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SEZIONE SCIENTIFICA

BREVI NOTE

A note on a global existence result of R. Conti

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Summary. - *If* $V(t, x) \to \infty$ *as* $|x| \to \infty$ *for each fixed* t, *and if* $\dot{V}(t, x) \leq 0$, *then all solutions of* $x' = f(t, x)$ *exist in the future. This corrects a previous result due to R. Conti.*

1. - Main results.

Let $|\cdot|$ denote any norm in Euclidean n-space R^n and let R denote R^1. Consider the ordinary differential equation

$$(E) \qquad x' = f(t, x) \qquad (' = d/dt)$$

where $f: R \times R^n \to R^n$ is continuous. Let (E) have uniqueness, i.e., for each (t_0, x_0) in $R \times R^n$ there is a unique solution $x(t; t_0, x_0)$ of (E), defined in a neighborhood of t_0, such that $x(t_0; t_0, x_0) = x_0$.

Let I be an interval, possibly unbounded, and let $V: I \times R^n \to R$. We say that V is *locally Lipschitz* if it is continuous on $I \times R^n$ and if for each (t, x) in $I \times R^n$ there is a neighborhood N of (t, x) and a constant $k > 0$ such that

$$|V(s, y) - V(s, z)| \leq k\,|y - z|$$

for all (s, y) and (s, z) in $N \cap (I \times R^n)$. Let

$$J = I - \sup I.$$

Define $\dot{V}: J \times R^n \to R$ by

$$\dot{V}(t, x) = \limsup_{h \to 0+} h^{-1}(V(t + h, x + hf(t, x)) - V(t, x)).$$

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If V is locally LIPSCHITZ on $I \times R^n$, then

$$\dot{V}(t, x) = \limsup_{h \to 0+} h^{-1}(V(t+h,\ x(t+h;\ t,\ x)) - V(t,\ x))$$

on the set $J \times R^n$, as is proved in [6, p. 3].

CONTI [1] has stated the following result.

THEOREM 1. - *Let* $V: R \times R^n \to R$ *be locally Lipschitz. Let*

(1) $$V(t,\ x) \to \infty \text{ as } |x| \to \infty \text{ for each fixed } t,$$

and

(2) $$\dot{V}(t,\ x) \leq \varphi(t,\ V(t,\ x)).$$

Suppose $\varphi: R \times R \to R$ *is continuous and for every real* t_0 *and* r_0 *the maximal solution* $r(t;\ t_0.\ r_0)$ *of the comparison equation*

(*CE*) $$r' = \varphi(t,\ r)$$

exists in the future (exists for all $t \geq t_0$*). Then every solution of* (*E*) *exists in the future.*

In the proof, CONTI showed that if, in fact, some solution $x(\cdot)$ of (E) fails to exist in the future, then

(3) $$|x(t)| \to \infty \text{ as } t \to \omega^-$$

for some finite ω; hence

(4) $$V(t,\ x(t)) \to \infty$$

as $t \to \omega^-$. Since $V(t,\ x(t))$ is a solution of

$$r' \leq \varphi(t,\ r),$$

it follows that

$$V(t,\ x(t)) \leq r(t;\ t_0,\ V(t_0,\ x(t_0))).$$

Therefore, from (4), $r(t;\ t_0,\ V(t_0,\ x(t_0))) \to \infty$ as $t \to \omega^-$, a contradiction to the existence assumption on (CE).

The flaw in this argument is that (3) does not immediately imply (4). This implication is immediate, however, if (1) is replaced by

(1*) $$V(t,\ x) \to \infty \text{ as } |x| \to \infty$$

uniformly in t for t in any compact set.

Indeed, Theorem 1 was proved under this stronger hypothesis, using the above argument, by LaSalle and Lefschetz [4, p. 108]. Furthermore, Kato and Strauss [3] have shown that if all solutions of (E) exist in the future, then there exists a locally Lipschitz $V: R \times R^n \to R$ satisfying (1*) and (2). Thus there would seem to exist a natural correspondence between existence in the future on one hand and locally Lipschitz V satisfying (1*) and (2) on the other.

The purpose of this note is threefold: first, to prove that Conti's result (Theorem 1) is true; then, to show by an example that a particular V satisfying (1) and (2) need not satisfy (1*), so that Kato and Strauss' result shows *some other* V must satisfy (1*) and (2); and finally, to prove the following theorem, thereby establishing conditions under which (1) and (1*) are equivalent.

Theorem 2. – *Let* $V: R \times R^n \to R$ *be locally Lipschitz, let* (1) *hold, and let*

$$\psi(t, V(t, x)) \leq \dot{V}(t, x) \leq \varphi(t, V(t, x)). \tag{5}$$

Suppose φ *and* ψ *are continuous, all maximal solutions of*

$$r' = \varphi(t, r)$$

exist in the future, and all maximal solutions of

$$r' = \psi(t, r)$$

exist in the past. Then all solutions of (E) *exist forever* (*past and future*) *and* V *satisfies* (1*).

Remark. – Kato and Strauss [3] proved a converse to Theorem 2, namely, if all solutions of (E) exist forever, then there exists a locally Lipschitz V satisfying (1*) and (5). Thus the natural correspondence mentioned earlier is not quite right. The natural correspondences seem to be between existence in the future on one hand and locally Lipschitz V satisfying (1) and (2) on the other, and between existence forever on one hand and locally Lipschitz V satisfying (1*) and (5) on the other.

2. – Proofs.

Proof of Theorem 1. – Suppose the result does not hold, so that some solution $x(\cdot)$ of (E) fails to exist in the future. This means that

$$|x(t)| \to \infty \text{ as } t \to \omega^-,$$

for some ω. Choose $t_0 < \omega$ sufficiently close to ω that some solution $\tilde{x}(\cdot)$ exists on $[t_0, \omega]$. Let L be the line segment

$$L = \{ z_\lambda = \lambda \tilde{x}(t_0) + (1-\lambda)x(t_0) : 0 \leq \lambda \leq 1 \},$$

and let

$$\lambda_* = \inf \{ \lambda : x(\omega; t_0, z_\mu) \text{ is finite for } \lambda < \mu \leq 1 \}.$$

From $z_1 = \tilde{x}(t_0)$ and continuous dependence, we see that $0 \leq \lambda_* < 1$.

We claim that $x(t; t_0, z_{\lambda_*})$ does not exist on $t_0 \leq t \leq \omega$. If it did, then $\lambda_* > 0$, and by continuous dependence, $x(\omega; t_0, z)$ would be finite for all z in a neighborhood of z_{λ_*}, contradicting the definition of λ_*. This establishes the claim.

Thus

$$|x(t; t_0, z_{\lambda_*})| \to \infty \text{ as } t \to \omega_*^- \tag{6}$$

for some $t_0 < \omega_* \leq \omega$. If the set

$$B = \{ x(\omega_*; t_0, z_\lambda) : \lambda_* < \lambda \leq 1 \}$$

were bounded, we could choose $\lambda_i \to \lambda_*$ such that

$$x(\omega_*; t_0, z_{\lambda_i}) \to v$$

for some $v \in R^n$. By continuous dependence and (6),

$$|x(t_i; t_0, z_{\lambda_i})| \to \infty \tag{7}$$

for some sequence $t_i \to \omega_*^-$. By local existence at (ω_*, v), the solution $x(t; \omega_*, v)$ exists on $\omega_* - \varepsilon \leq t \leq \omega_*$ for some $\varepsilon > 0$. Therefore by continuous dependence

$$x(t_i; t_0, z_{\lambda_i}) = x(t_i; \omega_*, x(\omega_*; t_0, z_{\lambda_i})) \to v,$$

contradicting (7). Hence B is unbounded.

Thus we can choose $x_i \in B$ such that $|x_i| \to \infty$, and

$$x_i = x(\omega_*; t_0, z_{\lambda_i})$$

for some $\lambda_* < \lambda_i \leq 1$. From (1)

$$V(\omega_*, x(\omega_*; t_0, z_{\lambda_i})) \to \infty \text{ as } i \to \infty. \tag{8}$$

Let $m_i(t) = V(t, x(t; t_0, z_{\lambda_i}))$ for $t_0 \leq t \leq \omega_*$.

Since the line segment L is compact and V is continuous,

$$r_0 = \sup \{ V(t_0, z) : z \in L \}$$

is finite. Then the maximal solution $r(t; t_0, r_0)$ of (CE) exists in the future. Now $m_i(\cdot)$ is a solution of the differential inequality

$$r' \leq \varphi(t, r)$$

(in the upper right-and derivative sense) and $m_i(t_0) \leq r_0$ for all i Therefore (see [2, p. 26] or [5, Theorem 9.5 and Remark 9.3])

$$m_i(t) \leq \varphi(t; t_0, r_0)$$

for all $t_0 \leq t \leq \omega_*$ and all i, a contradiction to (8) at $t = \omega_*$. This proves Theorem 1.

The following result can easily be proved with analogous arguments. It should be noted that $\dot{V}$ is still an upper right-hand derivative (see [5, Theorem 9.6]).

COROLLARY. - *Let* $V : R \times R^n \to R$ *be locally Lipschitz and satisfy* (1) *and*

$$\dot{V}(t, x) \geq \psi(t, V(t, x)).$$

Suppose $\psi : R \times R \to R$ *is continuous and for every real* t_0 *and* r_0 *the maximal solution* $\rho(t; t_0, r_0)$ *of* $r' = \psi(t, r)$ *exists in the past. Then every solution of* (E) *exists in the past.*

In the example below we construct a locally LIPSCHITZ function V satisfying (1) and (2), but not (1*).

EXAMPLE. - Consider (E) where $f(t, x) = -x|x|$ for x real. Since f is continuously differentiable, (E) has uniqueness. Furthermore, all solutions exist in the future (although not in the past) and all tend to zero as $t \to \infty$. In fact

$$x(s; t, y) = \begin{cases} y(y(s-t)+1)^{-1} & \text{if } y \geq 0, \\ -y(y(s-t)-1)^{-1} & \text{if } y < 0. \end{cases} \tag{9}$$

We define V on $[0, 1] \times R$ by

$$V(t, y) = \begin{cases} (x(1; t, y) - 1)^2(1-t)(y^4 + 2y^3) + x^2(1; t, y) \\ \qquad \text{if } 0 < t < 1 \text{ and } y > 0, \\ y^2 \text{ elsewhere on } [0, 1] \times R. \end{cases} \tag{10}$$

Using (9), we have

$$(11)\qquad V(t, y) = \begin{cases} [(yt-1)^2(1-t)(y^4+2y^3)+y^2][y(1-t)+1]^{-2} \\ \qquad \text{if } 0 < t < 1 \text{ and } y > 0, \\ y^2 \text{ elsewhere on } [0, 1] \times R. \end{cases}$$

Using (11), we see that $V(t, y) \to \infty$ as $|y| \to \infty$ for each fixed t, hence (1) holds. However

$$V(t, t^{-1}) \equiv 1 \text{ for } 0 < t < 1.$$

Thus $V(t, y) \to \infty$ as $|y| \to \infty$ non-uniformly in t for $t \in [0, 1]$, so that (1*) does not hold.

Clearly $\partial V/\partial y$ is continuous on $(0, 1) \times R$ and V is continuous on $[0, 1] \times R$. Furthermore, by a long but straightforward computation, $\partial V/\partial y$ is continuous on $[0, 1] \times R$. Thus V is locally LIPSCHITZ there. Therefore we may compute $\dot{V}$ on $[0, 1] \times R$ by taking the (upper right-hand) derivative of $V(s, x(s))$.

Let $0 \leq t < 1$ and $y \in R$. If $y \leq 0$, then for $s > t$,

$$V(s, x(s;\, t, y)) = x^2(s;\, t, y) \leq y^2.$$

Therefore

$$\dot{V}(t, y) \leq 0.$$

If $y > 0$, then using (10) for $s > t$, we have

$$V(s, x(s;\, t, y)) = x^2(1;\, t, x) +$$
$$+ (x(1;\, t, y) - 1)^2(1-s)(x^4(s;\, t, y) + 2x^3(s;\, t, y)).$$

Thus, from (E),

$$\dot{V}(t, y) = \frac{d}{ds} V(s, x(s;\, t, y))|_{s=t} =$$
$$= -(x(1;\, t, y) - 1)^2(y^4 + 2y^3) -$$
$$-(x(1,\, t, y) - 1)^2(1-t)(4y^5 + 6y^4) \leq 0,$$

so that $\dot{V}(t, y) \leq 0$ for all (t, y) in $[0, 1) \times R$.

Now we extend V to $R \times R$ by periodicity, i.e., so that

$$V(t+1, y) = V(t, y)$$

for all real t and y. Then V and $\partial V/\partial y$ are continuous on $R \times R$, hence V is locally LIPSCHITZ there. Furthermore, (1) holds but

(1*) fails for t in any compact interval of length greater than one. Since in this example (E) is autonomous, the solutions on $[i, i+1)$ are merely translates of those on $[0, 1)$ for every $i = 1, 2, \ldots,$. Hence

$$\dot{V}(t, y) \leq 0$$

on $R \times R$. Thus (2) holds with $\varphi \equiv 0$.

REMARK. – In the above example we defined V to have period 1 for convenience. The same type of construction can be used to prove that given any $\alpha > 0$, there exists a locally LIPSCHITZ V_α satisfying (1) and (2), but not satisfying (1*) on any compact interval of length greater than α. Thus the following question arises: is there a locally LIPSCHITZ V satisfying (1) and (2) such that (1*) fails on every compact interval of positive length? This question remains open.

PROOF OF THEOREM 2. – Because of Theorem 1 and its corollary, we need only prove that such V satisfy (1*). Suppose there were a locally LIPSCHITZ V satisfying (1) and (5) but not (1*). Then there exist $M > 0$, t_0 real, and sequences $\{x_i\}$ and $\{t_i\}$ such that $|x_i| \to \infty$, $t_i \to t_0$ monotonically, and

$$V(t_i, x_i) \leq M.$$

We shall assume $t_i \nearrow t_0$ and use the second inequality of (5); if it were the case that $t_i \searrow t_0$ we would use the first.

Since the solution $x(t; t_i, x_i)$ exists in the future for every i, we see that

$$x(t_0, t_i, x_i) \equiv y_i$$

is finite. The sequence $\{y_i\}$ is unbounded by the same argument as that used to prove B is unbounded in the proof of Theorem 1. Thus we may assume that $|y_i| \to \infty$ as $i \to \infty$. Therefore

$$V(t_0, y_i) \to \infty \text{ as } i \to \infty. \tag{12}$$

Let $m_i(t) = V(t, x(t; t_i, x_i))$ for $t_i \leq t \leq t_0$. Then $m_i(t_i) \leq M$ for every i. Now φ is bounded on the square

$$\{(t, r) : |t - t_0| \leq 1, |r - M| \leq 1\}$$

and $m_i(\cdot)$ is a solution (in the upper right-hand derivative sense) of

$$r' \leq \varphi(t, r).$$

for $t_i \leq t \leq t_0$. Thus, by comparison with solutions of $r' = \varphi(t, r)$, there exists $Q > 0$ such that $m_i(t) \leq Q$ for all i and $t_i \leq t \leq t_0$. This contradicts (12) at $t = t_0$, completing the proof.

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