BOLLETTINO UNIONE MATEMATICA ITALIANA

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Note on Schur's expansion of $\sin \pi x$.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **21** (1966), n.4, p. 353–357.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1966_3_21_4_353_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1966.

Note on Schur's expansion of $\sin \pi x$.

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Summary. - *An explicit formula is obtained for the coefficients in Schur's expansion*

$$\sin \pi x = \sum_{n=1}^{\infty} A_n (x(1-x))^n$$

1. The formula

$$\sin \pi x = \pi x(1-x) \prod_{n=1}^{\infty} \left(1 + \frac{x(1-x)}{n(n+1)}\right) \tag{1}$$

is proved in PÓLYA-SZEGÖ [5, p. 128, n. 227]. It follows from (1) that

$$\sin \pi x = \sum_{n=1}^{\infty} A_n (x(1-x))^n \tag{2}$$

and that

$$A_n > 0 \qquad (n = 1, 2, 3, ...)$$

This result is attributed to SCHUR.

It is evident from (1) that

$$A_1 = A_2 = \pi; \tag{3}$$

however it is not clear how a simple closed form can be obtained for A_n. Such a result can be derived in the following way. We recall (see for example 1, p. 203) that

$$\frac{x^{n+1} - y^{n+1}}{x - y} = \sum_{2r \leq n} (-1)^r \binom{n-r}{r} (x+y)^{n-2r} (xy)^r.$$

It follows that

$$\frac{x^{n+1} - (1-x)^{n-1}}{2x - 1} = \sum_{2r \leq n} (-1)^r \binom{n-r}{r} (x(1-x))^r. \tag{4}$$

Then by (4)

$$\frac{x\cos\pi x-(1-x)\cos\pi(1-x)}{2x-1}=\sum_{n=1}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}\,\frac{x^{2n+1}-(1-x)^{2n+1}}{2x-1}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}}{(2n)!}\sum_{r=0}^{n}(-1)^r\binom{2n-r}{r}(x(1-x))^r$$

$$=\sum_{r=0}^{\infty}(x(1-x))^r\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}.$$

Since $\cos\pi(1-x)=-\cos\pi x$, this reduces to

$$\cos\pi x=(2x-1)\sum_{r=0}^{\infty}(x(1-x))^r\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r},\tag{5}$$

so that

$$\sin\pi x=-\sum_{r=0}^{\infty}\frac{(x(1-x))^{r+1}}{r+1}\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}.\tag{6}$$

Comparing (6) with (2) we have

$$(r+1)A_{r+1}=-\pi\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}.\tag{7}$$

In the next place

$$\sum_{n=r}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}=\sum_{n=0}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}\tag{8}$$
$$-\sum_{n=0}^{r-1}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n-r}{r}=S-T,$$

say. Then

$$S=\sum_{n=0}^{\infty}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\sum_{s=0}^{r}\binom{2n}{s}\binom{-r}{r-s}$$

$$=\sum_{s=0}^{r}\binom{-r}{r-s}\sum_{2n\geq s}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n}{s}$$

$$=\sum_{2s\leq r}\binom{-r}{r-2s}\sum_{n\geq s}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n}{2s}$$

$$+\sum_{2s<r}\binom{-r}{r-2s-1}\sum_{n>s}(-1)^{n-r}\frac{\pi^{2n}}{(2n)!}\binom{2n}{2s+1}$$

$$=(-1)^r\sum_{2s\leq r}(-1)^s\binom{-r}{r-2s}\frac{\pi^{2s}}{(2s)!}\sum_{n=s}^{\infty}(-1)^{n-s}\frac{\pi^{2n-2s}}{(2n-2s)!}$$

$$+(-1)^r \sum_{2s<r} (-1)^s \binom{-r}{r-2s} \frac{\pi^{2s+1}}{(2s+1)!} \sum_{n=s+1}^{\infty} (-1)^{n-s} \frac{\pi^{2n-2s-1}}{(2n-2s-1)!}$$

$$= (-1)^{r+1} \sum_{2r\leq s} (-1)^s \binom{-r}{r-2s} \frac{\pi^{2s}}{(2s)!}.$$

On the other hand, since

$$T = \sum_{s=0}^{r-1} (-1)^{s-r} \frac{\pi^{2s}}{(2s)!} \binom{2s-r}{r}$$

and

$$\binom{2s-r}{r} = 0 \qquad (r < 2s < 2r),$$

it follows that

$$T = \sum_{2s\leq r} (-1)^{s-r} \frac{\pi^{2s}}{(2s)!} \binom{2s-r}{r} \qquad (r > 0).$$

Hence, provided $r > 0$,

(9) $$S - T = (-1)^{r+1} \sum_{2s\leq r} (-1)^s \frac{\pi^{2s}}{(2s)!} \left\{ \binom{-r}{r-2s} + \binom{2s-r}{r} \right\}.$$

Now it is easily verified that if $a \geq 0$, $b \geq 0$ and $a \equiv b \pmod 2$, then

$$\binom{-a}{b} + \binom{-b}{a} = (-1)^a \binom{a+b}{a}.$$

Indeed

$$\binom{-a}{b} + \binom{-b}{a} = (-1)^b \binom{a+b \quad 1}{b} + (-1)^a \binom{a+b-1}{b}$$

$$= (-1)^a \left\{ \binom{a+b-1}{a-1} + \binom{a+b-1}{a} \right\} = (-1)^a \binom{a+b}{a}.$$

Thus (9) becomes

(10) $$S - T = -\sum_{2s\leq r} (-1)^s \frac{\pi^{2s}}{(2s)!} \binom{2r-2s}{r} \qquad (r < 0).$$

Substituting from (8) and (10) in (7) we get

(11) $$(r+1)A_{r+1} = \pi \sum_{2s\leq r} (-1)^s \frac{\pi^{2s}}{(2s)!} \binom{2r-2s}{r}$$

which holds for $r \geq 0$. For example

$$A_2 = \pi, \qquad A_3 = \frac{\pi}{3}\left(6 - \frac{\pi^2}{2}\right), \qquad A_4 = \frac{\pi}{2}(10 - \pi^2),$$

$$A_5 = \frac{\pi}{5}\left(70 - \frac{15}{2}\pi^2 + \frac{\pi^4}{24}\right).$$

2. Logarithmic differentiation of (1) gives

$$\pi x(1-x)\cot \pi x = (1-2x)\left\{1 + \sum_{k=1}^{\infty}(-1)^{k-1}(x(1-x))^k \sigma_k\right\}, \tag{12}$$

where

$$\sigma_k = \sum_{n=1}^{\infty} \frac{1}{(n(n+1))^k}.$$

Put $w = x(1-x)$, so that

$$x = \frac{1}{2}(1 - \sqrt{1-4w}).$$

We recall [2, p. 101] that

$$x^n = w^n F\left(\frac{n}{2}, \frac{n+1}{2}; n+1; 4w\right) = \sum_{r=0}^{\infty} \frac{(n)_{2r}}{r!\,(n+1)!}.$$

Differentiating we get

$$x^n = (1-2x)\sum_{r=0}^{\infty} \frac{(n+2r+1)!}{r!\,(n+r)!} w^{n+r},$$

so that

$$(1-x)x^n = (1-2x)\sum_{r=0}^{\infty} \binom{n+2r-1}{r} w^{n+r}. \tag{13}$$

We recall also that

$$\pi x \cot \pi x = \sum_{n=0}^{\infty} (-1)^n \frac{(2\pi x)^{2n}}{(2n)!} B_{2n},$$

where B_{2n} denotes the BERNOULLI number in the even suffix notation. Then by (13)

$$\pi x(1-x)\cot \pi x = (1-2x)\sum_{n=0}^{\infty}(-1)^n \frac{(2\pi)^{2n}}{(2n)!} B_{2n} \cdot \sum_{r=0}^{\infty} \binom{2n+2r-1}{r} w^{2n+r}$$

$$= (1-2x)\sum_{k=0}^{\infty} w^k \sum_{2n \leq k} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} \binom{2k-2n-1}{k-2n} B_{2n}.$$

Comparison with (12) yields

$$\sigma_k = (-1)^{k-1} \sum_{2n \le k} (-1)^n \frac{(2\pi)^{2n}}{(2n)!} \binom{2k-2n-1}{k-1} B_{2n} \qquad (k \ge 1). \tag{14}$$

This result is due to GLAISHER [3]: it has been extended by P. K. MENON [4].

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*Pervenuta alla Segreteria dell'*U. M. I.
il 31 luglio 1965