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Representation of partially and simply ordered sets by terminating sequences

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Sunto. - *Si dimostra un teorema generale sulla rappresentazione degli insiemi parzialmente ordinati per mezzo di sequenze costituite da* 0, 1 *e u terminanti con* 0 *e ciascuna con almeno un elemento non nullo. Questo teorema conduce a una forma più forte dei noti teoremi di rappresentazione di Sierpinski e Popruzenko.*

In this paper we prove a general Theorem for representation of partially ordered sets by means of sequences made up of 0, 1, and u, terminating in 0's, and each with a last non-zero term. As shown below, this Teorem yields both a stronger form of SIERPINSKI's [1] and POPRUZENKO's [2] representation theorems.

DEFINITION 1. - *Let* (a_i) *and* (b_i) *be two sequences (of the same finite or transfinite type) made up of the numbers* 0, 1, *and the letter u. We say that* (a_i) *is less than or equal to* (b_i) *according to the principle of first numerical differences, and we denote this by:*

$$(1) \qquad (a_i) \preceq (b_i)$$

if (a_i) *is equal (identical) to* (b_i) *or if there exists an index j such that*

(i) $a_j = 0$ *and* $b_j = 1$

(ii) $a_i = 1$ *implies* $b_i = 1$ *for* $i < j$

(iii) $b_i = 0$ *implies* $a_i = 0$ *for* $i < j$

As usual, if $(a_i) \preceq (b_i)$ and $(a_i) \neq (b_i)$ then we write $(a_i) \prec (b_i)$.

LEMMA. - *Let* λ *be an ordinal and let* T_λ *be the set of all sequences of type* λ *made up of* 0, 1, *and u. Then* T_λ *is partially ordered by the principle of first numerical differences i.e.,* $(T_\lambda, \preceq)$ *is a partially ordered set.*

POOF. - Clearly for every element $(a_i)_{i<\lambda}$ of T_λ we have $(a_i)_{i<\lambda} \not\prec (a_i)_{i<\lambda}$ since (i) cannot hold in this case. Thus $\prec$ is irreflexive.

Next we show $\prec$ is transitive. Let $(a_i)_{i<\lambda} \prec (b_i)_{i<\lambda}$ and $(b_i)_{i<\lambda} \prec$ $\prec (c_i)_{i<\lambda}$. Then by (i), (ii), and (iii) there exists a j such that

$$a_j = 0 \quad \text{and} \quad b_j = 1 \tag{2}$$

$$a_i = 1 \quad \text{implies} \quad b_i = 1 \quad \text{for} \quad i < j \tag{3}$$

$$b_i = 0 \quad \text{implies} \quad a_i = 0 \quad \text{for} \quad i < j \tag{4}$$

and there exists a k such that

$$b_k = 0 \quad \text{and} \quad c_k = 1 \tag{5}$$

$$b_i = 1 \quad \text{implies} \quad c_i = 1 \quad \text{for} \quad i < k \tag{6}$$

$$c_i = 0 \quad \text{implies} \quad b_i = 0 \quad \text{for} \quad i < k \tag{7}$$

Clearly, in view of (2) and (5) we see that $j \neq k$. Thus it remains to consider the following two cases.

CASE 1. - If $j < k$ then by (2) we have

$$a_j = 0 \tag{8}$$

Also since $j < k$, and $b_j = 1$, by (6) we have

$$c_j = 1 \tag{9}$$

Now since $j < k$, if $i < j$ then $i < k$. Thus by (3) and (6) we have

$$a_i = 1 \quad \text{implies} \quad b_i = 1 \quad \text{and} \quad c_i = 1 \quad \text{for} \quad i < j \tag{10}$$

and by (7) and (4) we have

$$c_i = 0 \quad \text{implies} \quad b_i = 0 \quad \text{and} \quad a_i = 0 \quad \text{for} \quad i < j \tag{11}$$

From (8), (9), (10), and (11) it follows that $(a_i)_{i<\lambda} \prec (c_i)_{i<\lambda}$.

CASE 2. - If $k < j$ then by (5) we have

$$c_k = 1 \tag{12}$$

and since $k < j$ and $b_k = 0$ by (4) we have

$$a_k = 0 \tag{13}$$

Now since $k < j$, if $i < k$ then $i < j$. Thus by (3) and (6) we have

(14) $\qquad a_i = 1$ implies $b_i = 1$ and $c_i = 1$ for $i < k$

and by (7) and (4) we have

(15) $\qquad c_i = 0$ implies $b_i = 0$ and $a_i = 0$ for $i < k$

From (12), (13), (14), and (15) again it follows that $(a_i)_{i<\lambda} \prec (c_i)_{i<\lambda}$. Thus $\prec$ is a transitive relation.

We have shown that $\prec$ is an irreflexive and transitive relation, which implies that $(T_\lambda, \preceq)$ is a partially ordered set, as desired.

Let us also observe that in case two sequences have no term u, the ordering $\preceq$ as introduced in (1) reduces to the usual ordering by first differences.

THEOREM. - *Let $(P, \leq)$ be a partially ordered set of power $\aleph_\mu$. Then $(P, \leq)$ is isomorphic to a subset S of T_{ω_μ} ordered by the principle of first numerical differences such that for every element $(s_i)_{i<\omega_\mu}$ of S there exists a λ with $s_\lambda = 1$ and $s_i = 0$ for every $i > \lambda$, and for every ordinal $\tau < \omega_\mu$ there exists an element $(t_i)_{i<\omega_\mu}$ of S with $t_\tau = 1$ and $t_i = 0$ for every $i > \tau$.*

PROOF. - Let $(p_j)_{j<\omega_\mu}$ be a well-ordering of P. Consider a mapping f from P into T_{ω_μ} defined as follows:

$$f(p_j) = (a_i^j)_{i<\omega_\mu} \quad \text{for every element } p_j \text{ of } P$$

where

$$(16) \quad a_i^j = \begin{cases} 1 \text{ if } p_i \leq p_j \text{ and } i \leq j \\ 0 \text{ if } p_i > p_j \text{ or } i > j \\ u \text{ otherwise (i.e. if } p_i \text{ and } p_j \text{ are incomparable and } i \leq j) \end{cases}$$

We shall show that f is the desired isomorphism.

From (16) il follows that for every $j < \omega_\mu$ we have

(17) $\qquad a_j^j = 1$ and $a_i^j = 0$ for every $i > j$.

Taking $j = \lambda$ on the one hand, and $j = \tau$ on the other, we see that the range S of f satisfies the conditions of the Theorem.

Next we show that f is a one-to-one mapping.

Let $f(p_j) = f(p_k)$, i.e., $(a_i^j)_{i<\omega_\mu} = (a_i^k)_{i<\omega_\mu}$. Then in view of (16) we have

$$a_j^j = 1 \quad \text{implies} \quad a_j^k = 1 \quad \text{and} \quad p_j \leq p_k \tag{18}$$

$$a_k^k = 1 \quad \text{implies} \quad a_k^j = 1 \quad \text{and} \quad p_k \leq p_j. \tag{19}$$

Thus we see that $p_j = p_k$ and therefore f is one-to-one.

To prove that f preserves order in both directions we consider the following two cases.

CASE 1. - Let $p_j < p_k$, where $f(p_j) = (a_i^j)_{i<\omega_\mu}$ and $f(p_k) = (a_i^k)_{i<\omega_\mu}$. Then since $p_j < p_k$ in view of (16) we have

$$a_k^k = 1 \quad \text{and} \quad a_k^j = 0 \tag{20}$$

If $a_i^j = 1$ then $p_i \leq p_j$ and since $p_j < p_k$ we have $p_i < p_k$. Thus from (16) it follows that

$$a_i^j = 1 \quad \text{implies} \quad a_i^k = 1 \quad \text{for} \quad i < k \tag{21}$$

On the other hand, if $a_i^k = 0$ and $i < k$ then by (16) we must have $p_i > p_k$, since $i \not> k$, and since $p_j < p_k$ we have $p_i > p_j$. Thus from (16) it follows that

$$a_i^k = 0 \quad \text{implies} \quad a_i^j = 0 \quad \text{for} \quad i < k. \tag{22}$$

In view of (20), (21), and (22) we see that $(a_i^j)_{i<\omega_\mu} \prec (a_i^k)_{i<\omega_\mu}$ and thus $p_j < p_k$ implies that $f(p_j) \prec f(p_k)$.

CASE 2. - Let $f(p_j) = (a_i^j)_{i<\omega_\mu} \prec (a_i^k)_{i<\omega_\mu} = f(p_k)$. Then there exists an index h such that in view of (i) and (ii) .

$$a_h^j = 0 \quad \text{and} \quad a_h^k = 1 \tag{23}$$

$$a_i^j = 1 \quad \text{implies} \quad a_i^k = 1 \quad \text{for} \quad i < h. \tag{24}$$

From (16) and (23) it follows that

$$p_h \leq p_k \quad \text{and} \quad h \leq k \tag{25}$$

If $h \leq j$ then since by (23) we have $a_h^j = 0$ we see by (16) that $p_h > p_j$. But then by (25) it follows that $p_j < p_k$.

If $j < h$ then since by (16) we have $a_j^j = 1$ we see by (24) that $a_j^k = 1$ which implies $p_j \leq p_k$. But since $f(p_j) \neq f(p_k)$ it follows that $p_j \neq p_k$ and hence $p_j < p_k$.

Thus $f(p_j) \prec f(p_k)$ implies $p_j < p_k$. Hence f is an isomorphism as desired.

DEFINITION 2. - *A partially ordered set* $(P, \leq)$ *is said to be quasi-isomorphic to a partially ordered set* $(Q, \leq^*)$ *if there exists a one-to-one mapping* f *from* P *onto* Q *such that for every two elements* x *and* y *of* P *we have* $x \leq y$ *implies* $f(x) \leq^* f(y)$.

It is obvious that if $(P, \leq)$ is a simply ordered set then the above quasi-isomorphism reduces to an isomorphism.

A slight modification of the proof of the above theorem yields the following stronger version of the result of J. POPRUZENKO, [2].

COROLLARY 1. - *Let* $(P, \leq)$ *be a partially ordered set of power* $\aleph_\mu$. *Then* $(P, \leq)$ *is quasi-isomorphic to a set* H *of sequences of* 0 *and* 1 *of type* ω_μ *ordered by first differences, such that for every element* $(h_i)_{i<\omega_\mu}$ *of* H *there exists a* λ *with* $h_\lambda = 1$ *and* $h_i = 0$ *for every* $i > \lambda$, *and for every* $\tau < \omega_\mu$ *there exists an element* $(g_i)_{i<\omega_\mu}$ *of* H *with* $g_\tau = 1$ *and* $g_i = 0$ *for every* $i > \tau$.

PROOF. - In the definition of the sequences $(a_i^j)_{i<\omega_\mu}$ given by (16) replace u by 0, i.e.

$$a_i^j = \begin{cases} 1 \text{ if } p_i \leq p_j \text{ and } i \leq j \\ 0 \text{ otherwise} \end{cases} \tag{26}$$

Then (17) through (21) remain valid. On the other hand, (22) becomes the contrapositive of (21) and hence is valid. Clearly (17) through (22) imply that f is a quasi-isomorphism, as desired.

An obvious consequence of Corollary 1 is the following.

COROLLARY 2. *Every partial order in a set* P *can be extended to a simple order in the same set* P *preserving the original order among the elements of* P.

Since for a simply ordered set $(P, \leq)$ the quasi-isomorphism mentioned in Corollary 1 is an isomorphism, we have as an immediate consequence of Corollary 1 the following result of W. SIERPINSKI, [1].

COROLLARY 3. - *Let* $(P, \leq)$ *be a simply ordered set of power* $\aleph_\mu$. *Then* $(P, \leq)$ *is isomorphic to a set* H *of sequences of* 0 *and* 1 *of type* ω_μ *ordered by first differences such that for every element*

$(h_i)_{i<\omega_\mu}$ *of* H *there exists a* λ *with* $h_\lambda = 1$ *and* $h_i = 0$ *for every* $i > \lambda$, *and for every* $\tau < \omega_\mu$ *there exists an element* $(g_i)_{i<\omega_\mu}$ *of* H *with* $g_\tau = 1$ *and* $g_i = 0$ *for every* $i > \tau$.

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