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A characterization of the group of a plane cuspidal element

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Summary. - *In this paper, we discuss the projective invariants of a plane cuspidal element of order seven.*

1. - In this note it is shown that in the ordinary projective plane P_2 the configuration of a cuspidal element E_7, a point. and a line have a projective invariant which completely characterizes the subgroup of homographies of P_2 which leave the element E_7 invariant (Theorem 2). A geometrical construction of this invariant is also given.

We will indicate points of P_2 by the ordered triad (x^0, x^1, x^2). Non-homogeneous coodinates are introduced by the embedding

$$x = x^1/x^0, \ y = x^2/x^0.$$

2. - Consider a curve with a cusp at the point $O(1, 0, 0)$. If we choose the cuspidal tangent at O to be the line $y = 0$, the equation of this curve (in non-homogeneous coordinates) is necessarily of the form

$$(1) \qquad 0 = f(x, y) = y^2 - (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) - \varphi_4(x, y) - \ldots$$

where $\varphi_i(x, y)$ is a form in x and y of degree i. In the neighborhood of O we can represent this curve by the developments

$$x = t^2$$

$$y = t^3(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots)$$

where

$$\alpha_0 = a_{30}^2, \ 2\alpha_1 = a_{21}$$

and the coefficients α_i, $i \geq 2$, depend only on the coefficients of the terms of order > 3 (if they exist) in equation (1) [2].

The minimum number of intersections coincident with O of two curves having a cusp at O with the same tangent, therefore, depends only on the two coefficients a_{30}, a_{12}. In fact it is easy to see that two curves like (1) having different coefficients a_{30} have at O six points in common; if they have the same a_{30} but different a_{21} they have at O seven points in common; if a_{30}, a_{21} are the same for two curves, then they have at O at least eight common points. The totality of curves with the same cusp and cuspidal tangent as the curve (1) which, in addition, have an eight point contact with (1) at O can, consequently, be represented by a development of the form

$$y^2 = a_{30}x^3 + a_{21}xy^2 + \dots \tag{2}$$

where the symbol ... indicates that the remaining coefficients are arbitrary. The equivalence class of curves determined by such a development is classically called a cuspidal curvilinear differential element of order seven E_7 with center O. [1]

It is always possible, however, to reduce the representation (2) of an element E_7 to a more convenient form. To this end, consider the cubics which contain the element E_7. These cubics (which form a net) are represented by the equation

$$y^2 = a_{30}x^3 + a_{21}x^2y + \lambda xy^2 + \mu y^3. \tag{3}$$

It is well-known that a plane cubic having a double point with a coincident tangent has only one inflection point. By examining the Hessian of the net (3), it is easy to see that the flexs of these ∞^2 cubics lie on the line

$$3a_{30}x + a_{21}y = 0.$$

This line — the so-called associated line of the element E_7 — is completely determined by the element E_7. By choosing this line as the line $x = 0$ or, equivalently, choosing the coefficient $a_{21}=0$, we can reduce the representation (2) of the element E_7 to

$$y^2 = a_{30}x^3 + y^2\varphi_1(x, y) + \varphi_4(x, y) + \dots$$

The (non-singular) homographies of P_2 which leave this development invariant (or leave the element E_7 invariant) are of the form

$$x^{0\prime} = c_{11}^3 c_{22}^{-2} x^0 + c_{01}x^1 + c_{02}x^2$$
$$x^{1\prime} = c_{11}x^1$$
$$x^{2\prime} = c_{22}x^2.$$

Since a homography of this group maps a line not passing through the cusp, whose equation we can write in the form

$$x^0 - ax^1 - bx^2 = 0, \tag{4}$$

onto the line $x^{0\prime} = 0$, we can, without any loss of generality, express the above homographies in the more convenient form

$$x^{0\prime} = c^3(x^0 - ax^1 - bx^2)$$
(5) $$x^{1\prime} = cx^1$$
$$x^{2\prime} = x^2.$$

This group, which we denote by $G(E_7)$, depends on three parameters and, in this respect, is similar to the groups of the classical non-euclidean geometries of P_2.

3. - Since $G(E_7)$ is a G_3, a line r, not passing through the cusp, and a point P, not belonging to the cuspidal tangent, have an invariant; in fact, we have the following theorem:

THEOREM 1. - Given a line $r : [1 - ax - by = 0]$ and a point $P(x, y)$, $y \neq 0$, the function

(6) $$I(r, P) = x^3 / (y^2 - axy^2 - by^3)$$

is invariant under $G(E_7)$.

PROOF. - If $g \in G(E_7)$, then obviously $I(gr, gP) = I(r, P)$.

Let $G(I(r, P))$ be the group of homographies which leave $I(r, P)$ invariant. Obviously $G(E_7) \subset G(I(r, P))$. In the following, we show that $G(I(r, P)) \subset G(E_7)$ and, consequently, that $G(I(r, P)) = G(E_7)$. Preparatory to this we prove the following lemma:

LEMMA. - A homography f which leaves the line $r' : [x^0 = 0]$ invariant, and is such that $I(r', P) = I(r', fP)$, identically with respect to P, belongs to $G(E_7)$.

PROOF. - The homography f is necessarily of the form

$$x^{0\prime} = a_{00}x^0$$
$$x^{1\prime} = a_{10}x^0 + a_{11}x^1 + a_{12}x^2$$
$$x^{2\prime} = a_{20}x^0 + a_{21}x^1 + a_{22}x^2.$$

The conditions of the lemma imply that

$$a_{10}^3 = 3a_{10}a_{11}^2 = 0,$$
$$a_{12}^3 = 3a_{12}a_{11}^2 = 0,$$
$$a_{20}^2 = a_{21}^2 = 0,$$
$$a_{11}^3 = a_{22}^2 a_{00}.$$

which, in turn, imply that $f \in G(E_7)$.

A homography $h \in G(I(r, P))$ has the property that

$$I(r, P) = I(hr, hP).$$

There obviously exist homographies g_1, $g_2 \in G(E)$ which map r onto r'. Since g_1, g_2 leave (6) invariant, we have

$$I(g_1 r, g_1 P) = I(r, P) = I(hr, hP) = I(g_2 hr, g_2 hP).$$

The homography $f = g_2 h g_1^{-1}$ maps $g_1 r$ onto $g_2 hr$ and leaves (6) invariant, identically with respect to P and $g_1 P$; therefore, by the Lemma it belongs to $G(E_7)$. The equality $h = g_2^{-1} f g_1$ implies, consequently, that $h \in G(E_7)$. We, therefore, have the following results:

THEOREM 2. - A homography of P_2 that leaves $I(r, P)$ invariant is necessarily a transformation of the group $G(E_7)$.

COROLLARY. - The invariant (6) of a point and a line completely characterizes the group $G(E_7)$.

Let us call the invariant $I(r, P)$ the « distance » of the point P from the line r.

From (6) it is easy to see that the set of points equidistant from a fixed line, i.e., the equidistant curves with a given axis, lie on a cubic. In particular, the set of points P such that $I(r, P)=0$, r fixed, lie on the reducible cubic composed of the element E_7, counted three times. Since this observation is valid irrespective of the axis, we have a new geometrical significance of the associated line of the element E_7.

THEOREM 3. - The associated line of an element E_7 is the set of points null-distant from every generic (with respect to the element E_7) line of P_2.

If the point P is incident with the line r, it is easy to see that the expression (6) is without meaning. In fact, the configuration of an element E_7 and a line incident with a given point (or, more precisely, of an element E_7 and an ordinary curvilinear element E_1) does not have a projective invariant. In order to prove this, let us choose a system of reference in which the given line is represented by the equation $x^0 = 0$. Among the homographies leaving the element E_7 invariant, those which map $x^0 = 0$ onto itself are given in (5) by $a = b = 0$. A homography of this type, say f, maps a generic point which, by selecting the unit point of reference, we can always choose to be $P(0, 1, 1)$, onto $P'(0, c, 1)$. Since $P' = fP$ is a generic point of $x^0 = 0$, we are forced to conclude that this configuration does not have an invariant.

We propose to show that the invariant $I(r, P)$ is transitive with respect to $G(E_7)$, i.e., there exists a transformation of $G(E_7)$ which maps a given point-line pair r, P, onto any other point-line pair q, R, for which $I(r, P) = I(q, R)$. Clearly, there exists transformations of $G(E_7)$ which map r, q onto r': $[x^{0\prime} = 0]$. Therefore, choose f[resp., g] such that $fr = r'$[resp., $gr = r'$]. Since f, $g \in G(E_7)$, we have

$$I(r, P) = I(fr, fP) = I(gq, gR) = I(q, R).$$

There exists a unique transformation $h \in G(E_7)$ which leaves r' invariant and is such that $hfP = gR$. Therefore, the composition $g^{-1}hf$ maps r onto q, P onto R and has the property that $I(r, P) = I(g^{-1}hfr, g^{-1}hfP) = I(q, R)$. We state this result in the following theorem:

THEOREM 4. - The invariant (6) of a point and line is transitive with respect to the group $G(E_7)$.

CORALLARY - The invariant $I(r, P)$ is the only (independent) invariant of a point and line under transformations of $G(E_7)$.

4. - In order to give a geometrical construction of the invariant $I(r, P)$ of a point P and a line r, consider the net of cubics which contain the element E_7. The line r uniquely determines a cubic in this net; namely, that cubic which admits the given line as its inflectional tangent. If we write the equation of the line r in the form (4), this cubic is given by

$$y^2(1 - ax - by) = a_{30}x^3. \tag{7}$$

Any line s: $[x - \lambda y = 0]$ through the cusp 0 of the element E_7 intersects the cubic (7) in the point C whose non-homogeneous coordinates are

$$(\lambda(a_{30}\lambda^3 + a\lambda + b)^{-1},\ (a_{30}\lambda^3 + a\lambda + b)^{-1})$$

and, of course, in the point 0, counted twice. In addition the point $R = r \cap s$ given by

$$(\lambda(a\lambda + b)^{-1},\ (a\lambda + b)^{-1})$$

is well-determined. Given a generic point $P \in s$ we, therefore, can compute the cross ratio $S = (O, R; P, C)$. It is only a matter of calculation to verify that $S = a_{30}^{-1}I(r, P)$. The invariant $I(r, P) = a_{30}S$, consequently, can be interpreted geometrically as the cross ratio (aside from a numerical factor) of four well-determined points on the line joining the cusp 0 and the given point P.

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