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Partial sums of coefficients of well-poised hypergeometric series

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Summary. - *The sum of the first n terms of a ${}_5F_4(1)$ is obtained in terms of a terminating Saalschützian ${}_4F_3(1)$. This relation, which generalizes a result of Carlitz, is obtained as a special case of a well-known transformation for generalized hypergeometric series due to Whipple. Special cases of this formula are also discussed.*

1. We start with WHIPPLE's theorem on well-poised series [4]:

$$(1)\qquad {}_4F_3\left(\begin{matrix} t, & x, & y, & z \\ & u, & v, & w \end{matrix}\,\middle|\, 1\right) =$$

$$+\frac{\Gamma(v+w-t)\Gamma(1+x-u)\Gamma(1+y-u)\Gamma(1+z-u)}{\Gamma(1+y+z-u)\Gamma(1+z+x-u)\Gamma(1+x+y-u)\Gamma(1-u)}$$

$$\times\, {}_7F_6\left(\begin{matrix} a, & 1+\frac{1}{2}a, & w-t, & v-t, & x & y & z \\ \frac{1}{2}a, & v, & w, & 1+y+z-u, & 1+z+x-u, & 1+x+y-u \end{matrix}\,\middle|\, 1\right)$$

where $a = x+y+z-u$, $u+v+w-t-x-y-z=1$, and one of t, x, y, z is a negative integer.

This formula transforms a terminating Saalschützian ${}_4F_3(1)$ into a well-poised ${}_7F_6(1)$, and was used by BAILEY [1; 94] to find the sum of the first n terms of a Saalschützian ${}_3F_2(1)$.

Let

$${}_pF_q\left(\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix}\,\middle|\, x\right)_n = \sum_{r=0}^{n} \frac{(a_1)_r \cdots (a_p)_r}{r!\,(b_1)_r \cdots (b_q)_r}\, x^r.$$

Then setting $u = z+1$ and letting $x \longrightarrow -n$ where n is a positive integer, we obtain:

$${}_4F_3\left(\begin{matrix} -n, & y, & t, & z \\ & v, & w, & z+1 \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(v+w-t)\Gamma(y-z)\Gamma(z+1)\Gamma(n+1)}{\Gamma(y)\Gamma(n+z+1)\Gamma(y-z-n)}$$

$$\times\, {}_5F_4\left(\begin{matrix} a, & 1+\frac{1}{2}a, & w-t, & v-t, & z \\ & \frac{1}{2}a, & v, & w. & y-z-n \end{matrix}\,\middle|\, 1\right)_n,$$

where $a = y-n-1$, $v+w+n = y+t$.

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To preserve symmetry we write $b = w - t$, $c = v - t$, and $z = d$; we thus obtain $v = 1 + a - b$, $w = 1 + a - c$, and $y - z - n = 1 + a - d$; this yields:

$$(2)\qquad {}_5F_4\left(\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\middle| 1\right)_n$$
$$= \frac{(1+a)_n(1+d)_n}{n!\,(1+a-d)_n}\,{}_4F_3\left(\begin{matrix} -n, & 1+a+n, & 1+a-b-c, & d \\ & 1+a-b, & 1+a-c, & d+1 \end{matrix}\middle| 1\right).$$

We have thus expressed the first $n+1$ terms of a well-poised ${}_5F_4(1)$, with the second parameter in a special form, in terms of a terminating Saalschützian, and we note that the initial conditions are automatically satisfied.

2. We now apply a relation between terminating Saalschützian ${}_4F_3(1)$, given in [1; 56], to obtain a more elegant version of (2). The result is:

$$(3)\qquad {}_5F_4\left(\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\middle| 1\right)_n$$
$$= \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!\,(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$
$$\times\,{}_4F_3\left(\begin{matrix} -n, & 1+a+n, & b+c+d-a, & 1 \\ & 1+b, & 1+c, & 1+d \end{matrix}\middle| 1\right).$$

If $d = \frac{1}{2} + \frac{1}{2}a$ and $d = \frac{1}{2}a$ in (3), we obtain respectively:

$$(4)\qquad {}_4F_3\left(\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c \end{matrix}\middle| 1\right)_n$$
$$= \left(1 + \frac{2n}{a+1}\right)\frac{(1+a)_n(1+b)_n(1+c)_n}{n!\,(1+a-b)_n(1+a-c)_n}$$
$$\times\,{}_4F_3\left(\begin{matrix} -n, & 1+a+n, & \frac{1}{2}+b+c-\frac{1}{2}a, & 1 \\ & 1+b, & 1+c, & \frac{3}{2}+\frac{1}{2}a \end{matrix}\middle| 1\right),$$

$$(5)\qquad {}_3F_2\left(\begin{matrix}a, & b, & c\\ & 1+a-b, & 1+a-c\end{matrix}\middle|\,1\right)_n=\frac{(1+a)_n(1+b)_n(1+c)_n}{n!\,(1+a-b)_n(1+a-c)_n}$$

$$\times\,{}_4F_3\left(\begin{matrix}-n, & 1+a+n, & b+c-\frac{1}{2}a, & 1\\ & 1+b, & 1+c, & 1+\frac{1}{2}a\end{matrix}\middle|\,1\right),$$

and if $c=\frac{1}{2}+\frac{1}{2}a$, (5) reduces to

$$(6)\qquad {}_2F_1\left(\begin{matrix}a, & b\\ 1+a-b\end{matrix}\middle|\,1\right)_n=\left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n(1+b)_n}{n!\,(1+a-b)_n}$$

$$\times\,{}_4F_3\left(\begin{matrix}-n, & 1+a+n, & \frac{1}{2}+b, & 1\\ & 1+b, & \frac{3}{2}+\frac{1}{2}a, & 1+\frac{1}{2}a\end{matrix}\middle|\,1\right).$$

Now if we let $d\longrightarrow\infty$, (3) becomes:

$$(7)\qquad {}_4F_3\left(\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c\\ & \frac{1}{2}a, & 1+a-b, & 1+a-c\end{matrix}\middle|\,-1\right)_n$$

$$=(-1)^n\frac{(1+a)_n(1+b)_n(1+c)_n}{n!\,(1+a-b)_n(1+a-c)_n}\,{}_3F_2\left(\begin{matrix}-n, & 1+a+n, & 1\\ & 1+b, & 1+c\end{matrix}\middle|\,1\right),$$

and we deduce that:

$$(8)\qquad {}_3F_2\left(\begin{matrix}a, & 1+\frac{1}{2}a, & b\\ & \frac{1}{2}a, & 1+a-b\end{matrix}\middle|\,-1\right)_n=(-1)^n\left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n(1+b)_n}{n!\,(1+a-b)_n}$$

$$\times\,{}_3F_2\left(\begin{matrix}-n, & 1+a+n, & 1\\ & 1+b & \frac{3}{2}+\frac{1}{2}a\end{matrix}\middle|\,1\right).$$

$$(9)\qquad {}_2F_1\left(\begin{matrix}a, & b\\ 1+a-b\end{matrix}\middle|\,-1\right)_n$$

$$=(-1)^n\frac{(1+a)_n(1+b)_n}{n!\,(1+a-b)_n}\,{}_3F_2\left(\begin{matrix}-n, & 1+a+n, & 1\\ & 1+b, & 1+\frac{1}{2}a\end{matrix}\middle|\,1\right).$$

Finally, it follows from (9) that

$$(10)\quad {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}\middle| -1\right)_n = (-1)^n\left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n}{n!}\,{}_3F_2\left(\begin{matrix} -n,\ 1+a+n,\ 1 \\ \frac{3}{2}+\frac{1}{2}a,\ 1+\frac{1}{2}a \end{matrix}\middle| 1\right).$$

If we let $d \longrightarrow \infty$ in (2) we obtain a formula, equivalent to (7), which has already been given by BAILEY [2; 516]. He calls it a «curious result».

3. Starting from the formula (3), we choose the parameters so that the series on the right-hand side can be summed, and we use SAALSCHÜTZ's theorem [1; 9] in the form

$$(11)\qquad {}_3F_2\left(\begin{matrix} -n,\ a+n,\ b \\ c,\ 1+a+b-c \end{matrix}\middle| 1\right) = \frac{(c-b)_n(1+a-c)_n}{(c)_n(1+a+b-c)_n}.$$

From (3), we have

$$(12)\qquad {}_5F_4\left(\begin{matrix} a,\ 1+\frac{1}{2}a, & b, & c, & d \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\middle| 1\right)_n = \frac{(1+a)_n(1+b)_n(1+c)_n(1+d)_n}{n!\,(1+a-b)_n(1+a-c)_n(1+a-d)_n}$$

provided that $a = b+c+d$. This result has recently been obtained by CARLITZ [3].

Using (11), we also have

$$(13)\qquad {}_5F_4\left(\begin{matrix} a,\ 1+\frac{1}{2}a, & b, & c, & d \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d \end{matrix}\middle| 1\right)_n = \frac{(1+a)_n(1+b)_n}{n!\,(1+a-b)_n}$$

provided that $a = c+d-1$; but (13) reduces immediately to:

$$(14)\qquad {}_3F_2\left(\begin{matrix} a,\ 1+\frac{1}{2}a, & b \\ \frac{1}{2}a, & 1+a-b \end{matrix}\middle| 1\right)_n = \frac{(1+a)_n(1+b)_n}{n!\,(1+a-b)_n}$$

with no restriction on a.

From (4), we find that

$$(15)\qquad {}_4F_3\left(\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & 1+a-b, & 1+a & c \end{matrix}\middle| 1\right)_n = \left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n(1+b)_n(1+c)_n}{n!\,(1+a-b)_n(1+a-c)_n}$$

provided that $a = 1 + 2(b + c)$.

From (5), we have

$$(16)\qquad {}_3F_2\left(\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix}\middle| 1\right)_n = \frac{(1+a)_n(1+b)_n(1+c)_n}{n!\,(1+a-b)_n(1+a-c)_n}$$

with $a = 2(b + c)$.

If $b = -\frac{1}{2}$ then (6) reduces to

$$(17)\qquad {}_2F_1\left(\begin{matrix} a, & -\frac{1}{2} \\ \frac{3}{2}+a & \end{matrix}\middle| 1\right)_n = \left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n\left(\frac{1}{2}\right)_n}{n!\left(\frac{3}{2}+a\right)_n}$$

while, for $b = 1 + \frac{1}{2}a$ and $b = \frac{1}{2} + \frac{1}{2}a$, we obtain

$$(18)\qquad {}_2F_1\left(\begin{matrix} a, & 1+\frac{1}{2}a \\ & \frac{1}{2}a \end{matrix}\middle| 1\right)_n = \left(1+\frac{2n}{a+1}\right)\frac{(1+a)_n}{n!}$$

and

$$(19)\qquad {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}\middle| 1\right)_n = \frac{(1+a)_n}{n!},$$

by using SAALSCHÜTZ's theorem.

Also from (7), with $c = 1 + a - b$, or from either (8), with $b = \frac{1}{2} + \frac{1}{2}a$, or (9), with $b = 1 + \frac{1}{2}a$, we find that

$$(20)\qquad {}_2F_1\left(\begin{matrix} a, & 1+\frac{1}{2}a \\ & \frac{1}{2}a \end{matrix}\middle| -1\right)_n = (-1)^n\frac{(1+a)_n}{n!}.$$

Finally we note that the well-poised ${}_5F_4(1)$ on the left-hand side of (3) can be summed, when $n \to \infty$, by a formula in BAILEY's tract [1; 27]. But this series converges only when $1+a>b+c+d$ so that when this condition holds the limit of the right-hand side is:

$$\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-d)\Gamma(1+a-b-c)\Gamma(1+a-c-d)}.$$

when $n \to \infty$.

4. There exist well-known formulas which express the sum of n terms of an ordinary hypergeometric series with unit argument in terms of an infinite series of the type ${}_3F_2(1)$.

From. Eq. (2) of [1; 93], for example. with $f = 1 + a - b$ and n replaced by $n+1$, we have

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}a, & b \\ & 1+a-b\end{matrix}\middle| 1\right)_n \\ &= \frac{\Gamma(1+a+n)\Gamma(1+b+n)}{n!\,\Gamma(1+a+b+n)}\,{}_3F_2\left(\begin{matrix}a, & b & n+a-b+1 \\ & 1+a-b, & n+a+b+1\end{matrix}\middle| 1\right).\end{aligned}$$

A comparison of this relation with our formula (6) yields an interesting relation between two Saalschützian series, namely

$$\begin{aligned}(21)\qquad &{}_3F_2\left(\begin{matrix}a, & b & 1+a-b+n \\ & 1+a-b, & 1+a+b+n\end{matrix}\middle| 1\right) \\ &= \left(1+\frac{2n}{a+1}\right)\frac{\Gamma(1+a-b)\Gamma(1+a+b+n)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+a-b+n)} \\ &\times {}_4F_3\left(\begin{matrix}-n, & 1+a+n & \frac{1}{2}+b, & 1 \\ & 1+b, & \frac{3}{2}+\frac{1}{2}a, & 1+\frac{1}{2}a\end{matrix}\middle| 1\right).\end{aligned}$$

The series ${}_4F_3$ terminates while the series ${}_3F_2$ does not.

For $b = -\frac{1}{2}$, the ${}_4F_3(1)$ reduces to unity and we have

$$(22)\qquad {}_3F_2\left(\begin{matrix}a, & -\frac{1}{2}, & \frac{3}{2}+a+n \\ \frac{3}{2}+a, & \frac{1}{2}+a+n\end{matrix}\middle| 1\right) = \frac{\left(1+\frac{2n}{a+1}\right)\Gamma\left(a+\frac{1}{2}\right)}{\left(1+\frac{2n}{2a+1}\right)\Gamma\left(\frac{1}{2}\right)\Gamma(a+1)},$$

while if $b = 1 + \frac{1}{2}a$, and with the aid of SAALSCHÜTZ's theorem we find that

$$(23)\qquad {}_3F_2\left(\begin{matrix} a, & 1+\frac{1}{2}a, & n+\frac{1}{2}a \\ & \frac{1}{2}a. & n+\frac{3}{2}a+2 \end{matrix}\middle|\, 1\right)$$

$$= \left(1 + \frac{2n}{a+1}\right)\frac{\Gamma\left(n+\frac{3}{2}a+2\right)}{\Gamma(a+1)\Gamma\left(n+\frac{1}{2}a+2\right)}.$$

We have thus obtained the sum of two particular, non-terminating, Saalschützian series of the type ${}_3F_2(1)$, which are, at the same time, nearly-poised series of the second kind.

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