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On the Ritt order of an entire Dirichlet series

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Summary. - *The theorem on Ritt order of an entire Dirichlet series is suitably extended to the case of functions of infinite order.*

Let $f(s) = \Sigma\, a_n e^{\lambda_n s}$, $s = \sigma + it$ and $\{\lambda_n\}$ be a stricty increasing sequence of positive numbers. If this series is assumed to be absolutely convergent for all values of s, then $f(s)$ is bounded in any left half and defines an entire function.

Let $M(\sigma) = \underset{-\infty<t<\infty}{l \cdot u \cdot b} \mid f(\sigma + it) \mid$. The RITT order ρ of $f(s)$, $0 \leq \rho \leq \infty$, is defined as

$$\limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \rho .$$

Improving upon a theorem of RITT ([5], p. 78) and his own result AZPEITIA ([1], p. 722) has proved recently ([2], p. 275) the following theorem.

THEOREM A. - If μ and δ are given by

$$\mu = \limsup_{n \to \infty} \frac{\lambda_n \log \lambda_n}{\log | a_n |^{-1}}, \qquad \delta = \limsup_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n}$$

then

(*i*) $0 \leq \mu \leq \rho \leq \infty$,

and if $\delta\mu < 1$,

(*ii*) $0 \leq \rho \leq \mu(1 - \delta\mu)^{-1}$, $0 \leq \delta < \infty$, $0 \leq \mu < \infty$.

Our aim in this note is to extend these results for functions of infinite order i.e. functions for which

$$\limsup_{\sigma \to \infty} \frac{\log \log M(\sigma)}{\sigma} = \infty, \qquad \limsup_{\sigma \to \infty} \frac{\log \log \log M(\sigma)}{\sigma} \leq \infty \text{ etc.}$$

Accordingly we define ρ_q, μ_q and δ_q by

$$\rho \equiv \rho_q = \limsup_{\sigma \to \infty} \frac{l_q M(\sigma)}{\sigma}, \qquad q > 2,$$

$$\mu \equiv \mu_q = \limsup_{n \to \infty} \frac{\lambda_n l_{q-1} \lambda_n}{\log |a_n|^{-1}},$$

$$\delta \equiv \delta_q = \limsup_{n \to \infty} \frac{\log n}{\lambda_n l_{q-1} \lambda_n}$$

where

$$l_q(x) = \log [l_{q-1} x], \qquad l_1 x = \log x.$$

It may be remarked here that an analogous result was recently proved by D. SATO ([3], Theorem 1) for entire functions $f(z)$ defined by their TAYLOR's series.

THEOREM B. - With ρ, μ and δ defined as above, we have (*i*) $0 \leq \mu \leq \rho \leq \infty$ and if $\delta\mu < 1$ (*ii*) $0 \leq \rho \leq \mu(1-\delta\mu)^{-1}$, $0 \leq \delta < \infty$, $0 \leq \mu < \infty$.

In particular when $\delta = 0$, $\rho = \mu$.

PROOF. - We shall assume that $\rho < \infty$ and $\mu > 0$, for otherwise the result is trivial. By the definition of μ, we have for $\varepsilon > 0$,

$$|a_v| > [l_{q-2}\lambda_v]^{-\lambda_v/(\mu-\varepsilon)} \tag{1}$$

where $\{v\}$ is a monotonic increasing sequence of positive integers. Since for all σ and all $n = 1, 2, 3, \ldots$ we know that ([4], p. 170)

$$M(\sigma) \geq |a_n| \exp(\sigma\lambda_n)$$

it follows that if we take ν such that

$$\exp[\sigma_\nu(\mu - \varepsilon)] = a l_{q-2}\lambda_\nu, \qquad a > 1,$$

then, using (1)

$$\begin{aligned} M(\sigma) &> (l_{q-2}\lambda_\nu)^{-\lambda_\nu/(\mu-\varepsilon)} \times (a l_{q-2}\lambda_\nu)^{\lambda_\nu/(\mu-\varepsilon)} \\ &= a^{\lambda_\nu/(\mu-\varepsilon)} \\ &= a^{e_{q-2}[\exp\{\sigma_\nu(\mu-\varepsilon)\}/a]/(\mu-\varepsilon)} \end{aligned}$$

where

$$\exp x = \exp(e_{p-1}x), \qquad e_1 = \exp x.$$

Thus

$$l_q M(\sigma_\nu)/\sigma_\nu \geq \mu - \varepsilon, \qquad \nu \to \infty$$

or

$$\rho \geq \mu - \varepsilon.$$

being arbitrary we get $\rho \geq \mu$ which proves (1) of Theorem B.
On the otherhand we have.

$$(2) \qquad M(\sigma) \leq \sum_{n=0}^{n_0} | a_n | \exp(\sigma\lambda_n) + \sum_{n=n_0+1}^{\infty} | a_n | \exp(\sigma\lambda_n).$$

Consider

$$G(x) = \sigma x - (\mu + \varepsilon)^{-1} x l_{q-1} x.$$

The maximum of this function is obtained for a value of $x = \lambda^*$, such that

$$q l_{q-2} x < \exp\{\sigma(\mu + \varepsilon)\} < p l_{q-2} x, \qquad q = 1 - \varepsilon_1, \; p = 1 + \varepsilon_1.$$

Then for an index m such that

$$\lambda_m = \max_{i \geq 1} \{\lambda_i / l_{q-1}\lambda_i < (1 - h)^{-1} \log(p l_{q-2}\lambda^*)\},$$

we have

$$(3) \qquad \sigma \leq (\mu + \varepsilon)^{-1} \log(p l_{q-2}\lambda^*) \leq (1 - h)(\mu + \varepsilon)^{-1} l_{q-1}\lambda_n, \quad \text{for all } n > m,$$

where

$$\delta(\mu + \varepsilon) < h < \delta(\mu + \varepsilon) + \varepsilon < 1.$$

Thus from (2) and (3) we obtain on account of the definition of m

$$M(\sigma) \leq \sum_{n=0}^{n_0} | a_n | \exp(\sigma\lambda_n) + m \exp[O(\sigma e_{k-2}(q^{-1} l^{\sigma'}))]$$
$$+ \sum_{n=m+1}^{\infty} \exp[- h(\mu + \varepsilon)^{-1}\lambda_n l_{q-1}\lambda_n]$$

where $\sigma' = \sigma(\mu + \varepsilon)$.

Now $h > (\mu + \varepsilon)\delta$ implies the convergence of the last series to a sum $S(\varepsilon)$. The first term is a polynomial in e^{σ}. Therefore it follows that

$$\log M(\sigma) \leq \log m + O[\sigma e_{k-2}(q^{-1} e^{\sigma'})].$$

For all large m,

$$\log m = \log m\, (\lambda_m l_{q-1}\lambda_m)^{-1}(\lambda_m l_{q-1}\lambda_m)$$
$$< (1 + \delta) e_{k-1}\{(1 - h)^{-1} \log(p l_{q-2}\lambda^*\} \times (1 - h)^{-1} \log(p l_{q-2}\lambda^*)$$
$$< (1 + \delta)(1 - h)^{-1} \log(p q^{-1} e^{\sigma'}) \times e_{k-1}[(1 - h)^{-1} \log(p q^{-1} e^{\sigma'})].$$

Hence

$$\log M(\sigma) < (1 + \delta)(1 - h)^{-1} e_{k-1}[(1 - h)^{-1} \log (pq^{-1} e^{\sigma'})] \times \log (pq^{-1} e^{\sigma'}) \{1 + 0(1)\}.$$

Thus

$$l_q M(\sigma)/\sigma \leq (1 - h)^{-1}(\mu + \varepsilon) + o(1) \leq (\mu + \varepsilon) \{1 - \delta(\mu + \varepsilon) - \varepsilon\}^{-1}.$$

ε being arbitrary,

$$\rho \leq \mu(1 - \delta\mu)^{-1}.$$

This proves *(ii)* of Theorem B.

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