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SEZIONE SCIENTIFICA
BREVI NOTE

Borel summability of Fourier series

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Summary. - *Borel summability has been applied to Fourier Series by Sahney and to the conjugate series of a derived Fourier Series by Sinvhal. In this note the authors have extended a result due to Sahney.*

1. - Let $f(x)$ be a function integrable in the sense of LEBESGUE over the interval $(0, 2\pi)$ and periodic with period 2π outside this interval. Let the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

be the FOURIER series associated with the function $f(x)$. We further write

$$\varphi(t) = \frac{1}{2}[f(x+t) + f(x-t) - 2s]. \tag{1.2}$$

KNOPP [1] has defined the generalised Borel summability. STONE [2] was the first mathematician to apply Borel summability to Fourier series. SINVHAL [3] has applied this method to the Conjugate series of the Derived Fourier series. Recently SAHNEY [4] considered the Borel summability of the series (1.1) and has proved the following theorem.

THEOREM S: - If

$$\int_0^t |\varphi(u)|\, du = 0\left(\frac{t}{\log 1/t}\right) \tag{1.3}$$

then the series (1.1) is summable by Borel Means (also called summable–B) to the sum zero at the point $t = x$.

The object of this note is to generalise Sahney's result and to prove the following theorem.

THEOREM: - *If*

$$\int_0^t |\varphi(u)|\, du = 0\left(\frac{t}{(\log 1/t)^{\Delta}}\right) \quad \textit{for } 0 \leq \Delta \leq 1 \tag{1.4}$$

then the series (1.1) *is summable*-B *to the sum zero at the point* $t = x$.

2. - PROOF OF THE THEOREM:

It is well known that the n^{th} partial sum of the series (1.1) is given by

$$S_n(t) = \frac{1}{\pi}\int_0^{\pi} \varphi(t)\frac{\operatorname{Sin} nt}{t}\, dt.$$

Now following HARDY [5], the Borel Transform of $S_n(t)$ is given by

$$\sigma_p(t) = \frac{e^{-p}}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t}\left(\sum_{n=0}^{\infty} \frac{p^n \operatorname{Sin} nt}{L^n}\right) dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{\varphi(t)}{t} \cdot \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}}\, dt.$$

$$= \frac{1}{\pi}\left[\int_0^{1/p} + \int_{1/p}^{1/p^{\alpha}} + \int_{1/p^{\alpha}}^{\pi}\right]\frac{\varphi(t)}{t} \cdot \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}}\, dt.$$

where $0 < \alpha < 1/2$

$$= I_1 + I_2 + I_3, \text{ say.} \tag{2.1}$$

Now,

$$I_1 = \frac{1}{\pi}\int_0^{1/p} \frac{\varphi(t)}{t} \cdot \frac{\operatorname{Sin}(p \operatorname{Sin} t)}{\exp\{p(1-\cos t)\}}\, dt$$

$$= O(1)\int_0^{1/p} \frac{|\varphi(t)|}{t} \cdot O(pt)dt$$

$$= O(p)\int_0^{1/p} |\varphi(t)|\, dt$$

$$= O(p)\left[\sigma\left\{\frac{t}{(\log 1/t)^{\Delta}}\right\}\right]_0^{1/p}, \quad \text{by (1.4)}$$

$$= o\left[\frac{1}{(\log p)^{\Delta}}\right]$$

(2.2) $$= o(1), \text{ as } p \to \infty.$$

Next we have,

$$I_2 = \frac{1}{\pi}\int_{1/p}^{1/p^{\alpha}} \frac{\varphi(t)}{t} \cdot \frac{\text{Sin}\,(p\,\text{Sin}\,t)}{\exp\{p(1-\cos t)\}}\,dt$$

$$= \frac{1}{\pi \exp\{p \cdot 2\,\text{Sin}^2\,1/2p\}}\int_{1/\alpha}^{1/\alpha'} \frac{\varphi(t)}{t}\,\text{Sin}\,(p\,\text{Sin}\,t)dt,$$

for $0 < \alpha < \alpha' < \frac{1}{2}$ by the second mean value theorem.

Let, $\int_0^t |\varphi(t)|\,dt \equiv \Phi(t) = 0\left(\frac{t}{(\log 1/t)^{\Delta}}\right)$, by (1.4).

Therefore,

$$I_2 = O(1)\int_{1/p}^{1/p^{\alpha'}} \frac{|\varphi(t)|}{t}\,dt$$

$$= O(1)\left[\frac{\Phi(t)}{t}\right]_{1/p}^{1/p^{\alpha'}} + O(1)\int_{1/p}^{1/p^{\alpha'}} \frac{\Phi(t)}{t^2}\,dt$$

$$= O(1)\left[o\left(\frac{1}{(\log 1/t)^{\Delta}}\right)\right]_{1/p}^{1/p^{\alpha'}} + O(1)\int_{1/p}^{1/p^{\alpha'}} o\left[\frac{1}{t(\log 1/t)^{\Delta}}\right]dt, \quad \text{by (1.4)}$$

$$= o\left[\frac{1}{(\log p)^{\Delta}}\right] + o\left[-1/\Delta\,\{\log(\log 1/t)^{\Delta}\}\right]_{1/p}^{1/p^{\alpha'}}$$

$$= o\left[\frac{1}{(\log p)^{\Delta}}\right] + o\left[\log(\log 1/t)^{\Delta}\right]_{1/p}^{1/p^{\alpha'}}$$

$$= o(1) + o(\log \alpha')$$

(2.3) $$= o(1), \quad \text{as } p \to \infty.$$

Lastly, we have

$$I_3 = \frac{1}{\pi} \int_{1/p^\alpha}^{\pi} \frac{\varphi(t)}{t} \cdot \frac{\text{Sin}\,(p\,\text{Sin}\,t)}{\exp\{p(1-\cos t)\}}\,dt.$$

$$= \frac{p^\alpha}{\pi \exp\{p \cdot 2\,\text{Sin}^2\, 1/2p^\alpha\}} \int_{1/p^\alpha}^{\delta} \varphi(t)\,\text{Sin}\,(p\,\text{Sin}\,t)dt$$

where $1/p^\alpha < \delta < \pi$ by the second mean value theorem.

$$= O(1)\left[\frac{p^\alpha}{\exp\{p^{1-2\alpha}\}}\right] \int_{1/p^\alpha}^{\delta} |\varphi(t)|\,dt$$

by the continuity part of the integral, $\int |\varphi(t)|\,dt$, weget

$$= O(1)\left[\frac{p^\alpha}{\exp}\{p^{1-2\alpha}\}\right] \cdot O(1)$$

(2.4) $\qquad = o(1)$, as $p \to \infty$ and since $0 < \alpha < 1/2$.

Thus collecting (2.2), (2.3) and (2.4), we find that

(2.5) $$\sigma_p(t) = o(1), \text{ as } p \to \infty.$$

This completes the proof of the theorem.

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