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# On certain non-linear differential equations

G. G. LEGATOS (in Athens)

**Sunto.** - *Si consideri l'equazione differenziale:*

(*) $$\ddot{x} + (f(x) + g(x)\ddot{x})\dot{x} + p(t)h(x) = e(t)$$

H. A. ANTOSIEWICZ [*1, Th. 2*] *ha dimostrato che, sotto certe condizioni per le funzioni f, g, h, e e* $p(t) \equiv 1$, *ogni soluzione della* (*) *è limitata con la sua derivata prima. Una generalizzazione di questo risultato fu fatta dall'autore* [*2, Teor. 2*] *ed un'altra da* LIANG ZONG-CHAO [*3, Th. 7.1, 7.2, 7.3*]. *Nella presente nota si dimostrano certi teoremi che generalizzano teoremi corrispondenti in* [*1*], [*2*], [*3*].

**Abstract.** - *Consider the differential equation:*

(*) $$\ddot{x} + (f(x) + g(x)\ddot{x})\dot{x} + p(t)h(x) = e(t)$$

H. A. ANTOSIEWICZ [*1, Th. 2*] *proved that, under certain conditions for f, g, h, e and* $p(t) = 1$, *every solution of* (*) *as well as its derivative are bounded functions. A generalization of this result has been given by the author* [*2, Th. 1,2*] *and another by* LIANG ZONG CHAO [*3, Th. 7.1, 7.2, 7.3*]. *In the present paper we prove certain theorems which extend corresponding theorems in* [*1*], [*2*], [*3*].

**1.** Consider the differential equation

(1) $$\dddot{x} + (f(x) + g(x)\dot{x})\dot{x} + p(t)h(x) = e(t)$$

such that : *a*) $f$, $g$, $h$, are defined in $\mathbb{R} = (-\infty, +\infty)$ and $p$, $e$ in an interval $I = [t_0, +\infty)$ and are real-valued and continuous.

b) There exist solutions of (1), which are defined in $I$.

c) $p(t) > 0$ and there exists the derivative $\dot{p}(t)$, $t \in I$. We introduce the following notation:

$$a = a(x) = \exp \int_0^x g(u)du$$

$b = b(x)$. $x \in \mathbb{R}$ is a real-valued function whose derivative $b'(x)$, $x \in \mathbb{R}$ exists

$$T = T(x) = a(x)f(x) - b'(x)$$

$$F = F(t,\ x) = a^2(x)h(x) + \frac{b(x)}{p(t)}\, T(x),$$

$$H = H(t,\ x) = \int_0^x F(t,\ u)du$$

$$V = V(t,\ x,\ y) = \left(\frac{y^2}{2p(t)} + H(t,\ x)\right)^{1/2}$$

We now prove the following.

LEMMA 1. *If* $\{ x(t),\ y(t) \}$, $t \in I$ is a solution of the system:

$$\dot{x} = \frac{y - b}{a}\ ,\ \dot{y} = -T\dot{x} - aph + ae \tag{2}$$

then we have:

$$2w\dot{w} = -\frac{T}{ap}(y-b)^2 - abh - \frac{\dot{p}}{2p^2}y^2 - \frac{\dot{p}}{p^2}\int_0^x b(u)T(u)du + \frac{aey}{p}\ ,\ t \in I \tag{3}$$

where: $w(t) = V(t,\ x(t),\ y(t))$.

PROOF. - By differentiation of $w^2(t) = \dfrac{y^2(t)}{2p(t)} + H(t,\ x(t))$ we get:

$$2w\dot{w} = \frac{y}{p}\,\dot{y} - \frac{\dot{p}}{2p^2}\,y^2 + F\dot{x} + \int_0^x \frac{\partial F(t,\ u)}{\partial t}\, du,$$

so that, by (2), the relation (3) is proved.

We are now in the position to prove the following theorems concerning equation (1):

THEOREM 1. - Let the equation (1) satisfy the conditions

i) $\dot{p}(t) \geqq 0,\ t \in I,$

ii) $\dot{p}(t)\displaystyle\int_0^x b(u)T(u)du \geqq 0,\ (t,\ x) \in I x \mathbb{R},$

iii) $T(x) \geqq 0,\ x \in \mathbb{R},$

iv) there exists a constant $A$ with $a(x) \leqq A,\ x \in \mathbb{R}$, $v)$ $\dot{b}(x)h(x) \geqq 0,\ x \in \mathbb{R},$

vi) there exists a function $H^*(x),\ x \in \mathbb{R}$ with: $0 < H^*(x) \leqq$

$\leq H(t, x), (t, x) \in I \times R$ and $\lim_{|x| \to +\infty} H^*(x) = +\infty$,

vii) $\int_{to}^{+\infty} p^{-\frac{1}{2}}(t) \,|\, e(t) \,|\, dt < +\infty$,

viii) there exists a constant $m > 0$ with $p(t) \leq m$, $t \in I$.

Then, by i)–vii), every solution $x(t)$, $t \in I$ of (1) is bounded.

If, moreover, viii) holds, then also the derivative $\dot{x}(t)$, $t \in I$ is bounded.

PROOF. – The substitution $\dot{x} = \frac{y - b}{a}$ reduces the equation (1) to the system (2). If $\{x = x(t),\ y = y(t)\}$, $t \in I$ is a solution of (1), then, by LEMMA 1, relation (3) is valid and hence, applying i)–vi), we have:

$$2w\dot{w} \leq \sqrt{2} A p^{-\frac{1}{2}} \,|\, e \,|\, w, \ t \in I,$$

so that we can easily verify that the relation

$$\dot{w} \leq \frac{A}{\sqrt{2}} p^{-\frac{1}{2}} \,|\, e \,|\,, \ t \in I$$

holds Thus, using vii), we have:

(4) $$w(t) = V(t, x(t), y(t)) \leq C, \quad t \in I, \ C = \text{constant}.$$

By vi) we immediately get that $x(t)$, $t \in I$ is bounded, and recalling viii) and (3) we conclude that the derivative $\dot{x}(t)$, $t \in I$ is also bounded. This completes the proof.

The above theorem for $p(t) = 1$ is the Th. 1 of [2] and for $b(0) = 0$, $T(x) = 0$ is the Th. 7.1 of [3].

THEOREM 2. – *Let the equation* (1) *satisfy the conditions*

i) $\int_0^x a^2(u) h(u) du \geq 0$, $x \in \mathbb{R}$,

ii) $\int_{to}^{+\infty} \frac{|\dot{p}(t)|}{p(t)} dt < +\infty$ *and*

iii)–viii) *of TH.* 1. *Then we get the conclusion of TH.* 1.

PROOF. - As in TH. 1, using LEMMA 1, we obtain the relation (3), from this and the conditions iii)-v) we have:

$$2w\dot{w} \leqq \frac{|\dot{p}|}{p}\left(w^2 - \int_0^x a^2(u)h(u)du\right) + \sqrt{2}Ap^{-\frac{1}{2}}|e|w, \; t \in I.$$

Thus, by i),

$$2w\dot{w} \leqq \frac{|\dot{p}|}{p} w^2 + \sqrt{2}Ap^{-\frac{1}{2}}|e|w, \; t \in I,$$

so that the relation (4) of TH. 1 now follows, by ii) and vii). The proof proceeds further as in TH. 1.

The above theorem for $p(t) = 1$ is the TH. 1 of [2] and for $b(0) = 0$, $T(x) = 0$ is the TH. 7.2 of [3].

THEOREM 3. - *Let the equation* (1) *satisfy the condition* $p(t) \geqq \geqq c > 0$, $t \in I$, $c =$ *constant and* $\dot{p}(t) \leqq 0$, $t \in I$ *as well as the conditions* ii)-vii) *of TH.* 1. *Then every solution* $x(t)$, $t \in I$ *of* (1) *and its derivative are bounded.*

PROOF. - We easily get the relation:

$$w\dot{w} \leqq -\frac{\dot{p}}{p} \cdot \frac{y^2}{2p} + \frac{1}{\sqrt{2}} Ap^{-\frac{1}{2}}|e|w, \; t \in I.$$

Thus, recalling also i), we find:

$$\dot{w} \leqq -\frac{1}{c}\dot{p}w + \frac{1}{\sqrt{2}} Ap^{-\frac{1}{2}}|e|, \; t \in I.$$

From this we easily obtain the relation (4) of TH. 1 and the proof proceeds further as in TH. 1.

We remark that condition viii) of TH. 1 holds, since, by i), we have $p(t) \leqq p(to) \leqq m$, $t \in I$.

The above theorem for $p(t) = 1$ is the TH. 1 of [2] and for $b(0) = 0$, $T(x) = 0$ the theorem 7.3 of [3].

REMARK 2. - If we put in (1) $g(x)\dot{x} + \varphi(t, x, \dot{x})$ instead of $g(x)\dot{x}$, where $\varphi$ can be written in the form: $\varphi(t, x, v) = u(av + b)\, P(t, x, v)$ with $P(t, x, v) \geqq 0$ in $I \times \mathbb{R}^2$, and if, moreover, $\varphi$ is such that there exist solutions of the equation defined in $I$, then it is easily verified that the above theorems remain valid (cf. [2, Th. 2]).

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