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A Lattice Theoretical Analogue of the Schur Lemma

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Sunto. - *Si osserva che una proposizione analoga al noto lemma di Schur della teoria della rappresentazione dei gruppi è valida per una ampia classe di reticoli comprendente i reticoli relativamente complementati con elemento zero.*

Summary. - *We remark that a proposition analogous to the known Schur Lemma of the representation theory of groups holds for a wide class of lattices, including the relatively complemented lattices with zero element.*

1. - Introduction.

The SCHUR Lemma [1] is a rather trivial proposition which is of course very useful expecially within the representation theory of groups.

One of the forms in which it is usually formulated is the following:

PROPOSITION Ia. - Let X, X' be two linear spaces, $(A_\alpha)_{\alpha\in\mathfrak{A}}$ ($\mathfrak{A}$ set of indices) an irreducible family of linear mappings of X, $(A'_\alpha)_{\alpha\in\mathfrak{A}}$ a corresponding family of linear mappings of X'.

Let T be a linear mapping from X into X' such that:

$$\text{for all } \alpha\in\mathfrak{A} \qquad TA_\alpha = A'_\alpha T.$$

Then either $TX = \{O'\}$, $\{O'\}$ being the null-subspace of X', or T is one-to-one.

More generally and abstractly it may be stated:

PROPOSITION Ib. - Any homomorphism T of a simple group G with a set $\mathfrak{A}$ of operators into a group G' with the same set $\mathfrak{A}$

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(1) ISSAI SCHUR, *Neue Begründung der Theorie der Gruppencharaktere*, «Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften», Berlin 1905, 406-432.

of operators, either maps G onto the trivial subgroup $\{e'\}$ of G', e' being the neutral element of G', or it is one-to-one.

Clearly, in this form, the proposition appears as a particular case of the extension to groups with operators of the proposition that relates group homomorphisms to invariant subgroups (2).

Therefore the SCHUR Lemma states, for a type of structured set, (group with operators) that irreducibility implies simplicity (3), simplicity meaning that any homomorphic image of the structured set either is isomorphic to the given structured set or is trivial in the sense that it reduces to a one element set. As a particular case the non trivial (in the above sense) endomorphisms of an irreducible structured set of that type form a group.

In this note, we will point out that an analogous proposition holds for a wide class of lattices, including the lattice of the linear manifolds of any linear space and the lattice of the closed subspaces of an HILBERT space.

2. - A Lattice Theoretical Analogue of the Schur Lemma.

Clearly, the possibility of proving the Propositions Ia or Ib arises from the fact that, in the case of groups, any homomorphism such that the inverse image of the neutral element is the neutral element, is one-to-one.

It is well known that a similar situation occurs for lattices with zero element O whose principal ideals (segments of the form $[O, a]$) are complemented (therefore for any relatively complemented lattice as a particular case), the O element taking the place of the neutral element of groups (4).

So we may write down an analogue of Proposition Ia for these lattices.

PROPOSITION II. - (Lattice-theoretical Analogue of the SCHUR Lemma).

Let L, L' be two lattices with zero elements O, O' respectively. Let L be such that its principal ideals are complemented (let L be a relatively complemented lattice as a particular case).

(2) See for instance N. BOURBAKI, *Éléments de Mathématique I*, Livre II par. 6 n° 13 page 82, théor. 5.

(3) See G. BIRKHOFF, *On the structure of abstract algebras*, «Proc. of the Cambridge Mat. Soc», 31 433-454 (1953)

(4) See G. BIRKHOFF, *Lattice Theory*, «Am. Math. Soc. Coll. Publ.», Vol. XXV 1 61, Ch. II par. 6, Th. 3 (page 23), together with N. BOURBAKI I Livre II Ch. I par. 4 Th. 1 (page 49).

Let $(A_\alpha)_{\gamma \in \mathfrak{A}}$ ($\mathfrak{A}$ a set of indices) be an irreducible family of mappings of L, irreducible meaning here that L does not contain any ideal stable under the set of mappings $\{A_\alpha \mid \alpha \in \mathfrak{A}\}$ other than $\{O\}$ and L.

Let $(A'_\alpha)_{\alpha \in \mathfrak{A}}$ be a corresponding family of mappings of L', and let any one of them leave O' invariant.

Let T be an homomorphism of L into L', mapping O in O', such that:

$$\text{for all } \alpha \in \mathfrak{A} \qquad TA_\alpha = A'_\alpha T.$$

Then, either T is improper (that is $TL = \{O'\}$), or T is one-to-one.

PROOF. – First of all, we show that either $T^{-1}\{O'\} = \{O\}$ or T is the improper homomorphism.

In fact, as T is an homomorphism, $T^{-1}\{O'\}$ is a convex sublattice of L; by assumption, $O \in T^{-1}\{O'\}$: so $T^{-1}\{O'\}$ is an ideal of L.

Moreover $T^{-1}\{O'\}$ is stable under $\{A_\alpha \mid \alpha \in \mathfrak{A}\}$. Let in fact be $x \in T^{-1}\{O'\}$: we have $Tx = O'$.

From the assumptions that $TA_\alpha = A'_\alpha T$ and $A'_\alpha O' = O'$ for all $\alpha \in \mathfrak{A}$, it follows:

$$TA_\alpha x = A'_\alpha Tx = A'_\alpha O' = O'$$

that is $A_\alpha x \in T^{-1}\{O'\}$ whenever $x \in T^{-1}\{O'\}$.

Therefore, according to the irreducibility of (A_α) that has been assumed, we conclude that either $T^{-1}\{O'\} = \{O\}$. or $T^{-1}\{O'\} = L$.

The latter case corresponds, as it has been claimed before, to T improper.

In the former case it is easily shown that T is one-to-one. Let in fact x, $y \in L$, and $Tx = Ty = x'$. We have to show that $x = y$.

Let us put $z = x \cap y$. We have $Tz = x'$. We will prove that $z = x$ using the fact that $z \leq x$. In the same way, it will then be possible to prove that $z = y$ using the fact that $z \leq y$, and so the proposition will be proved.

Let us then show that $Tx = Tz = x'$ and $z \leq x$ implies $z = x$.

Let $\bar{z}$ be a complement of z in the ideal $[O, x]$ of the lattice L. We have $\bar{z} \cap z = O$, $\bar{z} \cup z = x$.

$z \cup \bar{z} = x$ implies $\bar{z} \leq x$. T being an homomorphism, it is an isotony. So it follows $T\bar{z} \leq x'$. Therefore $T\bar{z} \cap x' = T\bar{z}$.

From $z \cap \bar{z} = O$ it follows $T\bar{z} \cap x' = O'$.

From both we have $T\bar{z} = O'$.

The assumption $T^{-1}\{O'\} = \{O\}$ implies $\bar{z} = O$.

This conclusion, together with $z \cup \bar{z} = x$, leads to $\bar{z} \cup O = x$, that is $z = x$.

So the proof of proposition II is concluded. We add a few remarks.

REMARK 1. – Some of the assumptions of the proposition may even be weakened. It is in fact sufficient to assume that L' is a $\cap$-semilattice with zero element and that correspondently T is an homomorphism with respect to the $\cap$-semilattice structure.

REMARK 2. – The only property of each of the mappings A_α, A'_α, which is required in the proof, is that O' is invariant under each of the A'_α. The situation is therefore completely analogous to that which occurs in the case of groups.

REMARK 3. – Let us make in Proposition II the further assumption that the lattice L is complete and that the homomorphism T is a complete homomorphism, in the sense that it preserves g l.b. and l.u.b. of arbitrary sets (5): then it is possible to weaken the assumption about the irreducibility of $(A_\alpha)_{\alpha \in \mathfrak{A}}$, by supposing that the only principal ideals stable under $\{A_\alpha \mid \alpha \in \mathfrak{A}\}$ are $\{O\}$ and L. In fact it is easly shown that, under these assumptions about L and T, $T^{-1}\{O\}$ is a principal ideal (in fact it is a complete convex sublattice of L, therefore a segment; moreover it contains O). Then the proof proceeds as before.

We remark that this set of assumptions corresponds more closely to the situation that occurs in the theory of representations in HILBERT space.

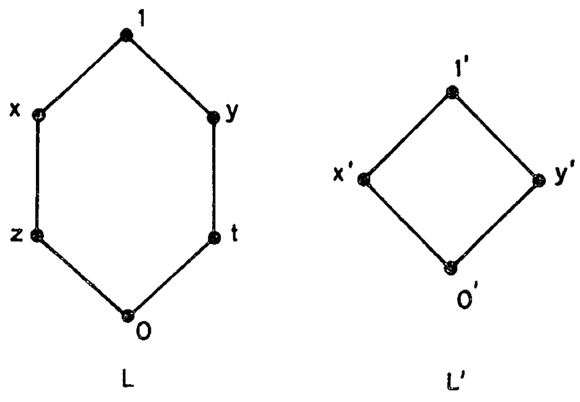

Fig. 1

(5) See G. BIRKHOFF, *Lattice Theory*, « Am. Math. Soc. Coll Publ. », Vol. XXV 1961, Ch IV, par. 1, Ex. 11, page 50.

REMARK 4. – Proposition II does not hold generally if the lattices are complemented, but not relatively complemented, even if the families (A_α) and (A'_α) contain only automorphisms. This is shown by the following counterexample: let the lattices be L and L' of Fig. 1: let (A_α) contain only the involutory automorphism $0 \leftrightarrow 0$, $1 \leftrightarrow 1$, $x \leftrightarrow y$, $z \leftrightarrow t$ and let (A'_α) contain only $0' \leftrightarrow 0'$, $1' \leftrightarrow 1'$, $x' \leftrightarrow y'$.

Let T be: $0 \to 0'$, $1 \to 1'$, $x \to x'$, $z \to x'$ $y \to y'$, $t \to y'$.

T clearly satisfies all the requirements of Proposition II, but still it is not one–to–one.

REMARK 5. – The study of the representations of groups in the group of the automorphisms of lattices of a certain type is an interesting one from the point of view of physics.

In fact, an axiomatic formulation of physics leads to a structure of an orthocomplemented lattice for the set of the physical properties, the order relation meaning phenomenological implication: groups of pysical symmetries correspond to groups of automorphisms of the lattice (6).

Irreducible representations of a symmetry group correspond to the simplest possible physical systems possessing that symmetry group.

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(6) G. BIRKHOFF, J. von NEUMANN, « Ann. of Math. », 37, 823-843 (1936) C. PIRON, *Axiomatique quantique,* « Helv. Phys Acta », 37. 439-468 (1964) G. EMCH, C. PIRON, *Symmetry in Quantum Theory,* « Journ. of Math. Phys. », 4, 469-473 (1963).