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SEZIONE STORICO-DIDATTICA

On the inversion of the order of integration

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Summary. - *A formula for the inversion of the order of integration between a real improper integral and a complex integral is extended so as to cover the case where a factor of the integrand is not continuous, yet integrable.*

In this note we estabish conditions under which the formula for the interchange of the order of integration

$$\int_C dz \int_a^\infty f(z,\ t)\Phi(t)dt = \int_a^\infty \Phi(t)dt \int_C f(z,\ t)dz$$

is valid, f being continuous with respect to the complex variable z along the smooth arc C, and with respect to the real variable t on every interval $[a, b]$ $(a < b)$, while Φ is only assumed to be RIEMANM integrable on every $[a, b]$. In some treatises on complex analysis this formula is established under the more restrictive assumption that $f(z, t)\Phi(t) = F(z, t)$ is continuous in z and in t. See reference 2, p. 97.

1. LEMMA. - *Let $f(x, y)$ be a continuous function of x and y on $a \leq x \leq b$, $c \leq y \leq d$, and suppose that $\Phi(x)$ is integrable on $[a, b]$. Then*

$$\int_{y_0}^{y_1} dy \int_a^b f(x,\ y)\Phi(x)dx = \int_a^b \Phi(x)dx \int_{y_0}^{y_1} f(x,\ y)dy \tag{1}$$

where y_0, y_1 are any two points in $[c, d]$.

PROOF. - See reference 1, p. 191.

2. THEOREM 1. - *Suppose that:*

1) *For each value of the real variable t on $[a, b]$ the function $f(z, t)$ is a continuous function of the complex variable z along the smooth arc* $C: z = z(\lambda) = x(\lambda) + iy(\lambda)$ $(\alpha \leq \lambda \leq \beta)$.

2) *For each value of $z \in C$, $f(z, t)$ is continuous with respect to t on $[a, b]$.*

3) *The function $\Phi(t)$ is integrable on $[a, b]$.*

Then

$$\int_C dz \int_a^b f(z, t)\Phi(t)dt = \int_a^b \Phi(t)dt \int_C f(z, t)dz \tag{2}$$

PROOF. - Let

$$f(z(\lambda), t) = u(x(\lambda), y(\lambda), t) + iv(x(\lambda), y(\lambda), t) = u + iv$$
$$dz = [x'(\lambda) + iy'(\lambda)]d\lambda = (x' + iy')d\lambda.$$

Then (2) becomes

$$\int_\alpha^\beta (x' + iy')d\lambda \int_a^b (u + iv)\Phi(t)dt = \int_a^b \Phi(t)dt \int_\alpha^\beta (u + iv)(x' + iy')d\lambda.$$

Each side can be separated into four real integrals, and corresponding integrals on the two sides are equal by the Lemma. For instance, we have

$$\int_\alpha^\beta x'(\lambda)d\lambda \int_a^b u(x(\lambda), y(\lambda), t)\Phi(t)dt = \int_a^b \Phi(t)dt \int_\alpha^\beta x'(\lambda)u(x(\lambda), y(\lambda), t)d\lambda$$

and similarly for the remaining pairs.

THEOREM 2. - *If* $f(z, t)$ and $\Phi(t)$ *satisfy the conditions of Theorem* 1 *for each interval* $[a, b]$ $(a < b)$, *and if in addition the integral*

$$\int_a^\infty f(z, t)\Phi(t)dt$$

converges uniformly for $z \in C$, then

$$\int_C dz \int_a^\infty f(z, t)\Phi(t)dt = \int_a^\infty \Phi(t)dt \int_C f(z, t)dz.$$

PROOF. - Given $\varepsilon > 0$ we have

$$\left| \int_b^\infty f(z, t)\Phi(t)dt \right| < \varepsilon$$

for b large enough, say $b > b_1$, and for all $z \in C$. Hence, by applying Theorem 1 we obtain, for $b > b_1$,

$$\left| \int_C dz \int_a^\infty f(z, t)\Phi(t)dt - \int_a^b \Phi(t)dt \int_C f(z, t)dz \right|$$

$$= \left| \int_C dz \int_a^\infty f(z, t)\Phi(t)dt - \int_C dz \int_a^b f(z, t)\Phi(t)dt \right|$$

$$= \left| \int_C dz \int_b^\infty f(z, t)\Phi(t)dt \right| < \varepsilon L(C),$$

where $L(C)$ denotes the lenght of C. This shows that (3) holds.

3. EXAMPLE. - Let $f(z, t) = 1/(t+z)^2$, $\Phi(t) = P_1(t) = t - [t] - \frac{1}{2}$. and let C be the line segment connecting 1 and z, where $z + |z| \neq 0$. It is not difficult to see that

$$\int_0^\infty \frac{P_1(t)dt}{(t+z)^2}$$

converges uniformly for $z \in C$. Since the other conditions in Theorem 2 are obviously satisfied, we have

$$\int_1^z dz \int_0^\infty \frac{P_1(t)dt}{(t+z)^2} = \int_0^\infty P_1(t)dt \int_1^z \frac{dz}{(t+z)^2} = (z-1)\int_0^\infty \frac{P_1(t)dt}{(t+1)(t+z)}.$$

REFERENCES

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[2] G. SANSONE - J. GERRETSEN, *Lectures on the theory of functions of a complex variable*, P. Noordhoff, 1960.

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