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On the Smallest Degree of a Surface Containing a Space Curve (*).

MARGHERITA ROGGERO - PAOLO VALABREGA

Sunto. – *Sia C una curva dello spazio di grado D contenuta in una superficie di grado r e non in una di grado $r-1$. Se C è integra, allora $r \leqslant \sqrt{6D-2}-2$; questo limite superiore, raggiunto in alcuni casi* (*cfr.* [5]), *non vale però per curve arbitrarie* (*cfr.* [?, 3 (iii)]). *Ogni curva C dello spazio* (*anche non ridotta o riducibile*) *può essere ottenuta come schema degli zero di una sezione non nulla di un opportuno fascio riflessivo F di rango* 2. *Mediante i fasci riflessivi, siamo in grado di estendere alle curve riducibili o non ridotte e di migliorare* (*anche nel caso delle curve integre*) *la precedente disegua-glianza relativa al grado minimo r di superfici contenenti C, in quanto tale grado è collegato ai livelli α e β delle prime due sezioni indipendenti di F. I nostri limiti superiori si ottengono introducendo, oltre al grado D della curva stessa, anche il numero $e = \max\{n/\omega_C(-n)$ ha una sezione globale che lo genera quasi ovunque$\}$ e la seconda classe di Chern c_2 di F. Più precisamente proveremo che, se $(e+4)/2 \geqslant \sqrt{D}$, allora $r \leqslant \sqrt{D}$; in caso contrario $r \leqslant \sqrt{6D+1}-1-(e+4)/4$. Inoltre, se C corrisponde alla prima sezione non nulla di F, si ha $r \leqslant 2\sqrt{3c_2+1+3c_1/4}-1$ ed anche $r \leqslant \sqrt{6D-(e+4)^2+1}-2$.*

Introduction.

Let C be a projective curve of degree D in $\mathbb{P}^3$. It is known (and easy to see) that, if C is integral, there is a surface of degree $r \leqslant \sqrt{6D-2}-2$ containing C (indeed, count the surfaces of degree $E(\sqrt{6D-2}-2)$, i.e. the integral part of $\sqrt{6D-2}-2$, and impose the condition that the surfaces pass through sufficiently many points of C: see [9], Remark 5.11). By the way, this upper bound is reached some times: in his paper [5] Hirschowitz introduces a family of smooth rational curves having the following property: if C is any such curve of degree D and m is an integer such that $Dm+1 \geqslant \binom{m+3}{3}$, then C does not lie on surfaces of degree m. This means that the smallest degree r of a surface containing C cannot be less than $E(\sqrt{6D-2}-2)$, hence it reaches $E(\sqrt{6D-2}-2)$.

The bound above is not valid for arbitrary curves. Beside the simple example of two skew lines, we can consider, on a smooth surface X of degree D in $\mathbb{P}^3$ with

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coordinates (x, y, z, t), the non-reduced curve C of degree D whose ideal is $(x^D, x^{D-1}y, \dots, y^D, f)$, $f = 0$ being an equation for X and $f(0, 0, z, t) \equiv 0$. Then C cannot belong to any surface of degree $D - 1$ (see [1,3], (iii)).

The problem of the smallest degree of a surface containing a curve C can be restated in terms of rank 2 reflexive sheaves. If C is an (almost) arbitrary curve, locally Cohen-Macaulay and almost everywhere locally complete intersection (but perhaps reducible and not reduced), it is known that C is the scheme of zeros of a non-zero section of $H^0 F(n)$, where F is a suitable normalized rank 2 reflexive sheaf (see [3]). More precisely, if $e = \max\{m/\omega_C(-m)$ has a global section generating it almost everywhere$\}$, then C is the scheme of zeros of a non-zero section of $F(n)$, where $2n + c_1 = e + 4$ and $D = c_2 + nc_1 + n^2$, c_1, c_2 being Chern classes for F, $c_1 = 0$ or -1. Moreover, if α and β are the levels of the first and second relevant section of F, then the smallest degree r of a surface containing C is $\alpha + \beta + c_1$ if $n = \alpha$, $n + \alpha + c_1$ otherwise ([8], Remark 1.4).

So the problem can be turned into a problem of sections of a reflexive sheaf.

Using reflexive sheaves we have been able to point out that the expected upper bound $E(\sqrt{6D-2} - 2)$ for the smallest degree r, although valid in some circumstances, is not a good one in general; a better bound depends not only upon the degree D, but also upon the number e or, which is the same, upon the levels α and β and the second Chern class c_2. In our previous paper [9] we found upper bounds for non-stable and semistable sheaves, both in the case $n = \alpha$ and in the case $n > \alpha$. The most difficult part of our proof concerned the case $n = \alpha$; moreover the bounds of [9] for $n > \alpha$ are sharp and, on the other hand, it is hopeless to find an upper bound better than the one given in [9] as far as non-stable sheaves are concerned and $n = \alpha$, considering the example above ([1]).

With these remarks in mind, we want to produce new upper bounds for r when F is semistable and $n = \alpha$, that is when C is minimal for a semistable F, and these results improve [9].

Precisely we want to show that, given any curve C which is minimal for a rank 2 semistable reflexive sheaf F, then:

$$r \leq \min\left(\sqrt{6c_2 + \tfrac{1}{2}(2\alpha + c_1)^2 + \tfrac{3}{2}c_1 + 1} - 1,\ \sqrt{6c_2 + \tfrac{3}{2}(2\alpha + c_1)^2 + \tfrac{3}{2}c_1 + 1} - \tfrac{1}{4}(2\alpha + c_1) - 1\right).$$

Note that the examples above by Hirschowitz come out to be sharp for this bound and for every degree D, but only for low α (precisely $\alpha = 2$ in this case). For large α it easy to see that any curve C having α as high as possible (i.e. $= E(\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - \frac{1}{2}c_1 - 1)$: see [3], Theorem 0.1) reaches the upper bound and that there are curves whose α is not maximal, but reaching the bound. However we do not know whether the bounds are sharp for all α, small or large with respect to c_2.

The last section of this paper contains an account of all results of [9] and of the previous sections, but they are stated in terms of curves and not of reflexive sheaves. So we obtain results which, given a curve C, link D, r and e, and allow to evaluate each one among the three numbers, provided that the other two numbers are known. Summing up, given a curve C which is the scheme of zeros of a section of a rank 2 reflexive sheaf (reducible and not reduced curves are included) then:

(i) $r \leqslant \sqrt{6D - (e+4)^2 + 1} - 1$, if $0 \leqslant e + 4 < 2\sqrt{D}$
(which means that F may be either semistable or non stable with $c_2 > 0$, but in the non stable case the minimal section must be excluded);

(ii) $r \leqslant \sqrt{D}$, if $e + 4 \geqslant 2\sqrt{D}$
(in this event F is necessarily non-stable);

(iii) $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{4}(e+4)$, if $e + 4 < 2\sqrt{D}$
(F may be either semistable or non-stable and C either minimal or not in this case).

We remark that for an integral curve, but not in general, the bound $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{4}(e+4)$ follows from Riemann-Roch.

Bounds (i) and (iii) hold in a common range; it is easy to see that for low n (say $\leqslant \sqrt{\frac{1}{2}c_2}$), the better bound is $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{4}(e+4)$, while for large n the better bound is $r \leqslant \sqrt{6D - (e+4)^2 + 1} - 1$.

We observe that our results are relevant also for sheaves in themselves; in fact the inequality $\alpha + \beta + c_1 \leqslant \sqrt{6c_2 + \frac{3}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1} - \frac{1}{4}(2\alpha + c_1) - 1$ for a semistable F has the following consequence:

for any $n \geqslant \sqrt{6c_2 + c_1 + 1} - 1$, $F(n)$ has sections whose schemes of zeros are curves (see Corollary 3.4 and [2], Theorem 0.1).

1. – Notations, definitions and the three main functions.

For a reflexive sheaf F on $\boldsymbol{P}^3$ (over an algebraically closed field of characteristic 0), $c_1 c_2$, c_3 are its Chern classes; F is normalized if $c_1 = 0$ or -1; $\alpha(F) = \alpha$ is the first integer such that $h^0 F(\alpha) \neq 0$, $\beta(F) = \beta$ is the first integer such that $h^0(\beta) > h^0 \mathcal{O}_{\boldsymbol{P}^3}(\beta - \alpha)$, while a and b are the same for a rank 2 vector bundle E on $\boldsymbol{P}^2$; in particular $a(F) := a(F_H)$, $b(F) := b(F_H)$ for F_H generic plane restriction of F; $\delta(F) = \delta = c_2 + c_1\alpha + \alpha^2$ is the degree of any curve C scheme of zeros of a non-zero section of $F(\alpha)$, while $\Delta = c_2 + c_1 a + a^2$ is the analogous of δ for a rank 2 vector bundle E on $\boldsymbol{P}^2$. Such curves C are called *minimal curves* of F and are the same for all twists of F (so also δ does not depend upon the twist).

$G^\vee$ means the dual of the sheaf G and V^* means the dual of the vector space V. F is called stable if $\alpha > 0$, semistable if $\alpha + c_1 \geqslant 0$, non-stable otherwise.

Unless otherwise stated a rank 2 reflexive sheaf F on $\boldsymbol{P}^3$ and a vector bundle E on $\boldsymbol{P}^2$ will always be normalized and non-split.

We recall here the following theorem, which we shall use many times in this paper:

THEOREM 1.1 ([4], Theorem 0.1). – *Let F be a rank 2 normalized reflexive sheaf such that $c_2 \geqslant 0$ and let t be an integer such that $t > \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 2 - \frac{1}{2}c_1$; then $t \geqslant \alpha$.*

So $\alpha \leqslant \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1 - \frac{1}{2}c_1$.

NOTATIONS 1.2. – *In this paper we shall consider the following three functions of c_1, c_2, α of a semistable sheaf F:*

$$f(c_1, c_2, \alpha) = \sqrt{6\delta + 1} - \frac{1}{4}(2\alpha + c_1) =$$

$$= \sqrt{6c_2 + \frac{3}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1} - \frac{1}{4}(2\alpha + c_1),$$

$$g(c_1, c_2, \alpha) = \sqrt{6\delta - (2\alpha + c_1)^2 + 1} = \sqrt{6c_2 + \frac{1}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1},$$

$$h(c_1, c_2, \alpha) = 2\sqrt{3\delta - \frac{3}{4}(2\alpha + c_1)^2 + 1} - 1 = 2\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 1.$$

From now on t will be a positive integer satisfying one among the following conditions:

(a) $t + \alpha + c_1 + 2 > f(c_1, c_2, \alpha)$;

(b) $t + \alpha + c_1 + 2 > g(c_1, c_2, \alpha)$;

(c) $t + \alpha + c_1 + 2 > h(c_1, c_2, \alpha)$.

REMARK 1.3. – We will prove that, for a semistable sheaf F, $t \geqslant \beta$, under whathever condition (a), (b) or (c).

First of all we observe that, if $c_2 = 1$, then $\alpha \leqslant 1$ because of theorem 1.1 above, hence $2\alpha + c_1 \leqslant 2$. Since, by [1], we have $\alpha + \beta + c_1 \leqslant c_2 + c_1\alpha + \alpha^2$, we obtain that $\beta = 1$. All three conditions above imply that $t \geqslant 1$, hence the claim is true in this event. So *we shall assume from now on that* $c_2 \geqslant 2$ and in particular that F is not a null-correlation bundle.

Moreover, we emphasize that $g \leqslant h$ for all α allowed by Theorem 1.1, and $f \leqslant g$ if and only if $\alpha + \frac{1}{2}c_1 \leqslant n$, $\sqrt{\frac{1}{3}c_2} \leqslant n \leqslant \sqrt{\frac{2}{3}c_2}$: see Figure 1.

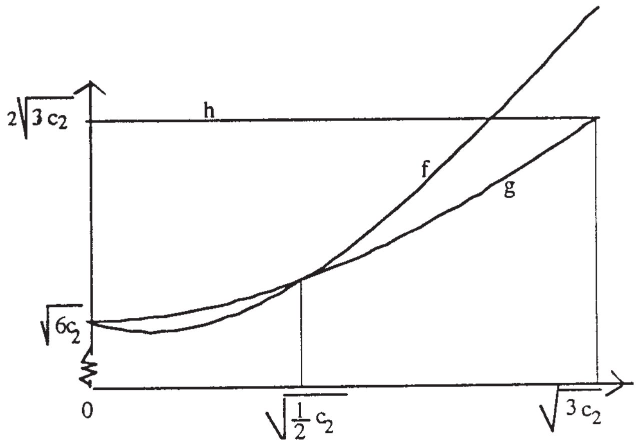

Fig. 1. – Pictures of the functions above with $c_1 = 0$ and fixed $c_2 \gg 0$.

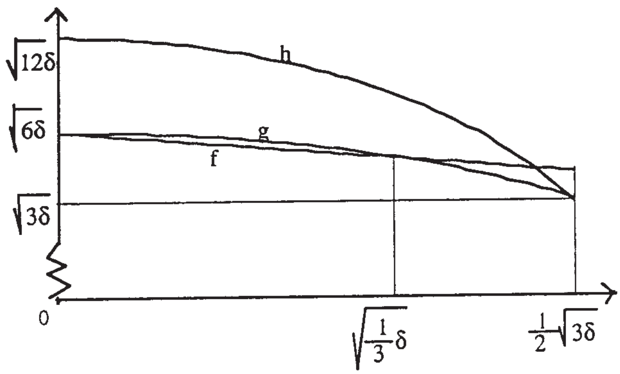

Fig. 2. – Pictures of the functions above with $c_1 = 0$ and fixed $\delta \gg 0$.

2. – Reduction steps.

It is known that an unstable surface for the reflexive sheaf F gives rise to a reduction step (see [4] and [9]). Here we prove three useful lemmas concerning reduction steps.

LEMMA 2.1. – *Let F be a semistable rank 2 reflexive sheaf, $x \neq 0$ an element of $H^2 F(t)^*$, f in $H^0 \mathcal{O}(d)$ an annihilator of minimal degree for x. If $r = t + 4 - d > 0$, then we have*:

(i) *X (the surface defined by $f = 0$) is an unstable surface for F and there is an exact sequence*

$$0 \to G(-\varepsilon) \to F \to I_{Z,X}(-r) \to 0 \tag{1}$$

where G is a normalized rank 2 reflexive sheaf with Chern classes c_1', c_2', c_3',

Z has codimension 1 *in X, k is the degree of the curve part of Z, and moreover*:

$$2\varepsilon = d - c_1 + c_1' ,$$

$$c_2' = c_2 - d(t+4+c_1-d) - \frac{1}{4}(d-c_1)^2 - \frac{1}{4}c_1' - k < c_2 ,$$

$$2\alpha + c_1 - d \leqslant 2\alpha' + c_1' \leqslant 2\alpha + c_1 + d \quad \textit{where} \quad \alpha' = \alpha(G) ,$$

$$2\beta + c_1 - d \leqslant 2\beta' + c_1' \quad \textit{where} \quad \beta' = \beta(G) .$$

(ii) *If $\alpha > \varepsilon$, then G is semistable.*

(iii) *If $t+4 > a+d$ then $a' = a(G) = a - \varepsilon$.*

(iv) *If $t+4 > \alpha + d$ then $\alpha' = \alpha - \varepsilon$, $\beta' \geqslant \beta - \varepsilon$ and*

$$\delta' = \delta(G) = \delta - d(c_1 + r + \alpha) - k < \delta .$$

PROOF. – It depends upon [9], Proposition 1.2 and Remark 1.3 except:

$$2\alpha + c_1 - d \leqslant 2\alpha' + c_1' \leqslant 2\alpha + c_1 + d \quad \text{and}$$

$$2\beta + c_1 - d \leqslant 2\beta' + c_1' \ \text{ when } \ 2\alpha' + c_1' \neq 2\alpha + c_1 - d$$

which are proved as follows.

Assume that $\alpha' < \alpha - \varepsilon$; then the cohomology sequence of (1) gives

$$0 \to H^0 G(\alpha') \to H^0 F(\alpha' + \varepsilon) = 0$$

which is absurd. Let now σ be a non-zero section of $H^0 F(\alpha)$ with image g in $H^0 I_{Z,X}(\alpha - r)$; then $fg = 0$, hence $f\sigma \neq 0$ is image of a non-zero element in $H^0 G(\alpha - \varepsilon + d)$. Therefore $\alpha' \leqslant \alpha - \varepsilon + d$, which implies the second claim. Assume now that $\beta' < \beta - \varepsilon$. If $s \in H^0 G(\alpha')$, $s' \in H^0 G(\beta')$ are indipendent sections as in lemma 1.1 of [8], then, if i is the canonical embedding $H^0 G(n-\varepsilon) \to H^0 F(n)$, $i(s)\, H^0 \mathcal{O}_{\mathbf{P}^3}(\beta' - \alpha')$ and $i(s')$ belong to $H^0 F(\beta' + \varepsilon) = \sigma H^0 \mathcal{O}_{\mathbf{P}^3}(\beta' + \varepsilon - \alpha)$, σ being a non-zero element of $H^0 F(\alpha)$. Then we have: $i(s) = \sigma h_1$, $i(s') = \sigma h_2$, hence $i(h_2 s) = i(h_1 s')$, which means $h_2 s = h_1 s'$, so contradicting Lemma 1.1 of [8].

LEMMA 2.2. – *Let F be a semistable rank* 2 *normalized reflexive sheaf and let t be a positive integer such that $H^2 F(t) \neq 0$. Then*

(a) *there is x in $H^2 F(t)^*$ whose image x_H in $H^1 F_H(-t - c_1 - 4)$ is $\neq 0$, H being a general plane*;

(b) *x being as in* (a), *let d be the smallest integer such that x is annihiled by an element of $H^0 \mathcal{O}_H(d)$ for all H. If moreover*

(i) $t + 3 > a + d$,

(ii) $t + 4 > 2d - a - c_1$,

then x has only one (*up to a unit*) *annihilator f in $H^0 \mathcal{O}_{\mathbf{P}^3}(d)$.*

PROOF. – It depends upon [4] and [9].

REMARK 2.3. – If $d \leqslant 2\alpha + c_1 + 1$ (for instance when $a < \alpha$), then (i) implies (ii).

LEMMA 2.4. – *Let F be a rank 2 semistable reflexive sheaf, H a general plane, $x_H \neq 0$ an element of $H^1 F_H(-t-c_1-4)$, f_H in $H^0 \mathcal{O}_H(d)$ an annihilator of minimal degree for x_H. If $t+3 > a+d$, then $d \leqslant \sqrt{2c_2 - 2a^2}$; if moreover $d \leqslant 2\alpha + c_1 + 1$ (for instance when $a < \alpha$), then $d \leqslant \sqrt{\frac{4}{3} c_2}$.*

PROOF. – First we observe that $-t-c_1-4+d-1 \leqslant -a-c_1-2$; then by Riemann-Roch we have $h^1 F_H(-t-c_1-4+d-1) \leqslant h^1 F_H(-a-c_1-2) \leqslant c_2 - a^2$, since $h^1 F_H(n)$ is increasing for $n \leqslant -2$. Moreover by the same definition of d, we have also $\frac{1}{2} d(d+1) \leqslant h^1 F_H(-t-c_1-4+d-1)$. By comparison we obtain that $d \leqslant \sqrt{2c_2 - 2a^2 + \frac{1}{4}} - \frac{1}{2} \leqslant \sqrt{2c_2 - 2a^2}$.

If moreover $d \leqslant 2\alpha + c_1 + 1$ (see also [4], Proposition 4.1 for the case $a < \alpha$), then $d \leqslant \min\left(\sqrt{2c_2 - 2a^2 + \frac{1}{4}} - \frac{1}{2},\ 2\alpha + c_1 + 1\right) \leqslant \sqrt{\frac{4}{3} c_2}$.

LEMMA 2.5. – *Let F be a semistable rank 2 normalized reflexive sheaf having $c_2 \geqslant 2$, t an integer such that $H^2 F(t) \neq 0$ and d as in Lemma 2.1. If t fulfils (a) or (b) or (c) of Notations 1.2, then $t+3 > a+d$ and $t+4+c_1 > 2d - a$.*

PROOF. – Choose x in $H^2 F(t)^*$ and its image x_H in $H^1 F(-t-c_1-4)$ as in Lemma 2.2, (a), and let d be the smallest degree of an annihilator of x_H. The inequality $t+3 > a+d$ follows from [4], Proposition 4.3, provided we show that

$$t + \alpha + c_1 + 2 \geqslant \sqrt{3c_2 + 1 + \frac{3}{4} c_1} + \frac{1}{2}(2\alpha + c_1).$$

Case 1: t fulfils (a) of Notations 1.2.

We must show that $\sqrt{6\delta + 1} \geqslant \sqrt{3c_2 + 1 + \frac{3}{4} c_1} + \frac{3}{4}(2\alpha + c_1)$. In fact, squaring both sides and simplifying we obtain:

$$\left(\sqrt{3c_2 + 1 + \frac{3}{4} c_1} - \frac{3}{4}(2\alpha + c_1)\right)^2 + \frac{3}{8}(2\alpha + c_1)^2 \geqslant 1\,.$$

The inequality holds if $2\alpha + c_1 \geqslant 2$ or $= 0$. If $2\alpha + c_1 = 1$, it depends upon the inequality $c_2 \geqslant \frac{1}{16}(1 + 2\sqrt{10} - 4c_1)$, always true because $c_2 \geqslant 2$.

Case 2: t fulfils (b) of Notations 1.2. First we see that

$$\sqrt{6\delta - (2\alpha + c_1)^2 + 1} \geqslant \sqrt{3c_2 + 1 + \frac{3}{4} c_1} + \frac{1}{2}(2\alpha + c_1).$$

In fact, squaring and simplifying, we obtain: $\left(\sqrt{3c_2+1+\frac{3}{4}c_1}-\frac{1}{2}(2\alpha+c_1)\right)^2 \geqslant 1$, true because of Theorem 1.1.

Case 3: t fulfils (c) of Notations 1.2. The claim follows from Remark 1.3 and case 2.

COROLLARY 2.6. – *Under the assumptions of Lemma* 2.5, *there exists a reduction step for* F *as described in Lemma* 2.1.

PROOF. – It depends upon Lemma 2.5, Lemma 2.1 and Lemma 2.2.

3. – Upper bounds for the second section of a reflexive sheaf.

In this section we discuss and prove our upper bounds for the level β of the second section of a **semistable** reflexive sheaf F.

THEOREM 3.1. – *Let* F *be a semistable rank 2 normalized reflexive sheaf having* $c_2 \geqslant 2$ *and* t *an integer as in Notations* 1.2. *If* $H^2F(t)=0$, *then* $t \geqslant \beta$.

PROOF. – Riemann-Roch gives:

$$h^0F(t) = 2\binom{t+3}{3} - c_1\binom{t+2}{2} - c_2(t+2) + \frac{1}{2}(c_3 - c_1c_2) + h^1F(t).$$

The inequality $t \geqslant \beta$ holds if $h^0F(t) > h^0\mathcal{O}_{\mathbf{P}^3}(t-\alpha)$. So it is enough to prove that the polynomial function

$$P(X) = X^3 - (6\delta+1)X + 3\delta(2\alpha+c_1)$$

is positive when $X = t+\alpha+c_1+2$. Since $P(X)$ is increasing for $X \geqslant \sqrt{\frac{1}{3}(6\delta+1)}$, it is enough to show that the right hand sides of Notations 1.2, (a), (b), (c) are not smaller than $\sqrt{\frac{1}{3}(6\delta+1)}$ and that $P(X) \geqslant 0$ whenever $X =$ any such right hand side.

Case 1: t fulfils (a). Then it depends upon [9, Lemma 4.1, $b) \Rightarrow a)$].

Case 2: t fulfils (b). Let us prove that $\sqrt{6\delta-(2\alpha+c_1)^2+1} \geqslant \sqrt{\frac{1}{3}(6\delta+1)}$; squaring both sides and simplifying, we obtain $4c_2 + \frac{2}{3} + c_1 \geqslant 0$, which is always true. Now we compute

$$P\left(\sqrt{6\delta-(2\alpha+c_1)^2+1}\right) = (2\alpha+c_1)\left(3\delta-(2\alpha+c_1)\sqrt{6\delta-(2\alpha+c_1)^2+1}\right);$$

since $2\alpha+c_1 \geqslant 0$, it is enough to prove that $3\delta \geqslant (2\alpha+c_1)\sqrt{6\delta-(2\alpha+c_1)^2+1}$.

Squaring and simplifying, we obtain:

$$\left(\left(3c_2+1+\frac{3}{4}\,c_1\right)-\frac{1}{4}\,(2\alpha+c_1)^2-1\right)^2 \geqslant (2\alpha+c_1)^2\,,$$

true by Theorem 1.1.

Case 3: t fulfils (c). The claim follows from Case 2 and Remark 1.3.

THEOREM 3.2. – *Let F be a semistable rank 2 normalized reflexive sheaf having $c_2 \geqslant 2$, t an integer as in Notations 1.2. If $H^2F(t) \neq 0$, then $t \geqslant \beta$.*

PROOF. – We proceed by induction on c_2 (the case $c_2 = 1$ has been proved in section 1). We assume that the statement is true for every rank 2 semistable normalized reflexive sheaf with second Chern class $< c_2$. By Corollary 2.6 there is a reduction step giving a new sheaf G with $c_2' = c_2(G) < c_2$ (Lemma 2.1). We will prove that G is semistable and $t' = t - \varepsilon$ fulfils at least one among the conditions of Notation 1.2, relatively to G. This will imply the claim for F, because the claim is true for G and moreover (Lemma 2.1) $t = t' + \varepsilon \geqslant \beta' + \varepsilon \geqslant \beta$.

We will see that:

– condition (a) for F and t implies condition (a) for G and t', provided that either $a = \alpha$ or that $a < \alpha < \sqrt{\frac{2}{3}\,c_2} - \frac{1}{2}\,c_1$;

– condition (b) for F and t implies condition (b) for G and t', provided that $\alpha \geqslant \sqrt{\frac{1}{3}\,c_2} - \frac{1}{2}\,c_1$;

Having in mind remark 1.3, we see that this is enough to cover all cases.

Step 1: F and t fulfil condition (a), i.e. $t + \alpha + c_1 + 2 > \sqrt{6\delta+1} - \frac{1}{4}\,(2\alpha + c_1)$. If $a = \alpha$, the claim is proved in [9]; hence we assume $a < \alpha$ and then $d < 2\alpha + c_1$. Moreover we assume $\alpha < \sqrt{\frac{2}{3}\,c_2} - \frac{1}{2}\,c_1$. We want to show that G and t' fulfil (a) of Notation 1.2 that is: $t' + \alpha' + c_1' + 2 > \sqrt{6\delta'+1} - \frac{1}{4}\,(2\alpha' + c_1')$. First of all we observe that $t + 4 > \alpha + d$. In fact $t + \alpha + c_1 + 2 > \sqrt{6\delta+1} - \frac{1}{4}\,(2\alpha + c_1) \geqslant 2\alpha + c_1 + d$; it is now enough to square and recall that $2\alpha + c_1 \leqslant 2\sqrt{\frac{2}{3}\,c_2}$ that is $c_2 > \frac{3}{8}\,(2\alpha + c_1)^2$ and $d \leqslant (2\alpha + c_1)$ (Lemma 2.4). Therefore $\alpha' > a'$ and so G is semistable. Using Lemma 2.1 (iv), it is enough to show that:

$$t' + \alpha' + c_1' + 2 = t + \alpha + c_1 + 2 - d > \sqrt{6\delta+1} - \frac{1}{4}\,(2\alpha + c_1) - d \geqslant \sqrt{6\delta'+1} - \frac{1}{4}\,(2\alpha' + c_1')\,,$$

i.e. that $\sqrt{6\delta+1} - \frac{5}{4}\,d \geqslant \sqrt{6\delta - 6(t + 4 + \alpha + c_1 - d)\,d + 1}$.

Squaring and simplifying, we see that it is enough to prove that: $6(t+4+c_1+\alpha-d)+\frac{25}{16}d-\frac{5}{2}\sqrt{6\delta+1}=\frac{5}{2}(t+\alpha+c_1+2)-\frac{5}{2}(d-2)+\frac{7}{2}(t+4+c_1+\alpha-d)+\frac{25}{16}d-\frac{5}{2}\sqrt{6\delta+1}\geqslant 0$. Since t fulfils (a) and $t+4>\alpha+d$, it is enough to show that $\frac{23}{8}(2\alpha+c_1)-\frac{15}{8}d+10\geqslant 0$, true because $d<2\alpha+c_1$.

Step 2: F and t fulfil condition (b), i.e.

$$t+\alpha+c_1+2>\sqrt{6\delta-(2\alpha+c_1)^2+1}=\sqrt{6c_2+\tfrac{1}{2}(2\alpha+c_1)^2+\tfrac{3}{2}c_1+1}\,.$$

Since the case $\alpha<\sqrt{\frac{2}{3}c_2}-\frac{1}{2}c_1$ follows from step 1 (see Remark 1.3), we assume that $\alpha\geqslant\sqrt{\frac{1}{3}c_2}-\frac{1}{2}c_1$ and so $d\leqslant\sqrt{\frac{4}{3}c_2}$ (see Lemma 2.4 and Lemma 2.5). If $a<\alpha$ we have $d\leqslant 2\alpha+c_1-1$ and therefore $\alpha'\geqslant\alpha-\varepsilon>0$, while, if $a=\alpha$, then $\alpha'\geqslant\alpha-\frac{1}{2}d\geqslant\sqrt{\frac{1}{3}c_2}-\frac{1}{2}c_1-\frac{1}{2}\sqrt{\frac{4}{3}c_2}\geqslant 0$; hence G is semistable.

First of all we remark that $t+4>d+\frac{1}{3}\alpha$, i.e. that $\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}-(d+\frac{4}{3}\alpha+c_1-2)>0$. The left side of the last inequality, as a function of $2\alpha+c_1$, is decreasing, when $2\alpha+c_1\leqslant 4\sqrt{6c_2+\frac{3}{2}}$, i.e. for all values of α allowed by Theorem 1.1, and moreover, when $2\alpha+c_1=2\sqrt{3c_2}$, it has the value $\sqrt{12c_2+\frac{3}{2}c_1+1}-d-\frac{4}{3}\sqrt{3c_2}-\frac{1}{3}c_1+2>0$ (recall that $d\leqslant\sqrt{\frac{4}{3}c_2}$ and $c_2\geqslant 2$).

Now we prove our claim under the following condition:

$$(*)\qquad 6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1\leqslant(2\alpha+c_1)^2+\frac{2}{3}d(2\alpha+c_1)+2d^2-1$$

In this event we have: $d\geqslant 2$ (Theorem 1.1). Let us prove that G and $t'=t-\varepsilon$ fulfil (c), i.e. that

$$t'+\alpha'+c_1'+3=t+\alpha+c_1+3-d\geqslant$$

$$\geqslant\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}-d\geqslant 2\sqrt{3c_2+\frac{3}{4}c_1+1}\,.$$

In fact, squaring, simplifying and using Lemma 2.1, we see that it is enough to prove that

$$6c_2-12d(t+4+c_1-d)-3d^2+6dc_1+3c_1+3\leqslant$$

$$\leqslant\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+d^2-2d\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+}\,.$$

Using (b), we see that it is enough to show that

$$6c_2-10d(t+4+c_1-d)+2d(\alpha+d-2)+3-4d^2+6dc_1+\frac{3}{2}c_1\leqslant\frac{1}{2}(2\alpha+c_1)^2\,.$$

Using the inequality $t+4>d+\frac{1}{3}\alpha$ it is enough to show that

$$6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1\leqslant(2\alpha+c_1)^2+2d(2\alpha+c_1)+2d^2-2+\frac{10}{3}c_1+4d\,,$$

which follows from $(*)$ since $d\geqslant 2$.

We now assume that

$$(**)\qquad 6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1>(2\alpha+c_1)^2+\frac{2}{3}d(2\alpha+c_1)+2d^2-1$$

and $2\alpha'+c_1'<2\alpha+c_1$ hold. In this event we show that G and t' fulfil (b), i.e. that

$$t'+\alpha'+c_1'+2=t+\alpha+c_1+2-d\geqslant$$

$$\geqslant\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}-d\geqslant\sqrt{6c_2'+\frac{1}{2}(2\alpha'+c_1()^2+\frac{3}{2}c_1'+1}\,.$$

Squaring and using Lemma 2.1, we see that it is enough to show that

$$d\left(6d(t+c_1+4-d)-2\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}+\frac{5}{2}d^2-3dc_1\right)\geqslant 0$$

or, using assumption (b) on t:

$$4\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}\geqslant\frac{7}{2}d+3(2\alpha+c_1)-12\,.$$

Squaring both sides and using $(**)$, we obtain:

$$16(2\alpha+c_1)^2+\frac{32}{3}(2\alpha+c_1)\,d+32d^2-16\geqslant$$

$$\frac{49}{3}d^2+21(2\alpha+c_1)\,d+9(2\alpha+c_1)^2+144-84d-72(2\alpha+c_1)$$

i.e. $7(2\alpha+c_1)^2-\frac{31}{3}(2\alpha+c_1)\,d+\frac{79}{4}d^2\geqslant 160-84d-72(2\alpha+c_1)$, which is true because the left side is positive and the right side is negative, except when $d=2\alpha+c_1=1$, in which case we have $7-\frac{31}{3}+\frac{79}{4}\geqslant 4$.

We now assume that $(**)$ and $2\alpha'+c_1'\geqslant 2\alpha+c_1$ hold. In this event we show that G and t' fulfil (b) i.e. that

$$t'+\alpha'+c_1'+2\geqslant t+\alpha+c_1+2-\frac{1}{2}d\geqslant$$

$$\sqrt{6c_2+\frac{1}{2}(2\alpha+c_1)^2+\frac{3}{2}c_1+1}-\frac{1}{2}d\geqslant\sqrt{6c_2'+\frac{1}{2}(2\alpha'+c_1)^2+\frac{3}{2}c_1'+1}\,.$$

Squaring and using Lemma 2.1, (in particular the fact that $2\alpha'+c_1'\leqslant 2\alpha+$

$c_1 + d)$, we see that it is enough to show that

$$6d(t+4) - d(2\alpha + c_1) + 3c_1 d - \frac{19}{4} d^2 > d\sqrt{6c_2 + \frac{1}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1}\,.$$

Using (b) on t, it is enough to show that

$$5\sqrt{6c_2 + \frac{1}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1} > \frac{19}{4} + 4(2\alpha + c_1)\,.$$

Squaring and using $(**)$, we see that it is enough to show that

$$9(2\alpha + c_1)^2 - \frac{64}{3} d(2\alpha + c_1) + \frac{439}{16} d^2 > 0\,,$$

which is always true.

THEOREM 3.3. – *Let F be a normalized rank 2 reflexive sheaf. Then*:

$$\beta + \alpha + c_1 \leqslant \sqrt{6c_2 + \frac{3(2\alpha + c_1)^2}{2} + \frac{3}{2}c_1 + 1} - \frac{(2\alpha + c_1)}{4} - 1\,,$$

$$\beta + \alpha + c_1 \leqslant \sqrt{6c_2 + \frac{(2\alpha + c_1)^2}{2} + \frac{3}{2}c_1 + 1} - 1 \quad \textit{if } F \textit{ is semistable}\,,$$

$$\beta + \alpha + c_1 \leqslant 2\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 2 \quad \textit{if } F \textit{ is semistable}\,.$$

PROOF. – If F is semistable, the claim follows from Remark 1.5, Theorem 3.1 and Theorem 3.2. The non-stable case is proved in [9], Remark 4.4. Observe that, for t and β integers, $t > x \Rightarrow t \geqslant \beta$ is the same as $\beta \leqslant x + 1$.

COROLLARY 3.4. – *Let F be a semistable reflexive sheaf and let s be a general section in $F(n)$. Then, whenever $n \geqslant \sqrt{6c_2 + c_1 + 1} - 1$, the scheme of zeros of s is a curve.*

PROOF. – Consider, for every positive c_2 and $c_1 = 0$ or -1, the function $g'(\alpha) = \sqrt{6c_2 + \frac{1}{2}(2\alpha + c_1)^2 + \frac{3}{2}c_1 + 1} - \alpha - c_1 - 1$; it is easy to see, by a straighforward computation, that it reaches its highest value, within the interval $-c_1 \leqslant \alpha \leqslant \sqrt{3c_2 + 1 + \frac{3}{4}c_1} - \frac{1}{2}c_1 - 1$ (see Theorem 1.1), when $\alpha = -c_1$, and such a value is exactly $\sqrt{6c_2 + c_1 + 1} - 1$. Then it follows from Theorem 3.3 that $\beta \leqslant \sqrt{6c_2 + c_1 + 1} - 1$ and for $n \geqslant \beta$ the general section of $F(n)$ gives rise to a curve (see [8], § 1, n. 2).

REMARK 3.5. – We observe that, if F is non-stable, then such a result does not exist. Indeed for every funtion $\phi(c_2)$ there are sheaves having $\beta > -\alpha > \phi(c_2)$: see [9], Example 5.12.

4. – Bounds for the degree of a surface containing a curve.

Theorem 3.3 can be traslated into the language of curves as follows.

THEOREM 4.1. – *Let r be the smallest degree of a surface containing a minimal curve C of the normalized semistable rank 2 reflexive sheaf F. Then* $r \leqslant 2\sqrt{3c_2 + 1 + \frac{3}{4}c_1} - 2$.

DEFINITION 4.2. – *Let C be a curve. Define*:

$$e = \max\{n/\omega_C(-n) \text{ has a section generating it almost everywhere}\}.$$

THEOREM 4.3. – *Let C be a curve of degree D lying on a surface of degree r and not less.*

If $\frac{1}{2}(e+4) \geqslant \sqrt{D}$, $r \leqslant \sqrt{D}$; *otherwise* $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{4}(e+4)$.

If moreover $0 < \frac{1}{2}(e+4) < \sqrt{D}$, *then*

$$r \leqslant \min\left(\sqrt{6D+1} - 1 - \frac{1}{4}(e+4), \sqrt{6D-(e+4)^2+1} - 1\right).$$

PROOF. – Let $F(n)$ be the sheaf corresponding to C through a section of $\omega_C(-e)$, where F is normalized: then $e + 4 = 2n + c_1$ and $D = c_2 + nc_1 + n^2$. From the assumption $\frac{1}{2}(e+4) \geqslant \sqrt{D}$ it follows that $n > 0$ and $4c_2 + c_1 \leqslant 0$ (it is enough to square both members), which means that F is non-stable and the result follows from [9], Proposition 5.3.

If $\frac{1}{2}(e+4) < \sqrt{D}$ and $n = \alpha$, then the claim follows from Theorem 3.3 above for the stable case and from [9], Remark 4.4 for the non-stable case.

Finally, if $0 < \frac{1}{2}(e+4) < \sqrt{D}$ and $n = \alpha > 0$, then the claim follows from Theorem 3.3 above; if $n > \alpha$ from the assumption it follows that

$$\min\left(\sqrt{6D+1} - \frac{1}{4}(e+4), \sqrt{6D-(e+4)^2+1} - 1\right) - 1 \geqslant$$

$$\min\left(\sqrt{6D+1} - \frac{1}{2}\sqrt{D}, \sqrt{2D+1}\right) - 1 \geqslant \sqrt{3D+1} - 1 \geqslant \sqrt{D}$$

(direct computation).

Then the claim follows from [9, Proposition 5.3 in the non-stable case, Proposition 5.8 in the stable case: if $n > \alpha$, then $r \leqslant \sqrt{3D+1} - 1$, which is a statement slightly stronger than 5.8].

REMARK 4.4. – Let C be an integral curve (so having $e \geqslant 0$) and assume that C is not contained in any surface of degree e. Then the claim

$$t > \sqrt{6D+1} - 2 - \frac{1}{4}(e+4) \quad \text{and} \quad t > e \Rightarrow t \geqslant r$$

follows (easily) from Riemann-Roch. In fact for $t > e$ we have $h^0 \omega_C(-t) = 0$ (not true for a non-integral curve); moreover $h^0 I_C(t) \geqslant h^0 \mathcal{O}_{\boldsymbol{P}^3}(t) - h^0 \mathcal{O}_C(t)$ and $p_a = 1 + \frac{1}{2} eD + \frac{1}{2} c_3$, hence $h^0 I_C(t) \geqslant \binom{t+3}{3} - h^0 \mathcal{O}_C(t) = \binom{t+3}{3} + \frac{1}{2} eD + \frac{1}{2} c_3 - tD \geqslant \binom{t+3}{3} + \frac{1}{2}(e+4)D - (t+2)D$. This last term becomes $\geqslant 0$ if we replace t with $\sqrt{6D+1} - 2 - \frac{1}{4}(e+4)$, provided that $e+4 > 0$.

REMARK 4.5. – The upper bound of Theorem 3.3 is lower than $\sqrt{6D-2} - 3$, D being the degree of the curve C, except when $\alpha = 1$ and $c_1 = -1$.

REMARK 4.6. – Theorem 4.3 can be used to obtain information on the number e, provided that both r and D are known. For the sake of simplicity, assume that $c_1 = 0$. Then we have:

- if $r > \sqrt{D}$, then $e + 4 < 2\sqrt{D}$,
- if $r \leqslant \sqrt{D}$, then $e + 4 < D + 1$.

In fact, if $r \leqslant \sqrt{D}$ and moreover it is known that C corresponds to a semistable sheaf F, then Theorem 4.3 says that

$$0 \leqslant e + 4 \leqslant \min\left(4\sqrt{6D+1} - 4(\sqrt{D}+1),\ \sqrt{5D - 2\sqrt{D}}\right).$$

If, on the contrary, $r \leqslant \sqrt{D}$ but C is the scheme of zeros of the first section of a non-stable F, then $e + 4 < 0$; if $r \leqslant \sqrt{D}$ and C is the scheme of zeros of a section of $F(n)$, where F is non-stable and $n \geqslant \beta$, then $e + 4 = 2n$ and $c_2 + n^2 = c_2 + n^2 + \alpha^2 - \alpha^2 \geqslant c_2 + \alpha^2 + (n-\alpha)(n+\alpha) \geqslant 2n$, true whenever $n + \alpha$ is at least 2, i.e. if C is not a plane curve. If C is a plane curve, then of course $e + 4 = D + 1$. So in any event $D + 1$ is an upper bound for $e + 4$.

Observe that, if C corresponds to the first section of a non-stable F, e can be arbitrarily negative, even when both r and D are given (see [1, 3, (iii)].

REMARK 4.7. – Observe that, if $\frac{1}{2}(e+4) \geqslant \sqrt{D}$, then the curve is not minimal for a non-stable sheaf F. For such a curve the other bound $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{4}(e+4)$ is not valid, also because it may be a negative number: see [9], Remark 5.5.

5. – Examples.

EXAMPLE 5.1. – Let F be a normalized rank 2 reflexive sheaf with $c_3 = 0$, $c_2 \geqslant 1$, whose cohomology is seminatural ([5, Theorem 2.3]. Then it can easily be seen that $\alpha = E(\sqrt{3c_2 + \frac{3}{4}c_1 + 1} - \frac{1}{2}c_1 - 1)$. Therefore we have:

$$h^0 F(\alpha) = \frac{1}{3}(\alpha+1)(\alpha+2)(\alpha+3) - (\alpha+2)\,c_2 + \frac{1}{2}(\alpha+1)(\alpha+2)\,c_1 > 1$$

and $\alpha = \beta$ (by the main theorem of [10]). So the upper bound of Theorem 3.3 is sharp for all $c_2 \geqslant 1$.

It easy to see that in this case the upper bound $\sqrt{6D - (e+4)^2 + 1} - 1$ is reached.

EXAMPLE 5.2. – Let F be a seminatural cohomology reflexive sheaf with $c_1 = c_3 = 0$, $c_2 = 5$; then $\alpha = \beta = 3$, hence a curve C scheme of zeros of a section of $F(3)$ lies on a surface of degree 6 and not less and has degree $D = 14$. Let now Y be the skew union of C and a general complete intersection of a plane surface of degree 5. It is easy to see that Y lies on a surface of degree 7 and not less and we have: $7 = E(\sqrt{6(14+5) - 36 + 1}) - 1$.

EXAMPLE 5.3. – $C =$ the skew union of $D > 2$ lines having maximal rank (true for the general union by [7]) gives a sharp example.

In fact C is the minimal curve of a reflexive sheaf F with $c_1 = 0$, $\alpha = 1$. Since C is (-2)-subcanonical of genus $1 - D$ and $h^0\mathcal{O}_C(n) \geqslant (n+1)D$, then $r > n$, where n is the largest integer such that $h^0\mathcal{O}_{\boldsymbol{P}^3}(n) \leqslant (n+1)\,D$, that is the integral part of $-\frac{5}{2} + \sqrt{6D + \frac{1}{4}}$. On the other hand $r \leqslant \sqrt{6D+1} - 1 - \frac{1}{2}$. Choosing $D = 10$ (or other infinitely many values of D), we get a sharp example.

EXAMPLE 5.4. – Let C be the skew union of $h \geqslant 2$ complete intersections of type (n, n). Then we have:

$$r \leqslant \min\left(\sqrt{6D+1} - 1 - \frac{1}{4}(e+4),\ \sqrt{6D - (e+4)^2 + 1} - 1\right) =$$

$$\min\left(\sqrt{6hn^2+1} - 1 - \frac{1}{2}\,n,\ \sqrt{6hn^2 - 4n^2 + 1} - 1\right)$$

and from $h \geqslant 5$ this is a better bound than the trivial one, i.e. hn.

EXAMPLE 5.5. – Let C be the skew union of $h \geqslant 2n$ complete intersection of type $(1, 2n-1)$.

Then we have:

$$r \leqslant \min\left(\sqrt{6D+1} - 1 - \frac{1}{4}\,(e+4),\ \sqrt{6D-(e+4)^2+1} - 1\right)$$

and this usually is a better bound than the trivial one, i.e. h.

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