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On a Characterization of Ultraspherical Polynomials

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Summary. - *It is shown that the only polynomials* $\{p_n(x)\}$ *with a generating function of the form*

$$e^{xz}\varphi(z\sqrt{1-x^2}) = \Sigma p_n(x)z^n/n!, \quad \varphi \text{ even}$$

are essentially the ultraspherical polynomials.

A well-know generating function for the normalized ultraspherical polynomials $F_n(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ is

$$\sum_{n=0}^{\infty} F_n(x)z^n/n! = 2^{\lambda-\frac{1}{2}}\,\Gamma\left(\lambda+\frac{1}{2}\right)e^{xz}\left[(1-x^2)^{\frac{1}{2}}z\right]^{\frac{1}{2}-\lambda}$$

$$J_{\lambda-\frac{1}{2}}\left[(1-x^2)^{\frac{1}{2}}z\right], \quad \lambda > -\frac{1}{2},$$

where J_α is the BESSEL function of the first kind of order α. [3; for definitions see 5]. This generating function is of particular interest since its zeros are real; this fact enabled SZEGÖ to establish TURÁN'S inequality for ultraspherical polynomials [6].

We consider the generating function for the polynomials $\{p_n(x)\}$:

$$e^{xz}\varphi(z\sqrt{1-x^2}) = \sum_{n=0}^{\infty} p_n(x)z^n/n!. \tag{1}$$

where $\varphi(u)$ is even and expandible in a TAYLOR series about the origin φ in even and $\varphi(0) \neq 0$. Differentiating both sides of (1) first with respect to x and then with respect to z yields

$$e^{xz}\varphi(z\sqrt{1-x^2}) - \frac{xe^{xz}\varphi'(z\sqrt{1-x^2})}{1-x^2} = \sum_{n=1}^{\infty} p'_n(x)z^{n-1}/n! \tag{2}$$

$$xe^{xz}\varphi(z\sqrt{1-x^2}) + e^{xz}\sqrt{1-x^2}\,\varphi'(z\sqrt{1-x^2}) = \sum_{n=0}^{\infty} p_{n+1}(x)z^n/n! \tag{3}$$

Eliminating φ' in (2) and (3), we obtain

$$e^{xz}\varphi(z\sqrt{1-x^2}) - (1-x^2)\sum_{n=1}^{\infty} p'_n(x)z^{n-1}/n! = \sum_{n=0}^{\infty} xp_{n+1}(x)z^n/n!$$

and with (1)

$$\sum_{n=0}^{\infty} p_n(x)z^n/n! - (1-x^2)\sum_{n=1}^{\infty} p'_n(x)z^{n-1}/n! = \sum_{n=0}^{\infty} xp_{n+1}(x)z^n/n!$$

Equating coefficients of like powers of z yields

$$(1 - x^2)p'_{n+1}(x) = (n+1)[p_n(x) - xp_{n+1}(x)],\quad n = 0, 1, 2, \ldots \tag{4}$$

The normalized ultraspherical polynomials $F_n(x)$ satisfy [5]

$$(1 - x^2)F'_{n+1}(x) = (n+1)[F_n(x) - xF_{n+1}(x)]. \tag{5}$$

We now show that if $\{p_n(x)\}$ is any sequence of polynomials satisfying (4), then $p_n(x) = C_n F_n(x)$, $n = 0, 1, 2, \ldots$, where C_n is a constant. We proceed by induction. Assume that $p_k(x) = C_k F_k(x)$, $k = 0, 1, 2, \ldots, n$. Then we obtain from (4)

$$(1 - x^2)p'_{n+1}(x) = (n+1)[C_n F_n(x) - xp_{n+1}(x)])$$

which together with (5) yields

$$(1 - x^2)[p'_{n+1}(x) - C_n F'_{n+1}(x)] = -(n+1)x[p_{n+1}(x) - C_n F_{n+1}(x)].$$

Letting $u_{n+1}(x) = p_{n+1}(x) - C_n F_{n+1}(x)$, we obtain the first order differential equation

$$(1 - x^2)u'_{n+1}(x) = -(n+1)xu_{n+1}(x). \tag{6}$$

The general solution of (6) is $u_{n+1}(x) = C'_{n+1}(1 - x^2)^{\frac{n+1}{2}}$. Hence $p_{n+1}(x) = C_{n+1}F_{n+1}(x) + C'_{n+1}(1 - x^2)^{\frac{n+1}{2}}$. Now $F_n(-x) = (-1)^n F_n(x)$ and since φ is even il follows from (1) that $p_n(-x) = (-1)^n p_n(x)$. Hence, necessarily $C'_{n+1} = 0$.

We have proved the

THEOREM: If $\{p_n(x)\}$ is a sequence of polynomials with generating function

$$e^{xz}\varphi(z\sqrt{1 - x^2}) = \sum_{n=0}^{\infty} p_n(x)z^n/n!,$$

$\varphi(u)$ even and expandible in TAYLOR series about $u = 0$, $\varphi(u) \neq 0$, then $p_n(x) = C_n F_n(x)$, $n = 0, 1, 2, \ldots$, C_n a constant, where $F_n(x)$ is the normalized ultraspherical polynomial of degree n and order λ.

It would be interesting to characterize the polynomials with generating function $e^{f(xz)}\varphi(z\sqrt{1 - x^2})$, where $f(u)$ is expandible in a TAYLOR series about the origin.

It is known that

$$\sum_{n=0}^{\infty} G_n(x)z^n/n! = \Gamma(\alpha + 1)e^z(xz)^{-\alpha/2} J_\alpha[2(xz)^{\frac{1}{2}}],$$

$$\alpha > -1,\ G_n(x) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0),$$

$L_n^{(\alpha)}(x)$ the LAGUERRE polynomial of degree n and order α [3]. It is also known that if $\{p_n(x)\}$ is a sequence of orthogonal polynomials such that $\Sigma p_n(x)\xi^n/n! = e^z\varphi(xz)$, then the $p_n(x)$ are essentially the LAGUERRE polynomials [4,2]. Furthermore, if $\{f_n(x)\}$ is a set of orthogonal polynomials such that $\Sigma f_n(x)t^n/n! = \Phi(2xt + t^2)$, then $f_n(x)$ is essentially either the ultraspherical or HERMITE polynomial [1].

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