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Problems of integral geometry of lattices in an Euclidean space E_3.

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Summary. - *In this paper the author establishes an integral formula referring to lattices in an Euclidean space E_3, by which some of S. Oshio's theorems* [3] *are found directly and some suggestive interpretations of these theorems are obtained, some new theorems being also given. The results thus obtained are applied to some usual lattices in the space E_3.*

L. A. SANTALÒ has worked up a systematical study of the problems of integral geometry referring to the lattices in an Euclidean plane and has obtained a number of general results [6], [8] which have been applied to the lattices built up by means of some regular figures.

Some of Santalo's results have been extended to lattices in the space E_3 and the space E_n by S. OSHIO [3], [4].

I shall prove here a general integral formula for the lattices in the space E_3, from which I deduce Oshio's results and give a number of new results that are applied to the lattices built up by means of some regular spatial figures.

DEFINITION. - We call a lattice of fundamenta domain in the space E_3, a sequence of domains $\alpha_0, \alpha_1, \alpha_2, \ldots$, which satisfies the following conditions:

1) Each point P in the space, belongs to one and only one α_i.

2) Each domain α_i can be superposed on α_0 by a motion T of the space, which superposes every α_h on an α_k, that is, by a motion which leaves invariant the lattice given.

The domain α_0 is called fundamental cell of the lattice and a domain α_i is called a cell of a lattice.

Consider now a fixed figure K_0, which may be a domain, a surface, a curve, a system of surfaces, a system of curves. a system of surfaces and curves or a system of points (the points are considered a spheres of null radius).

Suppose the figure K_0 is conditioned in the fundamental cell α_0 and let K be a mobile figure.

Let

$$I = \int_{K \cap K_0 \neq \varnothing} f(K_0 \cap K) dK. \tag{1}$$

where f is a integrable function of the figure $K_0 \cap K$ (in case where $K_0 \cap K = \varnothing$, we take $f = 0$), and dK is an elementary kinematic measure in the space E_3, that is

$$dK = |\sin\theta| [dP d\varphi d\theta d\psi], \tag{2}$$

P being a point rigidly linked to the region K, ($\cos\varphi \sin\theta$, $\sin\varphi \sin\theta$, $\cos\theta$) being the directoris cosinus of a direction rigidly linked to the figure K, and ψ being the angle of the rotation round the axis.

We can write

$$I = \sum_i \int_{\alpha_i} f(K_0 \cap K) dK.$$

We apply to the space E the motion T_i, which superposes the cell α_i on the fundamental cell α_0, that is $T_i\alpha_i = \alpha_0$. This motion transforms the figure K into the congruent figure $K^* = T_i K$.

Taking into account the invariance of the elementary kinematic measure, we have $dK^* = dK$, hence

$$I = \sum_i \int_{\alpha_0} f(K_0 \cap T_i^{-1} K^*) dK^* = \sum_i \int_{\alpha_0} f(K_0 \cap T_i^{-1} K) dK.$$

If we consider that the figure $K_0 \cap T_i^{-1} K$ is congruent with $T_i K_0 \cap K$, consequently we have

$$I = \int_{\alpha_0} [\sum_i f(T_i K_0 \cap K)] dK.$$

From here as well as from (1) we deduce the following formula

$$\int_{K_0 \cap K \neq \varnothing} f(K_0 \cap K) dK = \int_{\alpha_0} [\sum_i f(T_i K_0 \cap K)] dK. \tag{3}$$

Let us assume now that the figure K_0 is a domain D_0 of volume V_0, whose boundary ∂K_0 has area S_0, and the figure K is a domain D of volume V, whose boundary ∂K has area S. Denoting by $\chi(D_0)$

the Euler-Poincaré characteristic of the domain D_0 and

$$\bar{H} = \int_{\partial K_0} H_0 d\sigma_0 \tag{4}$$

where H_0 is the mean curvature of the surface ∂K_0 and $\partial\sigma_0$ is the area element on this surface. we write the main kinematic relationship of Blaschke [1] (1)

$$\int_{D \cap D_0 \neq \varnothing} \chi(D_0 \cap D) dK = 8\pi^2[V_0\chi(D) + V\chi(D_0)] + 2\pi(S_0\bar{H} + S\bar{H}_0). \tag{5}$$

If we take in (3), $f(D_0 \cap D) = \chi(D_0 \cap D)$, and taking into account (5), we have:

$$\int_{\alpha_0} \chi_{01} dK = 8\pi^2[V_0\chi(D) + V\chi(D_0)] + 2\pi(S_0\bar{H} + S\bar{H}_0), \tag{6}$$

where χ_{01} is the Euler-Poincaré characteristic of the intersection of D with the figures $T_i D_0$, that is with the lattice generated by the reproduction of D_0 in each cell α_i, the integer being extended over $P \in \alpha_0$, $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \psi \leq 2\pi$.

On the other hand, we have

$$\int_{\alpha_0} dK = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\psi \int_{P \in \alpha_0} dP = 8\pi^2 v_0, \tag{7}$$

where v_0 is the volume of the fundamental cell α_0.

From (6) and (7) we deduce the mean value of χ_{01}

$$M[\chi_{01}] = \frac{V_0\chi(D) + V\chi(D_0)}{v_0} + \frac{S_0\bar{H} + S\bar{H}_0}{4\pi v_0}.$$

Let us consider now as a figure K_0 a curve (Γ_0) of length L_0 and as a figure K a surface of area S. In that case, if we denote by n the number of intersection points of the surface K with the curve (Γ_0) we have Santalo's formula [5] (2)

$$\int_{K \cap \Gamma_0 \neq \varnothing} n dK = 4\pi^2 S L_0.$$

(1) Pag. 547.
(2) Pag. 39.

Taking, into account this relationship in (3), we get Oshio's formula [3] ([3])

$$\int_{\alpha_0} n dK = 4\pi^2 S L_0,$$

where n represents the number of intersection points between the surface K and the lattice generated by the reproduction of (Γ_0) in each cell α_i.

From here as well as from (7) we deduce:

$$M[n] = \frac{SL_0}{2v_0}. \tag{8}$$

From this formula it results, that it is always possible tofind a position of the surface, which has at least $\left[\frac{SL_0}{2v_0}\right]$ points common to the curve (Γ_0).

If we suppose that the surface K may be intersected by the lattice generated by (Γ_0) in n_1 or n_2 points and if we denote by p_1 and p_2 the probabilities that the surface are K is intersected by the lattice in n_1 and n_2 points respectively, formula (8) is written as follows:

$$n_1 p_1 + n_2 p_2 = \frac{SL_0}{2v_0}.$$

Considering $p_1 + p_2 = 1$, we have

$$p_1 = \frac{SL_0 - 2n_2 v_0}{2v_0(n_1 - n_2)}, \qquad p_2 = \frac{2n_1 v_0 - SL_0}{2v_0(n_1 - n_2)}.$$

Let us take a fixed surface K_0 of area S_0 and a mobile surface or area S and let $f(K_0 \cap K) = s$ (the length of the intersection curve of the two surfaces).

Taking into account Santalo's formula [7] ([4])

$$\int_{K_0 \cap K \neq \emptyset} s dK = 4\pi^3 S_0 S,$$

we get the formula obtained by Oshio in an other way [3] ([5])

$$\int_{\alpha_0} L_{01} dK = 4\pi^3 S_0 S,$$

([3]) Pag. 39.
([4]) Pag 352.
([5]) Pag. 42.

where L_{01} is the length of the intersection curve of the surface K with the surfaces $T_i K_0$, that is with the lattice generated by the reproduction of K_0 in each cell α_i.

From here as well as from (7) it results the mean value:

$$M[L_{01}] = \frac{\pi S_0 S}{2v_0}.$$

This formula tells us that it is always possible to find a position of the surface K, whose intersection with the surface K_0 has at least the length $\frac{\pi S_0 S}{2v_0}$.

Suppose that the domain D_0 and D in Blaschke's formula (5) are simply convex. In that case we have $\chi(D) = \chi(D_0) = 1$. Denoting by ν the number of simple convex domains, of which the domain $D_0 \cap D$ is formed up, we have $\chi(D_0 \cap D) = \nu$ and the formula (5) gives us:

$$\int\limits_{D \cap D_0 \neq \varnothing} \nu dK = 8\pi^2(V_0 + V) + 2\pi(S_0\bar{H} + S\bar{H}_0)$$

where $K = D \cap \partial D$.

Taking into account this relation in (3), we get

$$\int\limits_{\alpha_0} \nu dK = 8\pi^2(V_0 + V) + 2\pi(S_0\bar{H} + S\bar{H}_0),$$

where ν is the number of simp, x convex domains, of which the intersection between the domain D and the lattice generated by the reproduction of D in each cell α_i is formed α_3.

From here as well as from (7) the mean value results

$$M[\nu] = \frac{V_0 + V}{v} + \frac{S_0\bar{H} + S\bar{H}_0}{4\pi v_0}. \tag{9}$$

Thus it is always possible to find a position of the domain D, whose intersection with D_0 is formed up from at least $\left[\frac{V_0 + V}{v} + \frac{S_0\bar{H} + S\bar{H}_0}{4\pi v_0}\right]$ simple connex domains.

Supposing that the intersection between the domain D and the lattice generated by D is formed up from ν_1 or ν_2 simply convex domains with p_1 and p_2 respectively, the probabilities corresponding to formula (9), we write

$$\nu_1 p_1 + \nu_2 p_2 = \frac{V_0 + V}{v_0} + \frac{S_0\bar{H} + S\bar{H}_0}{4\pi v_0}$$

hence

$$p_1 = \frac{4\pi(V_0 + V - \nu_2 v_0) + S_0\bar{H} + S\bar{H}_0}{4\pi v_0(\nu_1 - \nu_2)}$$

$$p_2 = \frac{4\pi(V_0 + V - \nu_1 v_0) + S_0\bar{H} + S\bar{H}_0}{4\pi v_0(\nu_1 - \nu_2)}.$$

Considering $D_0 = \alpha_0 \bigcup \partial\alpha_0$ and denoting by η the number of simple connex domains, in which the domain D is divided by the lattice, formula (9) is written as follows:

$$M[\eta] = \frac{V_0 + V}{v_0} + \frac{s_0\bar{H} + S\bar{h}_0}{4\pi v_0}$$

where S_0 is the area of $\partial\alpha_0$ and has the measure of the set of the planes intersecting the boundary $\partial\alpha_0$.

Hence we have the theorem:

Any simple connex domain of volume V, whose boundary ∂D has area S, can be covered by

$$N \leq 1 + \frac{V}{v_0} + \frac{s_0\bar{H} + S\bar{h}_0}{4\pi v_0}$$

cells of a lattice whose fundamental cell has volume v_0 and whose boundary has area s_0.

Let us apply this result to a lattice formed of cubes of side a. In this case we have $v_0 = a^3$, $s_0 = 6a^2$.

To calculate h_0 taking into account BLASCHKE, formula [2] (6) which says that the measure of the set of the planes intersecting a convex polyhedron is equal to $\frac{1}{2}\Sigma\, l\varphi_l$, l being the length of an edge of the polyhedron, and φ_l the dyhedron angle corresponding to this edge, and the sum being extended to all the edges of the polyhedron. So we have $\bar{h}_0 = 3\pi a$.

Hence:

$$N \leq 1 + \frac{V}{a^3} + \frac{3S}{a^2} + \frac{3\bar{H}}{2\pi a}.$$

(6) Pag. 89.

This formula has been proved by SANTALO in the case of a topological sphere [9] (7).

Suppose that the figure K_0 is formed of n points, and the figure K is a body of volume V. Taking into account that a point can be considered as a sphere of null radius, we have $S_0 = V_0 = 0$, $\gamma(K_0) = m$ and Blaschke's formula (5) becomes

$$\int_{K \cap K_0 \neq \varnothing} n dK = 8\pi^2 m V,$$

where n is the number of points inside a position of K.

If we denote by $m*$ the number of points in the lattice generated by the reproduction of K_0 in each cell α_i, contained inside K, formula (3) gives us:

$$\int_{\alpha_0} m * dK = 8\pi^2 m V,$$

the formula proved in an other way by OSHIO [3] (8).

From here as well as from (7) it results:

$$M[m*] = \frac{mV}{v_0}. \tag{10}$$

Supposing that inside the body K we may have m_1^* and m_2^* points and denoting by p_1 and p_2 the corresponding probabilities, formula (10) is written

$$m_1^* p_1 + m_2^* p_2 = \frac{mV}{v_0}$$

hence:

$$p_1 = \frac{mV - m_2^* v_0}{v_0(m_1^* - m_2^*)}, \qquad p_2 = \frac{m_1^* v_0 - mV}{v_0(m_1^* - m_2^*)}.$$

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(8) Pag 38.

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