BOLLETTINO
UNIONE MATEMATICA ITALIANA

David F. Dawson

On subgroups of semigroups.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **21** (1966), n.2, p. 144–146.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1966_3_21_2_144_0>

On subgroups of semigroups

DAVID F. DAWSON, (Denton, Texas, U.S.A.)

Summary. - *Some conditions are given under which the set of left zeroids* [1] *of a semigroup S is a subgroup of S. In particular, one of the conditions implies that S is a group* (*Theorem* 2), *and this result is an extension in a different direction from the author's earlier extension* [3] *of a theorem of Gallarati.*

Throughout, S denotes a semigroup with a left zeroid [1] and L denotes the set of all left zeroids of S. In this note we determine some conditions under which L is a group. Two of the conditions imply that $S = L$ and S is a group (Theorems 2 and 5). We omit dual statements and their proofs.

THEOREM 1. - If L contains a regular element μ of S [2, p. 26] such that the element $x \in S$ for which $\mu = \mu x \mu$ is unique, then L is a group.

PROOF. - Since $\mu = \mu x \mu$, we have $x\mu = x\mu x\mu$, and so $x\mu$ is a left zeroid idempotent of S, since $x\mu \in S\mu = L$. Let $e = x\mu$. Suppose f is a left zeroid idempotent of S. There exists $k \in S$ such that $k\mu = f$. We note that f is a right identity of $Sf = L$. Thus $\mu k \mu = \mu f = \mu$. Therefore $k = x$ by the uniqueness property in the hypothesis, and so $f = e$. Hence S has only one idempotent in L, namely e. Therefore e is a right zeroid of S [4]. Thus L is the group of zeroids of S [1].

REMARK. - CLIFFORD and MILLER [1] showed that if S has a left zeroid and a right zeroid, then every left or right zeroid of S is a zeroid of S, and the set of zeroids of S forms a subgroup of S. We can easily show that if L is a group, then every left zeroid of S is a right zeroid of S. To this end, suppose L is a group, $\mu \in L$, and $x \in S$. Then $x\mu \in L$ and there exists $y \in L$ such that $\mu = (x\mu)\, y = x\, (\mu y)$ since L is a group. Thus μ is a right zeroid of S.

THEOREM 2. - Suppose $\mu \in L$ and for each $b \in S$ the solution $x \in S$ of $xb = \mu$ is unique and if there exists $y \in S$ such that $by = \mu$, then y is unique. Then S is a group.

PROOF. – Let $k \in S$ such that $k\mu\mu = \mu$ and let $e = k\mu$. Then $e \in L$ and $ee = e$ by the left uniqueness property, since $ee\mu = e\mu = \mu$. It follows from a lemma in [4] that Se is regular. Thus since $Se = L$, there exists $x \in S$ such that $\mu = \mu x \mu$. Suppose $y \in S$ such that $\mu = \mu y \mu$. By the left uniqueness property, $e = \mu x$ and $e = \mu y$. Since e is a right identity of Se and $\mu \in Se$, we have $\mu = \mu e = \mu\mu x$ and $\mu = \mu e = \mu\mu y$. Hence by the right uniqueness property, $x = y$. Thus by Theorem 1, L is a group, and by the remark following the proof of Theorem 1, μ is a zeroid element of S. Therefore by a theorem of GALLARATI [5], S is a group.

THEOREM 3. – If L contains a normal element, i.e., there exists $\mu \in L$ such that $\mu L = L\mu$, then L is a group.

PROOF. – Let μ be a normal element of L. We note that $L\mu \subseteq S\mu = L$. Suppose $\alpha \in L$. There exists $z \in S$ such that $z\mu\mu = \alpha$. Thus $\alpha \in L\mu$ since $z\mu \in L$. Hence $L \subseteq L\mu$. Therefore $\mu L = L\mu = L$. There exists $x \in S$ such that $x\mu\mu\mu = \mu$, and there exists $y \in L$ such that $(x\mu)\mu = \mu y$, since $L\mu = \mu L$. Hence $\mu = x\mu\mu\mu = \mu y \mu$, and so μ is a regular element of L. We note that $e = \mu y$ is a left zeroid idempotent of S. Suppose $\beta \in L$. There exists $t \in L$ such that $\beta = \mu t$ since $L = \mu L$. Thus $e\beta = e(\mu t) = (\mu y)(\mu t) = (\mu y \mu)t = \mu t = \beta$. Hence e is a left identity element of L. Recalling that $y \in L$, we can find $w \in S$ such that $w\beta = y$. Thus $(\mu w)\beta = \mu y = e$, and $\mu w \in L$ since L is an ideal of S [1]. Hence e is a left zeroid element of the semigroup L. Thus the semigroup L contains a left identity, with respect to which each element of L has a left inverse in L. Therefore L is a group.

COROLLARY. – If L contains a normal element, then every element of L is a normal element of S.

PROOF. – From the hypothesis and Theorem 3, we see that every element of L is a normal element of L. Let $\mu \in L$. We note that $\mu S \supseteq \mu L$. But L is an ideal of S [1], and so $\mu S \subseteq L$. But from the proof of Theorem 3, $L = \mu L$. Hence $\mu S = \mu L = L = S\mu$, and μ is a normal element of S. This completes the proof.

K. ISÉKI [6] defined a relation « $\leq$ » on the nonempty set of idempotents of S as follows: $e \leq f$ if and only if $ef = e$. We conclude with two theorems concerning ISÉKI's relation.

THEOREM 4. – If L contains an idempotent and S contains a unique least idempotent, then L is a group.

PROOF. – Let e be the unique least idempotent of S and let f be an idempotent in L. Suppose $e \in S - L$. Since $e \leq f$, we have

$ef = e$. But $ef \in Sf = L$, which contradicts the assumption that $e \in S - L$. Thus $e \in L$. Hence e is a zeroid of S [4], and so L is a group [1].

THEOREM 5. - If S is regular and contains a unique greatest idempotent e, then S is a group if $e \in L$.

PROOF. - Suppose f is an idempotent in L. Then f is a right identity of $Sf = L$, and so $ef = e$, which means that $e \leqq f$. But since «$\leqq$» is transitive and e is the unique greatest idempotent of S, we have $f = e$. Thus e is the only idempotent in L, and so e is a zeroid of S [4] and L is a group [1]. Suppose g is an idempotent in $S - L$. Then $g \leqq e$ and so $ge = g$. But $ge \in Se = L$ and we have a contradiction. Thus $S - L$ contains no idempotent. Hence e is the only idempotent of S. Suppose $S \neq L$. Let $b \in S - L$. Then there exists $x \in S$ such that $b = bxb$ since S is regular. But xb is an idempotent and so $xb = e$. Thus $b = be \in L$, which is a contradiction. Hence $S = L$ and S is a group.

REFERENCES

[1] A. H. CLIFFORD and D. D. MILLER, *Semigroups having zeroid elements*, Amer. J. Math., 70 (1948), pp. 117-125.

[2] A. H. CLIFFORD and G. B. PRESTON, *The algebraic theory of semigroups*, vol. 1, Mathematical Surveys, n. 7, American Mathematical Society, Providence, Rhode Island, 1961.

[3] D. F. DAWSON, *On a result of D. Gallarati concerning semigroups*, Boll. Un. Mat. Ital., (3), 19 (1964), pp. 446-447.

[4] D. F. DAWSON, *Semigroups having left or right zeroid elements*, Notices Amer. Math Soc., 12 (1965), p. 372, Abstract 65T 180.

[5] DIONISIO GALLARATI, *Un'osservazione sopra i semigruppi*, Boll. Un. Mat. Ital., (3), 18 (1963), pp. 279-280.

[6] K. ISÉKI, *Contribution to the theory of semigroups*. II, Proc. Japan Acad., 32 (1956), pp. 225-227.

*Pervenuta alla Segreteria dell'*U M.I.
l'11 gennaio 1966