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Combinatorial equivalents of certain theta formulas.

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Summary. - *The paper contains a combinatorial theorem equivalent to the theta formula* (5) *below.*

Put

$$\prod_{n=1}^{\infty}(1+x^{n}y^{n-1})(1+x^{n-1}y^{n})=\sum_{m,n=0}^{\infty}\alpha(m,\ n)x^{m}y^{n}, \tag{1}$$

so that $\alpha(m, n)$ is the number of partitions of (m, n) into distinct parts

$$(a,\ a-1),\quad (b-1,\ b)\qquad (a,\ b=1,\ 2,\ 3,\ \ldots). \tag{2}$$

The writer [1] showed that the JACOBI theta formola

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n+1}t)(1+q^{2n-1}t^{-1})+\sum_{n=-\infty}^{\infty}q^{n^2}t^{n} \tag{3}$$

is equivalent to

$$\alpha(m,\ n)=p\left\{n-\frac{1}{2}(n-m)(n-m+1)\right\}, \tag{4}$$

where $p(n)$ denotes, as usual, the number of unrestricted partitions of n. E. M. WRIGHT [4] has given a combinatorial proof of (4); CHEEMA [2] has given a combinatorial proof of a different formula that is also equivalent to (3).

It may be of interest to state combinatorial equivalents of other identities involving theta functions. For simplicity and concreteness we shall limit ourselves to the following identity [3, p. 468]:

$$\{3\}+\{4\}=\{3\}'+\{4\}', \tag{5}$$

where

$$\{r\}=\vartheta_r(w)\vartheta_r(x)\vartheta_r(y)\vartheta_r(z),$$

$$\{r\}'=\vartheta_r(w')\vartheta_r(x')\vartheta_r(y')\vartheta_r(z'),$$

and

(6)
$$\begin{cases} 2w' = -w + x + y + z \\ 2x' = w - x + y + z \\ 2y' = w + x - y + z \\ 2z' = w + x + y - z. \end{cases}$$

We recall that

$$\vartheta_3(x) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2ix})(1 + q^{2n-1}e^{-2ix}),$$

$$\vartheta_4(x) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1}e^{2ix})(1 - q^{2n-1}e^{-2ix}).$$

If we put

(7)
$$a = e^{2iw}, \quad b = e^{2ix}, \quad c = e^{2iy}, \quad d = e^{2iz},$$

(3) and (4) become

$$\prod_{n=1}^{\infty} \{(1 - q^{2n})^4 \prod_{a,b,c,d} (1 + q^{2n-1}a)(1 + q^{2n-1}a^{-1})\}$$

and

$$\prod_{n=1}^{\infty} \{(1 - q^{2n})^4 \prod_{a,b,c,d} (1 - q^{2n-1}a)(1 - q^{2n-1}a^{-1})\},$$

respectively. Thus (5) becomes

(8)
$$\begin{aligned} &\prod_{n=1}^{\infty} \prod_{a,b,c,d} (1 + q^{2n-1}a)(1 + q^{2n-1}a^{-1}) + \\ &+ \prod_{n=1}^{\infty} \prod_{a,b,c,d} (1 - q^{2n-1}a)(1 - q^{2n-1}a^{-1}) = \\ &= \prod_{n=1}^{\infty} \prod_{a,b,c,d} (1 + q^{2n-1}a')(1 + q^{2n-1}a'^{-1}) + \\ &+ \prod_{n=1}^{\infty} \prod_{a,b,c,d} (1 - q^{2n-1}a')(1 - q^{2n-1}a'^{-1}), \end{aligned}$$

where

(9)
$$a' = e^{2iw'}, \quad b' = e^{2ix'}, \quad c' = e^{2iy'}, \quad d' = e^{2iz'}.$$

We now put

(10)
$$q = xx_0x_1x_2x_3x_4$$

and

(11) $a = x_0 x_1 x_2^{-1}, \quad b = x_0 x_2 x_3^{-1}, \quad c = x_0 x_3 x_4^{-1}, \quad d = x_0 x_4 x_1^{-1}.$

Then by (6), (7), (9) and (11) we have

(12) $a' = x_0 x_1^{-1} x_2, \quad b' = x_0 x_2^{-1} x_3, \quad c' = x_0 x_3^{-1} x_4, \quad d' = x_0 x_4^{-1} x_1.$

Substituting from (10) and (11), the left member of (8) becomes

$$(13) \quad \prod_{n=1}^{\infty} \prod_{1,2,3,4} (1 + x^{2n-1} x_0^{2n} x_1^{2n} x_2^{2n-2} x_3^{2n-1} x_4^{2n-1}) (1 + x^{2n-1} x_0^{2n-2} x_1^{2n-2} x_2^{2n} x_3^{2u-1} x_4^{2n-1})$$
$$+ \prod_{n=1}^{\infty} \prod_{1,2,3,4} (1 - x^{2n-1} x_0^{2n} x_1^{2n} x_2^{2n-2} x_3^{2n-1} x_4^{2n-1}) (1 - x^{2n-1} x_0^{2n-2} x_1^{2n-2} x_2^{2n-1} x_3^{2n-1} x_4^{2n-1}),$$

where $\prod_{1,2,3,4}$ indicates that the product is extended over the cyclic permutations of 1, 2, 3, 4.

Similarly, by (12), the right member of (8) becomes

$$(14) \quad \prod_{n=1}^{\infty} \prod_{1,2,3,4} (1 + x^{2n-1} x_0^{2n} x_1^{2n-2} x_2^{2n} x_3^{2n-1} x_4^{2n-1}) (1 + x^{2n-1} x_0^{2n-2} x_1^{2n} x_2^{2n-2} x_3^{2n-1} x_4^{2n-1})$$
$$+ \prod_{n=1}^{\infty} \prod_{1,2,3,4} (1 - x^{2n-1} x_0^{2n} x_1^{2n-2} x_2^{2n} x_3^{2n-1} x_4^{2n-1} (1 - x^{2n-1} x_0^{2n-2} x_1^{2n} x_2^{2n-2} x_3^{2n-1} x_4^{2n-1}).$$

If we put

$$(15) \quad \prod_{n=1}^{\infty} \prod_{1,2,3,4} (1 + x^{2n-1} x_0^{2n} x_1^{2n} x_2^{2n-2} x_3^{2n-1} x_4^{2n-1}) (1 + x^{2n-1} x_0^{2n-2} x_1^{2n-2} x_2^{2n} x_3^{2n-1} x_4^{2n-1})$$
$$= \sum_{n, n_0, \ldots, n_4 = 0}^{\infty} \alpha(n, n_0, n_1, n_2, n_3, n_4) x^n x_0^{n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4},$$

it is clear that $\alpha(n, n_0, n_1, n_2, n_3, n_4)$ is the number of partitions of $(n, n_0, n_1, n_2, n_3, n_4)$ into distinct parts of the following kind:

$$(16) \quad \begin{cases} (u, u+\epsilon, u+\epsilon, u \quad \epsilon, u, u), & (u, u+\epsilon, u, u+\epsilon, u-\epsilon, u), \\ (u, u+\epsilon, u, u, u+\epsilon, u-\epsilon), & (u, u+\epsilon, u-\epsilon, u, u, u+\epsilon), \end{cases}$$

where $u = 1, 3, 5, \ldots$ and $\epsilon = \pm 1$. Thus (13) becomes

$$(13)' \quad 2 \sum_{n, n_0, \ldots, n_4 = 0}^{\infty} \alpha(2n, n_0, n_1, n_2, n_3, n_4) x^{2n} x_0^{n_0} \ldots x_4^{n_4}.$$

On the other hand, if we put

$$(17)\qquad \prod_{n=1}^{\infty}\prod_{1,2,3,4}(1+x^{2n-1}x_0^{2n}x_1^{2n-2}x_2^{2n}x_3^{2n-1}x_4^{2n-1})$$
$$(1+x^{2n-1}x_0^{2n-2}x_1^{2n}x_2^{2n-2}x_3^{2n-1}x_4^{2n-1})$$
$$=\sum_{n,n_0,\dots,n_4=0}^{\infty}\beta(n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)x^n x_0^{n_0}x_1^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4},$$

it is clear that $\beta(n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)$ is the number of partitions of $(n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)$ into distinct parts of the following kind:

$$(18)\qquad \begin{cases}(u,\ u-\epsilon,\ u+\epsilon,\ u-\epsilon,\ u,\ u), & (u,\ u\ \ \epsilon,\ u,\ u+\epsilon,\ u-\epsilon,\ u),\\ (u,\ u-\epsilon,\ u,\ u,\ u+\epsilon,\ u-\epsilon), & (u,\ u-\epsilon,\ u-\epsilon,\ u,\ u,\ u+\epsilon),\end{cases}$$

where $u = 1,\ 3,\ 5,\ \dots$ and $\epsilon = \pm 1$.

Note that in comparing (16) and (18) the only change is the sign of ϵ in the second blank.

Thus (14) becomes

$$(14)'\qquad 2\sum_{n,n_0,\dots,n_4=0}^{\infty}\beta(2n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)x^{2n}x_0^{n_0}\dots x_4^{n_4}.$$

Comparing (14)′ with (13)′ we get

$$(19)\qquad \alpha(2n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)=\beta(2n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4),$$

that is, the number of partitions of

$$(2n,\ n_0,\ n_1,\ n_2,\ n_3,\ n_4)$$

into distinct parts (18) is equal to the number of partitions into distinct parts (16).

If in place of (10) and (11) we take

$$q = x_0x_1x_2x_3x_4$$

and

$$a = x_1^{-1}x_2x_3x_4,\quad b = x_1x_2^{-1}x_3x_4,\quad c = x_1x_2x_3^{-1}x_4,\quad d = x_1x_2x_3y_4^{-1},$$

which implies

$$a' = x_1^2,\quad b' = x_2^2,\quad c' = x_3^2,\quad d' = {}_4^2,$$

we get another result like (16) concerning partitions of vectors $(n,\ n_1,\ n_2,\ n_3,\ n_4)$; however the «parts» are less symmetrical than (16) and (18).

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