sup_{\zeta \in B_d(\xi, 2r(\xi))} N_{r(\xi)}(\zeta) \to 0 \qquad \text{as } d(\xi) \to \infty .$$

Hence, for every $\xi \in \mathbb{H}^n$ satisfying $d(\xi) > R$, R sufficiently large, from (3.21) we obtain

$$\int_{B_d(\xi, r(\xi))} \Gamma_\xi V I_{r(\xi)} \leqslant \frac{1}{2} \int_{B_d(\xi, 2r(\xi))} \Gamma_\xi V w = \tag{3.22}$$

$$= \frac{1}{2} I_{r(\xi)}(\xi) + \frac{1}{2} \int_{B_d(\xi, 2r(\xi)) \backslash B_d(\xi, r(\xi))} \Gamma_\xi V w \leqslant \frac{1}{2} I_{r(\xi)}(\xi) + \frac{c}{r(\xi)^{Q-2}} \|V\|_q \|w\|_{q'} .$$

Recalling (3.11) and (3.13), from (3.19) and (3.22) we finally get

$$\frac{1}{2} I_{r(\xi)}(\xi) \leqslant \frac{c}{r(\xi)^s} + \frac{c}{r(\xi)^{Q-2}} \qquad \text{for } d(\xi) > R .$$

Therefore, since $r(\xi) = d(\xi)/4$, the estimate (3.18) gives

$$w(\xi) = O\left(\frac{1}{d(\xi)^s}\right), \qquad \text{as } d(\xi) \to \infty . \qquad \blacksquare$$

We are now able to establish the following «optimal» asymptotic behavior and then to prove Theorem 1.1.

POPOSITION 3.2. – *It is* $w(\xi) = O(d(\xi)^{2-Q})$*, as* $d(\xi) \to \infty$.

PROOF. – From (3.5), (3.9) and (3.10), choosing

$$s = \frac{Q}{Q^* - 1} = \frac{Q}{Q+2}\,(Q-2),$$

it follows that there exist $M, R > 0$ such that

$$f(\eta) \leqslant \frac{M}{d(\eta)^{s(Q^*-1)}} = \frac{M}{d(\eta)^Q} \qquad \text{if } d(\eta) > R\,.$$

Hence, for $d(\xi) > 2R$ we obtain

$$\sup_{\eta \in B_d(\xi,\, d(\xi)/2)} f(\eta) \leqslant \frac{2^Q M}{d(\xi)^Q} \tag{3.23}$$

which yields, also using (1.4) and (3.6),

$$w(\xi) = \int_{\{d(\xi,\,\eta) \geqslant d(\xi)/2\}} \Gamma(\xi, \eta) f(\eta) d\eta + \int_{\{d(\xi,\,\eta) < d(\xi)/2\}} \Gamma(\xi, \eta) f(\eta)\, d\eta \leqslant$$

$$\frac{c}{d(\xi)^{Q-2}} \|f\|_1 + \frac{c}{d(\xi)^Q} \int_{\{d(\xi,\,\eta) < d(\xi)/2\}} \frac{d\eta}{d(\xi, \eta)^{Q-2}} = \frac{c}{d(\xi)^{Q-2}}\,. \qquad \blacksquare$$

PROOF OF THEOREM 1.1. – It directly follows from (3.9), Lemma 2.4 and Proposition 3.2. $\blacksquare$

The following lemma provides an estimate of the derivative $\partial_t w$, which will play an important role in the next section. From now on we will suppose that Ω satisfies the boundary regularity condition (2.21). By Proposition 2.7 and Lemma 2.4 there exists $\alpha \in]0, 1[$ such that

$$f \in \Gamma^\alpha(\mathbb{H}^n) \tag{3.24}$$

where f is the function defined in (3.5). Then

$$w \in \Gamma^{2+\alpha}_{\text{loc}}(\mathbb{H}^n), \tag{3.25}$$

i.e. $XYw \in \Gamma^\alpha_{\text{loc}}(\mathbb{H}^n)$ for every $X, Y \in \{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ (see [FS]).

LEMMA 3.3. – *If Ω satisfies* (2.21) (*in particular if $\Omega = \Pi$ is a halfspace*) *then*

$$\partial_t w(\xi_0) = \int_{B_d(\xi_0,\, r)} \partial_t \Gamma(\xi_0, \xi')(f(\xi') - f(\xi_0))\, d\xi' + \int_{\mathbb{H}^n \setminus B_d(\xi_0,\, r)} \partial_t \Gamma(\xi_0, \xi') f(\xi')\, d\xi', \tag{3.26}$$

for every $\xi_0 \in \mathbb{H}^n$ and $r > 0$.

PROOF. – Let us fix $\xi_0 \in \mathbb{H}^n$ and $r > 0$. We define

$$B = B_d\left(\xi_0, \frac{r}{2}\right), \qquad B_0 = B_d(\xi_0, r),$$

$$f_0 = f\chi_{B_0}, \qquad f_1 = f - f_0,$$

where χ_{B_0} denotes the characteristic function of the set B_0. Let $\eta \in C^\infty([0, +\infty[)$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 0$ in $[0, 1]$, $\eta \equiv 1$ in $[2, +\infty[$, and for every $\varepsilon > 0$ let $\eta_\varepsilon = \eta(d/\varepsilon) \in C^\infty(\mathbb{H}^n)$; we will also denote $\eta_\varepsilon(\xi, \xi') = \eta_\varepsilon((\xi')^{-1} \circ \xi)$. We set

$$w_0 = \Gamma * f_0, \qquad w_1 = \Gamma * f_1, \qquad w_{0,\varepsilon} = (\Gamma\eta_\varepsilon) * f_0.$$

Since from (3.6) $f \in L^\infty$, it is immediate to verify that

$$w_{0,\varepsilon} \rightrightarrows w_0 \quad \text{in } B, \quad \text{as } \varepsilon \to 0^+ \tag{3.27}$$

(hereafter $\rightrightarrows$ will denote uniform convergence). Moreover $w_{0,\varepsilon} \in C^\infty(B)$ and, for every $\xi \in B$ and $\varepsilon < r/4$, we have

$$\partial_t w_{0,\varepsilon}(\xi) = \int_{B_0} \partial_t(\Gamma\eta_\varepsilon)(\xi, \xi') f_0(\xi')\, d\xi' = \tag{3.28}$$

$$\int_{B_0} \partial_t(\Gamma\eta_\varepsilon)(\xi, \xi')(f_0(\xi') - f_0(\xi))\, d\xi' - f_0(\xi) \int_{B_0} \partial_{t'}(\Gamma\eta_\varepsilon)(\xi, \xi')\, d\xi' =$$

$$\int_{B_0} \partial_t(\Gamma\eta_\varepsilon)(\xi, \xi')(f_0(\xi') - f_0(\xi))\, d\xi' - f_0(\xi) \int_{\partial B_0} \Gamma(\xi, \xi')\, \nu_t(\xi')\, d\sigma(\xi').$$

The last equality follows from the divergence theorem. We have denoted by σ the surface measure and by $\nu_t(\xi')$ the $(2n+1)$-th component of the outer unit normal. We set $\overline{w}_0 \colon B \to \mathbb{R}$,

$$\overline{w}_0(\xi) = \int_{B_0} \partial_t \Gamma(\xi, \xi')(f_0(\xi') - f_0(\xi))\, d\xi' - f_0(\xi) \int_{\partial B_0} \Gamma(\xi, \xi')\, \nu_t(\xi')\, d\sigma(\xi'). \tag{3.29}$$

From the Hölder continuity of f given by (3.24) and the estimate $|\partial_t \Gamma| \leq cd^{-Q}$, we obtain

$$\int_{B_0} |\partial_t \Gamma(\xi, \xi')(f_0(\xi') - f_0(\xi))|\, d\xi' \leq c \int_{B_0} d(\xi, \xi')^{\alpha - Q} < +\infty .$$

Moreover, from (3.28) and (3.29),

$$|\overline{w}_0(\xi) - \partial_t w_{0,\varepsilon}(\xi)| = \left| \int_{B_d(\xi, 2\varepsilon)} \partial_t(\Gamma(1-\eta_\varepsilon))(\xi, \xi')(f_0(\xi') - f_0(\xi))\, d\xi' \right| \leqslant$$

$$c \int_{B_d(\xi, 2\varepsilon)} \left(\Gamma(\xi, \xi') \|\eta'\|_\infty \frac{|\partial_t d(\xi, \xi')|}{\varepsilon} + |\partial_t \Gamma(\xi, \xi')| \right) d(\xi, \xi')^\alpha \, d\xi' \leqslant$$

$$c \int_{B_d(\xi, 2\varepsilon)} d(\xi, \xi')^{\alpha - Q} d\xi' = c \int_{B_d(0, 2\varepsilon)} d^{\alpha - Q},$$

for every $\xi \in B$ and $\varepsilon < r/4$. Hence

$$\partial_t w_{0,\varepsilon} \rightrightarrows \overline{w}_0 \quad \text{in } B, \quad \text{as } \varepsilon \to 0^+ . \tag{3.30}$$

From (3.27) and (3.30) we deduce

$$\partial_t w_0 = \overline{w}_0 \quad \text{in } B . \tag{3.31}$$

On the other hand, since $f \in L^1(\mathbb{H}^n)$ (see (3.6)), for every $\xi \in B$ we have

$$\partial_t w_1(\xi) = \int_{\mathbb{H}^n \setminus B_0} \partial_t \Gamma(\xi, \xi') f_1(\xi')\, d\xi' . \tag{3.32}$$

Moreover, the divergence theorem ensures that

$$\int_{\partial B_0} \Gamma(\xi_0, \xi')\, \nu_t(\xi')\, d\sigma(\xi') = cr^{2-Q} \int_{\partial B_0} \nu_t(\xi')\, d\sigma(\xi') = 0 . \tag{3.33}$$

Recalling that $w = w_0 + w_1$ and replacing ξ with ξ_0 in (3.29) and (3.32), from (3.29), (3.31), (3.32) and (3.33) we finally obtain (3.26). ■

PROPOSITION 3.4. – *If Ω satisfies* (2.21) (*in particular if $\Omega = \Pi$ is a halfspace*) *then*

$$|\partial_t w(\xi)| = O(d(\xi)^{2-Q}), \qquad \text{as } d(\xi) \to \infty .$$

PROOF. – We set $\beta = 4/(Q + 2 + \alpha)$ (α appears in (3.24)). From (3.6), (3.24) and (3.26), for every $\xi \in \mathbb{H}^n$ and $r > 0$, we get

(3.34) $\quad |\partial_t w(\xi)| \leqslant$

$$c \int_{B_d(\xi, r)} d(\xi, \xi')^{-Q} |f(\xi') - f(\xi)|^\beta \Big(\sup_{B_d(\xi, r)} f \Big)^{1-\beta} d\xi' + c \int_{\mathbb{H}^n \setminus B_d(\xi, r)} d(\xi, \xi')^{-Q} f(\xi')\, d\xi' \leqslant$$

$$c \Big(\sup_{B_d(\xi, r)} f \Big)^{1-\beta} \int_{B_d(\xi, r)} d(\xi, \xi')^{\alpha\beta - Q} d\xi' + cr^{-Q} \|f\|_1 = cr^{\alpha\beta} \Big(\sup_{B_d(\xi, r)} f \Big)^{1-\beta} + cr^{-Q} .$$

From (3.5) and Theorem 1.1 it follows that $f(\xi) = O(d(\xi)^{-Q-2})$ as $d(\xi) \to \infty$. Choosing $r = d(\xi)/2$ in (3.34), for large $d(\xi)$ we get

$$|\partial_t w(\xi)| \leqslant c d(\xi)^{\alpha\beta} d(\xi)^{(\beta-1)(Q+2)} + c d(\xi)^{-Q} = c d(\xi)^{2-Q} + c d(\xi)^{-Q} \leqslant c d(\xi)^{2-Q}. \qquad \blacksquare$$

4. – Proof of Theorem 1.2.

It is not restrictive to prove Theorem 1.2 under the assumption

$$\Pi = \Pi_1 := \{\xi = (x, y, t) \in \mathbb{H}^n \,|\, x_1 > 0\}. \tag{4.1}$$

Indeed, as it has been noticed in [U], for every halfspace Π with boundary parallel to the t-axis, there exist a left translation τ and a rotation ϱ around the t-axis, such that $\Pi = \varrho\tau\Pi_1$. Moreover the operators $\Delta_{\mathbb{H}^n}$ and $|\nabla_{\mathbb{H}^n}|$ are invariant with respect to τ and ϱ.

Throughout this section we will then assume (4.1) and denote by u a (fixed) nonnegative weak solution of (1.12).

The main step in the proof of Theorem 1.2 will be Proposition 4.4 where we find the estimate (1.13):

$$|\partial_t u| \leqslant MU.$$

As in the previous section we will denote by w the function

$$w(\xi) = (\Gamma * f)(\xi) = \int_{\mathbb{H}^n} \Gamma(\xi, \xi') f(\xi')\, d\xi', \qquad f = u^{(Q+2)/(Q-2)}.$$

Moreover v will denote the $\Delta_{\mathbb{H}^n}$-harmonic part of u, i.e.

$$v = w - u. \tag{4.2}$$

From (3.8), (3.25) and Remark 2.10 it follows that v is a classical solution of

$$\begin{cases} \Delta_{\mathbb{H}^n} v = 0 & \text{in } \Pi, \\ v = w & \text{in } \partial\Pi. \end{cases} \tag{4.3}$$

As a consequence, since the operators Δ_{H^n} and ∂_t commute, $u = 0$ on $\partial\Pi$ and $\partial\Pi$ is invariant with respect to the Euclidean translations which are parallel to the t-axis, we have

$$\begin{cases} \Delta_{\mathbb{H}^n}(\partial_t v) = 0 & \text{in } \Pi, \\ \partial_t v = \partial_t w & \text{in } \partial\Pi. \end{cases}$$

Our main idea is to use these properties for representing $\partial_t v$ as a fixed point for the following mean value operator modeled on the geometry of $\Pi = \Pi_1$. For every $\omega \in L^1_{\text{loc}}(\Pi)$ we define

$$T\omega\colon\ \Pi \to \mathbb{R}\,, \qquad (T\omega)(\xi) = (M_{r(\xi)}\,\omega)(\xi) \tag{4.4}$$

where M_r is the mean value operator introduced in (3.2) and

$$r(\xi) \equiv r(x_1, \ldots, x_n, y_1, \ldots, y_n, t) := \frac{x_1}{2}\,. \tag{4.5}$$

PROPOSITION 4.1. – *T is a linear operator with the following properties.*

(1) *T maps $L^1_{\text{loc}}(\Pi)$ into $C(\Pi)$. Hence we can define, by induction,*

$$T^{k+1}\omega = T(T^k(\omega)) \qquad \forall \omega \in L^1_{\text{loc}}(\Pi)\,, \quad \forall k \in \mathbb{N}\,.$$

(2) *T is an increasing operator, i.e.:*

$$(\omega_1, \omega_2 \in L^1_{\text{loc}}(\Pi),\ \omega_1 \leq \omega_2) \Rightarrow (T\omega_1 \leq T\omega_2)\,.$$

(3) *If $\omega \in C^2(\Pi)$ and $\Delta_{\mathbb{H}^n}\omega = 0$ then $T\omega = \omega$.*

(4) *If $\omega \in C^2(\Pi)$ and $\Delta_{\mathbb{H}^n}\omega \leq 0$ then $(T^k\omega)_{k \in \mathbb{N}}$ is a decreasing sequence.*

(5) *The operators T and ∂_t commute. More precisely if there exists $\partial_t\omega \in C(\Pi)$ then there exists also $\partial_t(T\omega) \in C(\Pi)$ and it is*

$$\partial_t(T\omega) = T(\partial_t\omega)\,.$$

PROOF. – (1) We prove that, for every fixed $\omega \in L^1_{\text{loc}}(\Pi)$ and $\xi_0 \in \Pi$,

$$|T\omega(\xi) - T\omega(\xi_0)| \leq |M_{r(\xi)}\,\omega(\xi) - M_{r(\xi)}\,\omega(\xi_0)| + |M_{r(\xi_0)}\,\omega(\xi_0) - M_{r(\xi)}\,\omega(\xi_0)| \to 0\,, \quad \text{as } \xi \to \xi_0\,.$$

On the one hand (3.2) gives

$$|M_{r(\xi)}\,\omega(\xi) - M_{r(\xi)}\,\omega(\xi_0)| =$$

$$\frac{1}{\alpha_Q r(\xi)^Q}\left|\int_{B_d(\xi,\, r(\xi))} \psi(\xi, \eta)\,\omega(\eta)\,d\eta - \int_{B_d(\xi_0,\, r(\xi))} \psi(\xi_0, \zeta)\,\omega(\zeta)\,d\zeta\right| =$$

$$\frac{1}{\alpha_Q r(\xi)^Q}\left|\int_{B_d(\xi_0,\, r(\xi))} \psi(\xi_0, \zeta)(\omega(\xi \circ \xi_0^{-1} \circ \zeta) - \omega(\zeta))\,d\zeta\right| \leq$$

$$\frac{1}{\alpha_Q r(\xi)^Q}\int_{B_d(\xi_0,\, r(\xi))} |\omega(\xi \circ \xi_0^{-1} \circ \zeta) - \omega(\zeta)|\,d\zeta\,.$$

The continuity of r (see (4.5)) and the L^1-continuity theorem ensure that the far right hand term in the previous inequalities goes to zero as $\xi \to \xi_0$. On the other hand, the continuity of r yields

$$\lim_{\xi \to \xi_0} M_{r(\xi)} \omega(\xi_0) = M_{r(\xi_0)} \omega(\xi_0).$$

It then follows that $T\omega \in C(\Pi)$.

(2), (3) immediately follow by comparing (4.4), (3.1) and (3.2).

(4) From (3.1) and (4.4) we obtain $T\omega \leqslant \omega$. This fact and (2) yield $\omega \geqslant T^k \omega \geqslant T^{k+1} \omega$, for every $k \in \mathbb{N}$.

(5) We set $F\colon \Pi \times B_d(0, 1) \to \mathbb{R}$,

$$F(\xi, \xi') = (\omega \circ \tau_\xi \circ \delta_{r(\xi)})(\xi') = \omega(z + r(\xi)\, z', t + r(\xi)^2 t' + 2r(\xi)(\langle x', y\rangle - \langle x, y'\rangle));$$

see (1.2) and (1.3) for notations. From (4.5) we have $\partial_t r \equiv 0$ so that

$$\partial_t F(\xi, \xi') = ((\partial_t \omega) \circ \tau_\xi \circ \delta_{r(\xi)})(\xi'). \tag{4.6}$$

By means of a change of variable we obtain

$$T\omega(\xi) = \frac{1}{\alpha_Q r(\xi)^Q} \int_{B_d(\xi, r(\xi))} (\psi \circ \tau_\xi^{-1}) \omega = \frac{1}{\alpha_Q} \int_{B_d(0, 1)} (\psi \circ \delta_{r(\xi)})(\omega \circ \tau_\xi \circ \delta_{r(\xi)}) =$$

$$\frac{1}{\alpha_Q} \int_{B_d(0, 1)} \psi(\xi')\, F(\xi, \xi')\, d\xi'.$$

Hence, from (4.6) and the continuity of $\partial_t \omega$, we get

$$\partial_t (T\omega)(\xi) = \frac{1}{\alpha_Q} \int_{B_d(0, 1)} \psi((\partial_t \omega) \circ \tau_\xi \circ \delta_{r(\xi)}) =$$

$$\frac{1}{\alpha_Q r(\xi)^Q} \int_{B_d(\xi, r(\xi))} (\psi \circ \tau_\xi^{-1})\, \partial_t \omega = T(\partial_t \omega)(\xi).$$

Using (1), we can write $\partial_t (T\omega) = T(\partial_t \omega) \in C(\Pi)$. ■

REMARK 4.2. – *We emphasize that the assertion* (5) *of the previous proposition holds since $\partial \Pi$ is parallel to the t-axis and we can choose $r(\xi)$ not depending on t* (*see* (4.5)).

LEMMA 4.3. – *$T^k u$ is a decreasing sequence which is pointwise convergent to zero in Π* as $k \to \infty$.

PROOF. – Since $\Delta_{\mathbb{H}^n} u \leqslant 0$, from Proposition 4.1-(4) we deduce the existence of a function $h\colon \Pi \to \mathbb{R}$ such that

$$T^k u \searrow h. \tag{4.7}$$

Since $u \geqslant 0$, using Proposition 4.1-(2) we obtain

$$0 \leqslant h \leqslant u \,. \tag{4.8}$$

Hence $h \in L^1_{\mathrm{loc}}(\Pi)$. Recalling (4.4) and (3.2), we see that (4.7) implies

$$h(\xi) = \lim_{k \to \infty} (T^{k+1} u(\xi)) = \lim_{k \to \infty} \left(\frac{1}{\alpha_Q r(\xi)^Q} \int_{B_d(\xi,\, r(\xi))} \psi(\xi, \xi')\, T^k u(\xi')\, d\xi' \right) =$$

$$\frac{1}{\alpha_Q r(\xi)^Q} \int_{B_d(\xi,\, r(\xi))} \psi(\xi, \xi')\, h(\xi')\, d\xi' = Th(\xi)$$

for every $\xi \in \Pi$. Therefore, Proposition 4.1-(1) provides

$$h = Th \in C(\Pi)\,. \tag{4.9}$$

Moreover, since $u \in C(\overline{\Pi})$, $u \equiv 0$ in $\partial \Pi$ and $u(\xi) \to 0$ as $d(\xi) \to \infty$ (see Remark 2.10), (4.8) yields

$$h \in C(\overline{\Pi})\,, \qquad h \equiv 0 \ \text{ in } \partial\Pi\,, \qquad h(\xi) \to 0 \ \text{ as } d(\xi) \to \infty\,. \tag{4.10}$$

Let us now assume by contradiction that h is not identically 0. From (4.8) and (4.10) there exists $\xi_0 \in \Pi$ such that

$$h(\xi_0) = \max_{\overline{\Pi}} h > 0\,.$$

Hence

$$A = h^{-1}(\{h(\xi_0)\})$$

is closed and nonempty. Moreover for every $\xi \in A$, (4.9) yields

$$0 = h(\xi) - Th(\xi) = \frac{1}{\alpha_Q r(\xi)^Q} \int_{B_d(\xi,\, r(\xi))} \psi(\xi, \xi') (\max_{\overline{\Pi}} h - h(\xi'))\, d\xi'\,.$$

Then $h(\xi') = \max_{\overline{\Pi}} h = h(\xi_0)$ for every $\xi' \in B_d(\xi, r(\xi))$, i.e. $B_d(\xi, r(\xi)) \subseteq A$. Therefore A is also open. This yields $A = \Pi$ since Π is connected. In other words $h \equiv h(\xi_0) > 0$ in Π, contradicting (4.10). Hence it has to be $h \equiv 0$ in Π. ∎

PROPOSITION 4.4. – *It is* $|\partial_t u(\xi)| = O(d(\xi)^{2-Q})$, *as* $d(\xi) \to \infty$, $\xi \in \Pi$.

PROOF. – From (4.3) and Proposition 4.1-(3), it is $Tv = v$. Lemma 4.3, (4.2) and the linearity of T give

$$T^k w = T^k v + T^k u = v + T^k u \searrow v\,. \tag{4.11}$$

Proposition 4.1-(5) and (4.11) yield

$$\int_{\Pi} \Phi T^k(\partial_t w) = \int_{\Pi} \Phi \partial_t (T^k w) = -\int_{\Pi} (\partial_t \Phi)\, T^k w \xrightarrow{k\to\infty} -\int_{\Pi} v \partial_t \Phi = \int_{\Pi} \Phi \partial_t v \tag{4.12}$$

for every $\Phi \in C_0^\infty(\Pi)$. From Proposition 3.4 and the continuity of $\partial_t w$ (see (3.25)), there exists $M > 0$ such that

$$-M\Gamma \leqslant \partial_t w \leqslant M\Gamma \tag{4.13}$$

where Γ is the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero (see (1.4)). Thus, for every $k \in \mathbb{N}$,

$$-M\Gamma = T^k(-M\Gamma) \leqslant T^k(\partial_t w) \leqslant T^k(M\Gamma) = M\Gamma \quad \text{in } \Pi\,, \tag{4.14}$$

by means of Proposition 4.1-(2),(3). From (4.12) and (4.14) we finally obtain

$$\left| \int_{\Pi} \Phi \partial_t v \right| \leqslant M \int_{\Pi} \Phi \Gamma \qquad \forall \Phi \in C_0^\infty(\Pi)\,, \qquad \Phi \geqslant 0$$

which implies

$$|\partial_t v| \leqslant M\Gamma \quad \text{in } \Pi\,. \tag{4.15}$$

Collecting (4.2), (4.13) and (4.15) we conclude that

$$|\partial_t u| \leqslant 2M\Gamma \quad \text{in } \Pi\,. \qquad \blacksquare$$

PROOF OF THEOREM 1.2. – It follows from Proposition 2.9 and Proposition 4.4, by using the Rellich-Pohozaev type integral identity proved in [GL2]. We use the same arguments as in the proof of Theorem 2.4 of that paper. For the sake of completeness, we next explain the changes we need to make to the proof of [GL2].

We assume $\Pi = \Pi_1$ (see (4.1)). Let us introduce the following notation for the point $\xi \in \mathbb{H}^n$: $\xi = (z, t) = (x_1; \hat{z}, t)$, where $\hat{z} = (x_2, \ldots, x_n, y)$. The outer unit normal to $\partial\Pi$ is

$$N = (-1; 0, 0)\,. \tag{4.16}$$

Let P be the vector field

$$P = -\partial_{x_1} + 2y_1 \partial_t \equiv (-1; 0, 2y_1)\,. \tag{4.17}$$

Then $\langle P, N\rangle = 1$ on $\partial\Pi$ and Π is τ-starshaped with respect to $(-1; 0, 0)$ (see Definition 2.2 in [GL2]). We also remark that, in the notation of [GL2], $P = P^N$. Setting $B_R = B_d(0, R)$ for every $R > 0$, using the integral identity (2.7) of [GL2] and proceeding as on page 83 of the same paper, we obtain

$$(4.18)\qquad \int_{B_R \cap \partial \Pi} |\nabla_{\mathbb{H}^n} u|^2 \, d\sigma = \int_{B_R \cap \partial \Pi} |\nabla_{\mathbb{H}^n} u|^2 \langle P, N\rangle \, d\sigma =$$

$$\int_{\Pi \cap \partial B_R} \left(\left(|\nabla_{\mathbb{H}^n} u|^2 - \frac{2}{Q^*} u^{Q^*} \right) \langle P, \nu\rangle - 2\langle A\nabla u, \nu\rangle \, Pu \right) d\sigma .$$

Here $\nu = \nabla d / |\nabla d|$ is the outer unit normal to ∂B_R, σ denotes the surface measure and A is the matrix

$$\begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \\ 2y & -2x & 4|z|^2 \end{pmatrix}$$

which allows to write $\varDelta_{\mathbb{H}^n}$ in the following divergence form

$$\varDelta_{\mathbb{H}^n} = \operatorname{div}(A\nabla), \qquad \nabla = \text{ gradient operator in } \mathbb{R}^{2n+1}.$$

By means of assumption (4.1) and Proposition 2.9 the function u belongs to $C^\infty(\overline{\Pi})$. Using Proposition 4.4 we obtain the following important estimate

$$(4.19)\qquad |Pu| = |-X_1 u + 4y_1 \partial_t u| \leqslant |\nabla_{\mathbb{H}^n} u| + c d^{3-Q} \qquad \text{in } \Pi .$$

Since

$$|\langle P, \nu\rangle(\xi)| = \left| \left\langle P, \frac{\nabla d}{|\nabla d|} \right\rangle (\xi) \right| = \frac{|\langle (-1; 0, 2y_1), d(\xi)^{-3}(|z|^2 z, t/2)\rangle|}{|\nabla d(\xi)|} \leqslant \frac{1}{|\nabla d(\xi)|}$$

and

$$|\langle A\nabla u, \nu\rangle| = \frac{|\langle A\nabla u, \nabla d\rangle|}{|\nabla d|} = \frac{|\langle \nabla_{\mathbb{H}^n} u, \nabla_{\mathbb{H}^n} d\rangle|}{|\nabla d|} \leqslant \frac{|\nabla_{\mathbb{H}^n} u|}{|\nabla d|} ,$$

(4.19) and (4.18) yield

$$(4.20)\qquad \int_{B_R \cap \partial \Pi} |\nabla_{\mathbb{H}^n} u|^2 \, d\sigma \leqslant c \int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{H}^n} u|^2 + u^{Q^*}}{|\nabla d|} \, d\sigma + c \int_{\Pi \cap \partial B_R} \frac{|\nabla_{\mathbb{H}^n} u| d^{3-Q}}{|\nabla d|} \, d\sigma .$$

By Federer's coarea formula (see [Fe]), for every $g \in L^1(\mathbb{H}^n)$ it holds

$$(4.21)\qquad \int_{\mathbb{H}^n} g = \int_0^{+\infty} \left(\int_{\partial B_r} \frac{g}{|\nabla d|} \, d\sigma \right) dr .$$

Letting $g = |\nabla_{\mathbb{H}^n} u|^2 + u^{Q^*}$, (4.21) implies that there exists a sequence $(R_k)_{k \in \mathbb{N}}$ such that $R_k \to +\infty$ and

$$\int_{\Pi \cap \partial B_{R_k}} \frac{|\nabla_{\mathbb{H}^n} u|^2 + u^{Q^*}}{|\nabla d|} d\sigma = o\left(\frac{1}{R_k}\right), \qquad \text{as } k \to +\infty . \tag{4.22}$$

Moreover, letting g be the characteristic function of the set B_R, (4.21) yields

$$\int_0^R \left(\int_{\partial B_r} \frac{d\sigma}{|\nabla d|} \right) dr = \int_{B_R} d\xi = cR^Q$$

and, by differentiation,

$$\int_{\partial B_R} \frac{d\sigma}{|\nabla d|} = cQR^{Q-1} . \tag{4.23}$$

From (4.20), (4.22) and (4.23) we finally obtain

$$\int_{B_{R_k} \cap \partial \Pi} |\nabla_{\mathbb{H}^n} u|^2 d\sigma \leqslant$$

$$o\left(\frac{1}{R_k}\right) + \frac{c}{R_k^{Q-3}} \left(\int_{\Pi \cap \partial B_{R_k}} \frac{|\nabla_{\mathbb{H}^n} u|^2}{|\nabla d|} d\sigma \right)^{1/2} \left(\int_{\Pi \cap \partial B_{R_k}} \frac{1}{|\nabla d|} d\sigma \right)^{1/2} \leqslant$$

$$o\left(\frac{1}{R_k}\right) + \frac{1}{R_k^{Q-3}} \left(o\left(\frac{1}{R_k}\right)\right)^{1/2} R_k^{(Q-1)/2} = o\left(\frac{1}{R_k}\right) + \frac{1}{R_k^{(Q-4)/2}} o(1).$$

Since $Q = 2n + 2 \geqslant 4$, as k goes to infinity we obtain $\nabla_{\mathbb{H}^n} u \equiv 0$ in $\partial \Pi$ and we can conclude that $u \equiv 0$ in Π as in [GL2]. ■

A. – Appendix.

In this appendix we briefly give more detailed proof of how we get global Hölder continuity for weak solutions of the Dirichlet problem (1.1) in Proposition 2.7.

THEOREM A.1. – *Let Ω be an open subset of $\mathbb{H}^n$ satisfying the boundary regularity condition* (2.21). *If $h \in S_0^1(\Omega) \cap L^\infty(\Omega)$ has distributional Kohn Laplacian $\Delta_{\mathbb{H}^n} h \in L^\infty(\Omega)$ then, setting $h = 0$ outside Ω, there exists $\alpha \in]0, 1[$ such that*

$$h \in \Gamma^\alpha(\mathbb{H}^n) .$$

PROOF. – The proof consists in adapting Moser's iteration technique, as presented in [GT], Chapter 8. We only need to replace the Euclidean distance with the Heisenberg distance d, to choose *ad hoc* cut-off functions modeled on d and to use a suitable version of the John-Nirenberg's Theorem adapted to the homogeneous structure of $\mathbb{H}^n$ (see [B]). ∎

Note added in proof. Theorem 1.2 holds true even il Π is a halfspace with boundary transverse to the t-axis. This case is studied in a forthcoming paper: F. UGUZZONI, *A non-existence theorem for a semilinear Dirichlet problem involving critical exponent on halfspaces of the Heisenberg group.*

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