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ABSTRACT ALMOST-PERIODIC FUNCTIONS AND FUNCTIONAL EQUATIONS

LUIGI AMERIO

CHAPTER I

ALMOST-PERIODIC FUNCTIONS IN BANACH SPACES

1. Definition of almost-periodic function. Elementary properties.

The general theory of almost-periodic functions with complex values, created by Harald Bohr [1] in his two classical papers published on Acta Mathematica in 1925 and 1926, has been greatly developed by Weyl, De la Vallée-Poussin, Bochner, Stepanov, Wiener, Bogoliubov, Levitan. An important class of these functions had already been studied, at the beginning of the century, by Bohl and by Esclangon.

Bohr's theory was then, in a particular case, extended by Muckenhaup [2] and, subsequently, by Bochner [3] and by Bochner and von Neumann [4] to very general abstract spaces. The extension to Banach spaces has, in particular, revealed itself of great interest, in view of the fundamental importance of these spaces in theory and applications.

To this extension will be devoted the first chapter of the present paper. In chapter 2 we shall deal with the applications to a p. partial, or, more generally, abstract differential equations, linear or non linear. This means, essentially, the extension of the classical theorems of Bohr-Neugebauer and of Fayard [5] on
ordinary linear differential equations. Such extension can be made, as will be seen, following a procedure whose nature will be quite clear already in the problem of the integration of a.p. functions.

Let $X$ be a Banach space; if $x \in X$, we shall indicate by $\|x\|$ or by $\|x\|_X$, the corresponding norm.

Let $J$ be the interval $-\infty < t < +\infty$ and

$$x = f(t)$$

a continuous function, defined on $J$ and with values in $X$: an application, in other words, $t \rightarrow f(t)$, from $J$ to $X$. Continuity will obviously be intended in the strong sense (i.e. $\lim_{\tau \to 0} f(t + \tau) = f(t)$ means that $\|f(t + \tau) - f(t)\| \to 0$).

When $t$ varies on $J$ the point $x = f(t)$ describes, in the $X$ space, a set which is called the range of the function $f(t)$, indicating it by $\mathcal{R}(f(t))$.

A set $E \subseteq J$ is said to be relatively dense (r.d.) if there exists a number $l > 0$ (inclusion length) such that every interval $a \sim a + l$ contains at least one point of $E$.

We shall now say that the function $f(t)$ is almost-periodic (a.p.) if to every $\varepsilon > 0$ there corresponds a r.d. set $|\tau|_\varepsilon$, such that

$$\sup_{t} \|f(t + \tau) - f(t)\| \leq \varepsilon, \quad \forall \tau \in |\tau|_\varepsilon.$$

Each $\tau \in |\tau|_\varepsilon$ is called an $\varepsilon$-almost period of $f(t)$; to the set $|\tau|_\varepsilon$ therefore corresponds an inclusion length $l_\varepsilon$ and it is clear that, when $\varepsilon \to 0$, the set $|\tau|_\varepsilon$ becomes rarified, while (in general) $l_\varepsilon \to +\infty$.

The above definition was given by Bochner and is an obvious extension of the definition adopted by Bohr for his theory of a.p. functions. It is, undoubtedly, in itself a very significant definition: its real depth can actually be understood only «a posteriori», from the beauty of the theory constructed on it and the importance of its applications.

The theory of a.p. functions with values in a Banach space is, in the way it is treated by Bochner [3], similar to Bohr's theory of numerical a.p. functions: new developments arise, as is natural, in connection with questions on compactness and boundedness. These questions (which have been recently studied especially in Italy) are of interest particularly in the integration of a.p. functions and, more generally, in the integration of abstract a.p. differential equations.
It is obvious that a continuous periodic function is also a.p. The almost-periodicity condition is however much less restrictive than that of periodicity: for instance, *all trigonometric polynomials*

\[ P(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t} \quad (a_k \in X, \lambda_k \in J) \]

are a.p. functions; not only, but, as we shall see later, the class of a.p. functions coincides with the closure, with respect to the uniform convergence on \( J \), of the set of such polynomials.

Let us now indicate the first properties of a.p. functions, which can be easily deduced from their definition. In what follows we shall omit the proofs, except at some fundamental or typical point. We add that when we say that \( f(t) \) is uniformly continuous, or bounded, or that the sequence \( |f_n(t)| \) converges uniformly etc., we *always* mean that this occurs on the whole interval \( J \).

When, for sake of clarity, it may be necessary to state in which space \( f(t) \) takes its values, we shall say, for instance, that \( f(t) \) is \( X \)-continuous, or \( X \)-a.p., instead of continuous, or a.p. etc.

\[ \begin{align*}
I &- f(t) \text{ a.p. } \Rightarrow f(t) \text{ uniformly continuous (u.c.).} \\
II &- f(t) \text{ a.p. } \Rightarrow \mathcal{R}_{f(t)}, \text{ relatively compact (r.c.).}
\end{align*} \]

This means that the closure \( \overline{\mathcal{R}_{f(t)}} \) is compact. It may be noted that, in the numerical case (or, equivalently, when \( X \) is Euclidean) property II reduces to Bohr's \( f(t) \text{ a.p. } \Rightarrow \mathcal{R}_{f(t)} \text{ bounded} \). In a general Banach space however, the r.c. sets are bounded sets of a very particular nature. The fact that the range \( \mathcal{R}_{f(t)} \) is r.c. is equivalent to the following: \( \forall \varepsilon > 0 \), there exists a finite number of points \( f(t_1), \ldots, f(t_n) \) such that

\[ \mathcal{R}_{f(t)} \subseteq \bigcup_{k=1}^{n} (f(t_k), \varepsilon) \]

(where \( (x, \varepsilon) \) denotes the open sphere with centre \( x \) and radius \( \varepsilon \)); equivalently, every sequence \( |f(t_n)| \) contains a convergent subsequence (in other words, for \( \mathcal{R}_{f(t)} \), the principle of Bolzano–Weierstrass holds).

\[ \begin{align*}
III &- f_n(t) \text{ a.p. } (n = 1, 2, \ldots), f_n(t) \rightarrow f(t) \text{ uniformly } \Rightarrow f(t) \text{ a.p.}
\end{align*} \]

The class of a.p. functions is therefore *closed* with respect to the topology of uniform convergence.
IV - \( f(t) \) a.p., \( f'(t) \) uniformly continuous \( \Rightarrow \) \( f'(t) \) a.p.

\( \forall - x = f(t)x - \text{a.p.}, y = g(x) \) with values in \( Y \) (Banach) and continuous on \( \mathbb{R}_+(t) \) \( \Rightarrow \) \( g(f(t)) \) \( \text{a.p.} \)

In particular:
\[
f(t) \text{ a.p., } k > 0 \Rightarrow \|f(t)\|^k \text{ a.p.}
\]

2. - Bochner's criterion.

The class of a.p. functions has been characterized by Bochner by means of a compactness criterion, which plays an essential rôle in the theory and in applications. The starting point consists in considering, together with a given function \( f(t) \), the set of its translates \( \{f(t + s)\} \) and its closure \( \overline{|f(t + s)|} \) with respect to uniform convergence.

We shall prove Bochner's criterion by the following analysis.

Let \( G \) be the Banach space of continuous and bounded functions \( f(t) \), from \( J \) to \( X(G = C(J; X) \cap L^\infty(J; X)) \), with norm corresponding to uniform convergence: if \( \tilde{f} \) is the point of \( G \) which corresponds to the function \( f(t) \), it will therefore be
\[
\tilde{f} = \{f(t); t \in J\}, \quad \|\tilde{f}\| = \text{Sup} \|f(t)\|.
\]

Let us now consider, together with \( f(t) \), the set of the translates \( f(t + s), \forall s \in J \). If
\[
\tilde{f}(s) = \{f(t + s); t \in J\},
\]
we have defined an application, \( s \rightarrow \tilde{f}(s) \), from \( J \) to \( G \); furthermore, \( \tilde{f}(0) = \tilde{f} \).

We shall call transformation of Bochner the operation by which we pass from \( f(t) \) to \( \tilde{f}(s) \): \( \tilde{f}(s) \) will be called the Bochner transform of \( f(t) \), using also the notation
\[
\tilde{f}(s) = \mathcal{B}(f(t)).
\]

Bearing in mind the definition of \( G \), it is clear that the transformation just defined is linear; moreover, the correspondence between \( f(t) \) and \( \tilde{f}(s) \) is one-to-one, (note that \( \tilde{f}(0) = \|f(t); t \in J\| \), that is \( f(t) \) is the function corresponding to the value \( \tilde{f}(0) \)).

The range \( \mathcal{R}(\tilde{f}, s) \), in \( G \), of the transform \( \tilde{f}(s) \) has the following important properties.
a) \( \mathcal{R}(\mathcal{F}) \) is a spherical line: more precisely

\[
\| \tilde{f}(s) \| = \sup_{t} \| f(t + s) \| = \sup_{t} \| f(t) \| = \| \tilde{f}(0) \| ;
\]

\[\text{(2.1)}\]

\[\gamma) \mathcal{R}(\mathcal{F}) \text{ r.c. } \iff \tilde{f}(s) \text{ a.p.}\]

This property, that identifies almost-periodicity with relative compactness, is of the greatest importance.

From theorem II it is obvious that \( \tilde{f}(s) \text{ a.p. } \Rightarrow \mathcal{R}(\mathcal{F}) \) r.c.

Let us now prove that \( \mathcal{R}(\mathcal{F}) \) r.c. \( \Rightarrow \tilde{f}(s) \) uniformly continuous. Suppose this is not true; there exist then a constant \( \varphi > 0 \) and two sequences \( |s''_n|, |s''''_n| \) such that

\[\text{(2.3)}\]

\[
\lim_{n \to \infty} (s''''_n - s''_n) = 0, \quad \| \tilde{f}(s''''_n) - \tilde{f}(s''_n) \| \geq \varphi.
\]

From the property \( \gamma \) and the second of (2.3) it follows

\[\text{(2.4)}\]

\[
\| \tilde{f}(s''''_n - s''_n) - \tilde{f}(0) \| \geq \varphi.
\]

As the range \( \mathcal{R}(\mathcal{F}) \) is r.c. it is possible to extract from \( |s''''_n - s''_n| \) a subsequence (again indicated by \( |s''''_n - s''_n| \)) such that

\[\text{(2.5)}\]

\[
\lim_{n \to \infty} \tilde{f}(s''_n - s''_n) = \tilde{g}
\]

and, by (2.4),

\[\text{(2.6)}\]

\[
\| \tilde{g} - \tilde{f}(0) \| \geq \varphi.
\]

On the other hand

\[
\tilde{g} = |g(t); t \in J| \in G
\]

and from (2.5) we get

\[
\lim_{n \to \infty} \| f(s''_n - s''_n) - \tilde{g} \| = \lim_{n \to \infty} \sup_{t} \| f(t + s''_n - s''_n) - g(t) \| = 0.
\]

It results then, \( \forall t \in J \),

\[
\lim_{n \to \infty} f(t + s''_n - s''_n) = g(t)
\]
and therefore, for the first of (2.3), \( f(t) \) being continuous,
\[
f(t) = g(t),
\]
that is \( \| \hat{f} - \hat{g} \| = 0 \), against (2.6).

We now prove that \( \tilde{f}(s) \) is a.p. For this, it is sufficient to show that, \( \forall \varepsilon > 0 \), the set \( | \tau |_\varepsilon \) of the \( \varepsilon \)—almost-periods is r.d.

As the range \( \mathcal{R}_{\tilde{f}(s)} \) is r.c., there exist, in correspondence of the given \( \varepsilon, \nu \) values \( \tilde{f}(s_1), \ldots, \tilde{f}(s_\nu) \) such that
\[
\mathcal{R}_{\tilde{f}(s)} \subseteq \bigcup_{k=1}^{\nu} (\tilde{f}(s_k), \varepsilon).
\]

Let us put \( l = 2 \max | s_h | \) and fix \( a \in J \) arbitrarily; we consider now the interval \( (a - \frac{l}{2}, a + \frac{l}{2}) \) of length \( l \). The point \( \tilde{f}(a) \) belongs to one of the \( \nu \) spheres \( (\tilde{f}(s_k), \varepsilon) \); let us suppose that
\[
\tilde{f}(a) \in (\tilde{f}(s_k), \varepsilon)
\]
and consider the value
\[
\tau = a - s_k.
\]

It will be
\[
a - \frac{l}{2} \leq \tau \leq a + \frac{l}{2}
\]
and, furthermore, by the property \( \beta \) and (2.8),
\[
\| \tilde{f}(s + \tau) - \tilde{f}(s) \| = \| \tilde{f}s_k + \tau - \tilde{f}(s_k) \| = \| \tilde{f}(a) - \tilde{f}(s_k) \|.
\]

From (2.7), (2.10) it follows that, \( \forall s \in J \),
\[
\| \tilde{f}(s + \tau) - \tilde{f}(s) \| < \varepsilon,
\]
that is \( \tau \) is an \( \varepsilon \)—almost—period for \( \tilde{f}(s) \). For (2.9), the set \( | \tau |_\varepsilon \) is also r.d., \( a \) being arbitrary and \( l \) independent of \( a \).

The almost—periodicity of \( \tilde{f}(s) \) is therefore proved.

\( \delta \) \( f(t) \) a.p. \( \iff \tilde{f}(s) \) a.p.

This property follows immediately from (2.2), as
\[
\mathop{\sup}_t \| f(t + \tau) - f(t) \| = \mathop{\sup}_s \| \tilde{f}(s + \tau) - \tilde{f}(s) \|.
\]

Properties \( \delta \) and \( \gamma \) express Bochner's criterion:

VI - \( f(t) \) a.p. \( \iff \mathcal{R}_{\tilde{f}(s)} \) r.c.
In other words:

Let \( f(t) \) be continuous, from \( J \) to \( X \). A necessary and sufficient condition for \( f(t) \) to be a.p. is that from every sequence \( \{s_n\} \) it may be possible to extract a subsequence \( \{l_n\} \) such that the sequence \( \|f(t + l_n)\| \) be uniformly convergent.

A very important consequence of Bochner's criterion is that the sum \( f(t) + g(t) \) of two \( X-a.p. \) functions is \( X-a.p. \); the product \( \varphi(t)f(t) \) of \( f(t) \), \( X-a.p. \), by a numerical a.p. function \( \varphi(t) \), is a.p.

Therefore: The set of \( X-a.p. \) functions \( f(t) \) is a vector space. The corresponding set, constituted by the Bochner transforms \( \tilde{f}(s) \), forms a subspace (closed, by theorem II) of the space \( G \).

Observation I. - As we have already pointed out, Bochner's transformation sets up a correspondance between a function \( f(t) \) and its transform \( \tilde{f}(s) = \mathbb{B}(f(t)) \), for which the «principle of conservation of distance» holds (property \( \beta \)). If we assume that \( f(t) \) itself satisfies a condition more general than \( \beta \), but essentially of the same nature, we obtain a sufficient condition for almost-periodicity due to Bochner [6], and utilized in the study of the homogeneous wave equation. It is well worth noting that such a condition is suggested by the «principle of conservation of energy», which holds for the solutions of that equation.

VII. - Let us assume that \( f(t) \), from \( J \) to \( X \), satisfies the following conditions:

1) \( f(t) \) is continuous and its range \( \mathbb{R}_{f(t)} \) is r.c.;
2) \( \forall \tau \in J \)

\[
\inf_t \|f(t + \tau) - f(t)\| \geq \sigma \sup_t \|f(t + \tau) - f(t)\|,
\]

\( \sigma \) being a constant \( >0 \) independent of \( \tau \).

\( f(t) \) is then a.p.

Let us observe that, if \( \|f(t + \tau) - f(t)\| \) is independent of \( t \) (that is if the principle of «conservation of distance» holds) 2) is satisfied with \( \sigma = 1 \). The proof of theorem VII was obtained by Bochner in the same way as the proof (given in property \( \gamma \)) that the set \( |\tau|_\varepsilon \) of \( \varepsilon \)-almost-periods is r.d.

Observation II. - Let \( f(t) \) be \( X-a.p. \). We shall say that a sequence \( \{s_n\} \) is regular (with respect to \( f(t) \)) if \( |f(t + s_n)| \) is uniformly convergent: in other words, if the sequence \( |\tilde{f}(s_n)| \) is convergent
Let us call \( S_f \) the set of all regular sequences with respect to \( f(t) \). Obviously \( S_f = S_{\tilde{f}} \).

Given a second function \( g(t), Y - a.p., \) it is important, (as will also appear from what follows) to recognise when \( S_f \subseteq S_g \).

To solve this problem we shall make use of the translation function (defined by Bochner):

\[
(2.12) \quad v_f(\tau) = \sup_t \| f(t + \tau) - f(t) \| = \| \tilde{f}(\tau) - \tilde{f}(0) \| \quad (\tau \in J).
\]

In an analogous way, consider the function

\[
v_g(\tau) = \sup_t \| g(t + \tau) - g(t) \| = \| \tilde{g}(\tau) - \tilde{g}(0) \|.\]

We now construct, in the following way, a function \( \omega_{f,g}(\varepsilon) \), which will be called comparison function of \( f \) with \( g: \forall \varepsilon, \) with \( 0 < \varepsilon < \sup \, v_f(\tau), \) we set

\[
\omega_{f,g}(\varepsilon) = \sup_{v_f(\tau) \leq \varepsilon} v_g(\tau).
\]

\( \omega_{f,g}(\varepsilon) \) is therefore the supremum of the translation function \( v_g(\tau) \) when \( \tau \) varies in the set of the \( \varepsilon \)-almost-periods of \( f(t) \): each \( \varepsilon \)-almost-period of \( f(t) \) is also a \( \omega_{f,g}(\varepsilon) \)-almost-period of \( g(t) \). It follows that \( \omega_{f,g}(\varepsilon) \) is a monotonic, non decreasing function of \( \varepsilon \):

\[
(2.13) \quad \omega_{f,g}(0 + ) \geq 0
\]

therefore exists and the following proposition can be proved:

VIII. \( S_f \subseteq S_g \iff \omega_{f,g}(0 + ) = 0. \)

It follows that, in order that all sequences \( \{ s_n \} \), regular with respect to \( f(t) \), be regular with respect to \( g(t) \), it is necessary and sufficient that the comparison function \( \omega_{f,g}(\varepsilon) \) be infinitesimal with \( \varepsilon \): it is, in other words, required that, taken an arbitrary sequence \( \{ \tau_n \} \) of \( \varepsilon_n \)-almost-periods of \( f(t) \), with \( \varepsilon_n \to 0, \) \( \tau_n \) be, \( \forall n, \) a \( \varepsilon_n \)-almost-period for \( g(t) \) (\( \sup_t \| g(t + \tau_n) - g(t) \| = \sigma_n \)) with \( \sigma_n \to 0. \)


The harmonic analysis of a.p. functions extends to these the theory of Fourier expansions of periodic functions.

In the numerical case, this analysis was made by Bohr, Weyl, De La Vallée Poussin, Bochner, who constructed at first the
FOURIER series associated to a given $f(t)$: subsequently, it was proved (approximation theorem) that this series is, in some convenient sense, summable to the value $f(t)$. Starting point of the theory is the *theorem of the mean*.

Bogoliubov [7] has, instead, followed an apposite procedure, proving at first directly the *approximation theorem* and later deducing the theorem of the mean and the FOURIER expansion.

For a general Banach space, the first procedure has been generalised by Bochner [3], Bochner and von Neumann [4], Kopec [8], Zaidman [9], the second by Amerio [10].

In what follows we shall keep to this second point of view, as the direct proof of the approximation theorem can be easily obtained, even for Banach spaces.

First of all, we observe that, $\forall a \in X, \lambda \in J$, the function

$$ae^{i\lambda t}$$

is periodic. From this follows the almost-periodicity of all trigonometric polynomials

$$P(t) = \sum_{k=1}^{m} a_k e^{i\lambda_k t}$$

and, consequently, of every $f(t)$ which is the limit of a uniformly convergent sequence of trigonometric polynomials. The approximation theorem enables us to prove that, in this way, it is actually possible to obtain all a.p. functions.

IX. - If $f(t)$ is a.p. there exists, $\forall \varepsilon > 0$, a trigonometric polynomial $P_\varepsilon(t)$ such that

$$\sup_{t} \| f(t) - P_\varepsilon(t) \| \leq \varepsilon.$$

The *theorem of the mean* can be easily deduced from the approximation theorem.

X. - If $f(t)$ is a.p. there exists the mean value

$$\mathfrak{M}(f(t)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t) dt.$$

We observe, first of all, that if

$$\Delta(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0, \end{cases}$$
it results

(3.5) \[ \mathcal{M}(e^{i\omega t}) = \Delta(\omega). \]

From (3.1), (3.4), (3.5) follows that the mean value exists for all trigonometric polynomials \( P(t) \):

(3.6) \[ \mathcal{M}(P(t)) = \sum_{k=1}^{m} a_k \Delta(\lambda_k). \]

From this relation and (3.2), (3.3) follows immediately.

Let us now observe that, if \( f(t) \) is a.p., also \( f(t) e^{-it \lambda} \) is a.p. \( \forall \lambda \in J \). The function

\[ a(\lambda; f) = \mathcal{M}(f(t)e^{-it \lambda}) \]

is therefore defined on \( J \). \( a(\lambda; f) \) takes its values in \( X \), as does \( f(t) \): we shall call this function the Bohr transform of the a.p. function \( f(t) \).

It can be seen at once that \( a(\lambda; f) = 0 \) on the whole of \( J \), with the exclusion, at most, of a sequence \( \{\lambda_n\} \).

It is, in fact, sufficient to construct a sequence of approximation polynomials \( |P_l(t)| \) such that

\[ \sup_{t} \| f(t) - P_l(t) \| \leq \frac{1}{l} \quad (l = 1, 2, \ldots). \]

If then

\[ P_l(t) = \sum_{k=1}^{n_l} a_k e^{i\lambda_k t}, \]

we have, for \( \lambda \notin \{\mu_n\} \), observing that \( a(\lambda; P_l) = 0 \ \forall l \),

\[ \| a(\lambda; f) \| = \| a(\lambda; P_l) + a(\lambda; f - P_l) \| = \| a(\lambda; f - P_l) \| \leq \frac{1}{l} \]

and, consequently, \( a(\lambda; f) = 0 \).

The values \( \lambda_n \) (and obviously \( \lambda_n \in \{\mu_k\} \)) for which \( a(\lambda_n; f) = 0 \) are called the characteristic exponents of \( f(t) \). If we put

\[ a(\lambda_n; f) = a_n, \]
we can associate to $f(t)$ the Fourier series

$$f(t) \sim \sum_{n=1}^{\infty} a_n e^{i\lambda_n t}.$$  

The $a_n$ are called the Fourier coefficients of $f(t)$.

It can be seen that it is possible to choose the approximating polynomials $P(t)$ in such a way that the characteristic exponents $\lambda_k$ belong to the sequence $|\lambda_n|$; hence

$$P(t) = \sum_{k=1}^{n} a_{ik} e^{i\lambda_k t}.$$  

Moreover one proves the fundamental uniqueness theorem:

XI. - If $f(t)$ and $g(t)$ X-a.p., $a(\lambda; f) = a(\lambda; g) \Rightarrow f(t) = g(t)$.

The correspondence between almost-periodic functions and their Bohr transforms is therefore one-to-one. A property of the transform $a(\lambda; f)$ is given by the following proposition.

XII - $a(\lambda'; f) = 0 \Rightarrow \lim_{\lambda \to \lambda'} a(\lambda; f) = 0$, that is the Bohr transform is continuous at all points in which it vanishes.

Furthermore,

$$\lim_{\lambda \to \infty} a(\lambda; f) = 0.$$  

Observation I. - It results

$$\lim_{n \to \infty} a_n = 0$$  

that is the sequence of Fourier coefficients of an a.p. function is infinitesimal as $n$ diverges.

If

$$\sup_t \|f(t)\| = M$$

we also have, by (3.7),

$$\|a_n\| = \|a(\lambda_n; f)\| \leq M.$$
If $X$ is a Hilbert space, property (3.9) can be made much more precise. It can in fact be deduced without difficulty (from the approximation theorem) that Parseval's equality holds:

$$ (3.12) \quad \mathcal{N}(\|f(t)\|^2) = \sum_{n=1}^{\infty} \|a_n\|^2. $$

Observation II. — To Bochner we owe the construction of a notable approximation polynomial, suggested by Fejer's theorem on the Cesàro-summability of the Fourier series of periodic functions.

Let, first of all, $L = (\beta_1, \beta_2, \ldots)$ be a base for the sequence $\lambda_n$ of characteristic exponents of the function $f(t)$; this means that the real numbers $\beta_n$ are linearly independent (i.e. $\sum_1^n c_\lambda \beta_\lambda = 0$, $c_\lambda$ integers $\Rightarrow c_\lambda = 0$) and that each $\lambda_n$ is a linear combination with rational coefficients of a finite number of the $\beta_\lambda$. It is clear that a base always exists: if the $\lambda_n$ are linearly independent, we can take $L = |\lambda_n|$; if not, $L$ can be obtained by eliminating those $\lambda_n$ which are linear combinations of the preceding ones.

If the base $L$ is infinite, Bochner's polynomial $\sigma_m(t)$ is defined by the formula

$$ (3.13) \quad \sigma_m(t) = \sum_{\lambda_1, \ldots, \lambda_m} \left( 1 - \frac{|v_{\lambda_i}|}{(m!)^2} \right) \ldots \left( 1 - \frac{|v_{\lambda_m}|}{(m!)^2} \right) \alpha \left( \sum_{\lambda=1}^{m} \frac{v_{\lambda} \beta_{\lambda}}{m!}; f \right) e^{i\left( \sum_{\lambda=1}^{m} \frac{v_{\lambda} \beta_{\lambda}}{m!} \right) t}, $$

while, if the base is finite ($L = (\beta_1, \ldots, \beta_q)$), it is, for $m \geq q$,

$$ (3.14) \quad \sigma_m(t) = \sum_{\lambda_1, \ldots, \lambda_q} \left( 1 - \frac{|v_{\lambda_i}|}{(m!)^2} \right) \ldots \left( 1 - \frac{|v_{\lambda_q}|}{(m!)^2} \right) \alpha \left( \sum_{\lambda=1}^{q} \frac{v_{\lambda} \beta_{\lambda}}{m!}; f \right) e^{i\left( \sum_{\lambda=1}^{q} \frac{v_{\lambda} \beta_{\lambda}}{m!} \right) t}. $$

It can be proved (and the proof is easily deduced from theorem IX) that

$$ (3.15) \quad \lim_{m \to \infty} \sigma_m(t) = f(t) $$

uniformly.
We observe that in (3.13), (3.14)
\[ a \left( \sum_{k=1}^{m} \frac{\gamma_{k}}{m!} ; f \right) = 0 \]
only if \( \frac{\sum_{k=1}^{m} \gamma_{k}}{m!} = \lambda_{j} \), characteristic exponent.

Relations (3.13) can therefore be written in the form

\[ \sigma_{m}(t) = \sum_{k=1}^{N_{m}} \gamma_{m k} a_{k} e^{i \lambda_{k} t} \]

where the convergence factors \( \gamma_{m k} \) depend only from \( m \) and the exponents \( \lambda_{k} \) (but not from the coefficients \( a_{k} \)).

Observation III. – In observation II of § 2 we found that the facts that \( S_{f} \subseteq S_{g} \) and that the comparison function \( \omega_{f,g}(\varepsilon) \) was infinitesimal, as \( \varepsilon \to 0 \), were connected.

The harmonic analysis of a.p. functions expresses the condition \( S_{f} \subseteq S_{g} \) by means of a relation between the characteristic exponents \( |\lambda_{n}| \) of \( f(t) \) and those, \( |\mu_{n}| \), of \( g(t) \).

XIII. – \( S_{f} \subseteq S_{g} \) if, and only if, each exponent \( \mu_{n} \) is a linear combination with integer coefficients of a finite number of exponents \( \lambda_{k} \), that is

\[ \mu_{n} = \sum_{k=1}^{q_{n}} c_{n k} \lambda_{k} \]

(\( c_{n k} \) integers).

Moreover the almost-periods and the characteristic exponents of an a.p. function are explicitly connected by the following propositions.

XIV. – Let \( |\lambda_{n}| \) be the sequence of characteristic exponents. To every fixed arbitrary \( \varepsilon > 0 \), there correspond a positive integer \( N_{\varepsilon} \) and a number \( \delta_{\varepsilon} > 0 \) such that every solution \( \tau \) of the system of inequalities

\[ |e^{i \lambda_{n} \tau} - 1| \leq \delta_{\varepsilon} \quad (n = 1, \ldots, N_{\varepsilon}) \]

is an \( \varepsilon \)-almost-period of \( f(t) \).

(It may be noted that, by a theorem of Kronecker, system (3.18) is compatible for every \( \delta_{\varepsilon} > 0 \).)

For (3.8) it is, in fact,

\[ \| f(t + \tau) - f(t) \| \leq \frac{2}{l} \sum_{i=1}^{n_{l}} a_{i} \| e^{i \lambda_{n} \tau} - 1 \| . \]
Chosen \( l \) in such a way that \( \frac{2}{l} \leq \frac{\varepsilon}{2} \), let us impose on \( \tau \) to satisfy the system of inequalities

\[
| e^{i\lambda_n \tau} - 1 | \leq \frac{\varepsilon}{2} \left( \sum_{k=1}^{n_i} \| a_{i_k} \| \right)^{-1} = \delta_i \quad (k = 1, \ldots, n_i).
\]

It will then be \( \| f(t + \tau) - f(t) \| \leq \varepsilon \) \( \forall t \in J \), which proves our thesis.

XV. - Let \( |\lambda_n| \) be the sequence of characteristic exponents. Then every \( \varepsilon \)-almost-period, \( \tau \), satisfies the system of inequalities

(3.19) \[
| e^{i\lambda_n \tau} - 1 | \leq \varepsilon \| a_n \|^{-1}.
\]

In fact \( \| a_n(e^{i\lambda_n \tau} - 1) \| = \| \mathcal{M}(f(t + \tau) - f(t))e^{-i\lambda_n \tau} \| \leq \varepsilon. \)

4. - Weakly almost-periodic functions.

Given the BANACH space \( X \), we shall call \( X^* \) its dual space (constituted by the linear functionals continuous on \( X \)). If \( x \in X \), \( x^* \in X^* \), we shall indicate by \( \langle x^*, x \rangle \) the complex value that, through the functional \( x^* \), corresponds to \( x \), and by \( \| x^* \| \) the norm of \( x^* \).

We shall say that \( f(t) \), with values in \( X \), is weakly almost periodic (w.a.p.) if, \( \forall x^* \in X^* \), the numerical function

\[
\langle x^*, f(t) \rangle
\]

is a.p. [11].

As may be seen, the definition given here has, with respect to that of an a.p. function, the same relation as the definition of weakly continuous function has with respect to that of continuous function.

It is clear (as \( \langle x^*, x \rangle = \langle x^* \| x \| \rangle \)) that \( f(t) \) a.p. \( \Rightarrow \) f(t) w.a.p. In order to indicate that \( |x_n| \) is a sequence converging weakly to \( x \) (i.e. if \( \langle x^*, x_n \rangle \to < x^*, x > \), \( \forall x^* \in X^* \)) we shall make use of the following notations

\[
x_n \xrightarrow{\ast} x \text{, or } \lim_{n \to \infty}^* x_n = x.
\]

\( x \) is called the weak limit (which, if it exists, is also unique)
of the sequence \(|x_n|\). Let us remember that, in an arbitrary Banach space, a sequence \(|x_n|\) can be scalarly convergent (i.e. \(\lim_{n \to \infty} \langle x^*, x_n \rangle\) exists and is finite \(\forall x^* \in X^*\)) without necessarily being weakly convergent, that is without there being an \(x\) which is its weak limit. If this circumstance is not present (i.e. if scalar convergence implies weak convergence) the space \(X\) is said to be semicomplete (reflexive, and, in particular, Hilbert spaces are semicomplete).

Let us now indicate some properties of w.a.p. functions.

XVI. - \(f(t)\) w.a.p. \(\Rightarrow\) \(\mathcal{R}_{f(t)}\) bounded and separable.

Where necessary, we can therefore assume that \(X\) is separable.

XVII - \(f(t)\) w.a.p. \((n = 1, 2, ...)\), \(f_n(t) \to f(t)\) uniformly \(\Rightarrow\) \(f(t)\) w.a.p. \(f_n(t) \to f(t)\) uniformly means that, \(\forall x^* \in X^*, \langle x^*, f_n(t) \rangle \to \langle x^*, f(t) \rangle\) uniformly).

XVIII - \(f(t)\) w.a.p., \(f(t + s_j) \to f(t)\) \(\forall t \in J \Rightarrow\) that the convergence is uniform.

XIX - Let \(X\) be semicomplete and \(f(t)\) weakly continuous. Then \(f(t)\) w.a.p. \(\iff\) \(\forall s_n\) there exists \(l_n \leq |s_n|\) such that \(|f(t + l_n)|\) is uniformly weakly convergent.

This proposition extends Bochner's criterion to w.a.p. functions (with, however, a restrictive hypothesis on the nature of the space \(X\)).

As we have already observed, \(f(t)\) a.p. \(\Rightarrow\) \(f(t)\) w.a.p. It is interesting to note that the property that has to be added to weak almost-periodicity to obtain almost-periodicity is one of compactness (by theorem XVI, a w.a.p. function is bounded).

The following theorem can, in fact, be proved.

XX - \(f(t)\) w.a.p. and \(\mathcal{R}_{f(t)}\) r.c. \(\iff\) \(f(t)\) a.p.

Observation. - Let us assume that \(X\) is a separable Hilbert space and let \(|z_n|\) be a complete orthonormal sequence. Then, if \(f(t)\) takes its values in \(X\), it results, \(\forall t \in J\),

\[
(4.1) \quad f(t) = \sum_{n=1}^{\infty} \varphi_n(t) z_n
\]

where \(\varphi_n(t) = \langle f(t), z_n \rangle\) is the scalar product of \(f(t)\) by \(z_n\).

By (4.1) we have

\[
(4.2) \quad \| f(t) \| = \sum_{n=1}^{\infty} | \varphi_n(t) |^2.
\]
Let us now prove the following propositions.

**XXI** - \( f(t) \) w.a.p. \( \iff \sum_n |\varphi_n(t)|^2 \leq M^2 < +\infty. \)

**XXII** - \( f(t) \) a.p. \( \iff \sum_n |\varphi_n(t)|^2 \) uniformly convergent.

The necessity of the condition expressed by XXI is evident. For its sufficiency, observe that, chosen \( y \in X \) arbitrarily, it results

\[
y = \sum_1^\infty \eta_n \varphi_n (\eta_n = (y, \varphi_n), \sum_1^\infty |\eta_n|^2 = \|y\|^2).
\]

Consider the scalar product \((f(t), y)\); to prove our thesis it is sufficient to show that it is a.p. Now we have

\[
(4.3) \quad (f(t), y) = \sum_1^\infty \varphi_n(t) \eta_n
\]

and the series on the right hand side (constituted by a.p. functions) converges uniformly, since it is, by Schwarz's inequality,

\[
| \sum_p \eta_p \varphi_p(t) | \leq | \sum_p |\eta_p|^2 |^{1/2} | \sum_p |\varphi_p(t)|^2 |^{1/2} \leq M | \sum_p |\eta_p|^2 |^{1/2}.
\]

To prove that the condition expressed by XXII is necessary, we observe that, \( f(t) \) being a.p., the range \( \mathcal{R}_f(t) \) is r.c. This implies that, \( \forall \varepsilon > 0 \), there exist a finite number of points \( f(t_1), \ldots, f(t_v) \) such that

\[
\mathcal{R}_f(t) \subset \bigcup_{k=1}^v (f(t_k), \varepsilon).
\]

Let us now fix an index \( m \) such that

\[
\| f_m(t_k) \| = | \sum_m |\varphi_m(t_k)|^2 |^{1/2} \leq \varepsilon \quad (k = 1, \ldots, v)
\]

Chosen arbitrarily \( \bar{t} \in J \), we have, for a certain \( t \), \( \|f(\bar{t}) - f(t)\| < \varepsilon \), and it results

\[
| \sum_m |\varphi_m(\bar{t})|^2 |^{1/2} = \| f_m(\bar{t}) \| \leq \| f_m(t_j) \| + \| f_m(\bar{t}) - f_m(t_j) \| \leq \| f_m(t_j) \| + \| f(\bar{t}) - f(t) \| < 2\varepsilon.
\]

As \( \bar{t} \) is arbitrary, the thesis is proved.

It is obvious that the condition is sufficient because the uni-
form convergence of the series \( \sum_{n=1}^{\infty} |\varphi_n(t)|^2 \) is equivalent to the uniform convergence of the series (of a.p. functions) on the right hand side of (4.1).

It may be noted that, by (4.2), if the series of a.p. functions

\[
\sum_{n=1}^{\infty} |\varphi_n(t)|^2
\]

converges, the norm \( \| f(t) \| \) is a.p.

We may now ask if the converse is true, that is if, \( X \) being a HILBERT space, \( f(t) \) w.a.p., \( \| f(t) \| \) a.p. \( \Rightarrow f(t) \) a.p. (on this subject it may be noted that, in a HILBERT space, the following proposition holds: \( f(t) \) weakly continuous, \( \| f(t) \| \) continuous \( \Rightarrow f(t) \) continuous).

The answer to this question is however negative, as may be shown by examples. It is necessary to extend the hypothesis of almost-periodicity of the norm to a whole family of functions; precisely to the family associated to \( f(t) \) by Bochner's criterion.

Let \( f(t) \) be w.a.p. and \( S_r \) indicate the family of sequences \( s_k \) regular with respect to \( f(t) \): in other words, such that

\[
\lim_{n \to \infty} f(t + s_n) = f_i(t)
\]

uniformly.

We obtain in this way a family \( \Phi_r = \{ f_i(t) \} \) of w.a.p. functions and the following theorem can be proved

XXIII - \( f(t) \) w.a.p., \( \| f_i(t) \| \) a.p. \( \forall i \in \Phi_r \Rightarrow f(t) \) a.p.

5. - Integration of a.p. functions.

If \( f(t) \) is an a.p. function with values in a BANACH space \( X \), we will write, in what follows,

\[
F(t) = \int_{-\infty}^{t} f(\eta) \, d\eta.
\]

The problem of the integration of a.p. functions in BANACH spaces is of notable interest also because it serves, so to say, as a model for classifying BANACH spaces in relation to the theory of abstract a. p. equations.

If \( X \) is Euclidean, then Bohr's theorem holds: \( F(t) \) bounded \( \Rightarrow F(t) \) a.p.
For the general case (X arbitrary Banach space), the almost-periodicity of F(t) has been proved by Bochner [3] under the hypothesis that \( R_{\mathcal{F}(t)} \) is r.c.

This condition is obviously much more restrictive than that of boundedness; it can not however be substituted in the general case by latter, as can be shown on examples (Amerio [12]). Nevertheless Amerio [12] has proved that Bohr's enunciation remains unaltered if the space X is uniformly convex (it holds therefore in Hilbert spaces, in \( L^p \) and \( L^p \), with \( 1 < p < +\infty \)).

Let us prove the following theorems.

XXIV - (Bochner) — X arbitrary, \( f(t) \) a. p., \( R_{\mathcal{F}(t)} \) r.c. \( \Rightarrow \) \( F(t) \) a. p.

XXV - (Amerio) — X uniformly convex, \( f(t) \) a. p., \( F(t) \) bounded \( \Rightarrow \) \( F(t) \) a. p.

a) Proof of theorem XXIV. As \( R_{\mathcal{F}(t)} \) is r.c., \( F(t) \) is bounded:

\[
(5.2) \quad \text{Sup}_t || F(t) || = M < +\infty.
\]

Furthermore, \( \forall x^* \in X^* \),

\[
| < x^*, F(t) > | = | < x^*, \int_0^t f(\eta) d\eta > | = \\
= | \int_0^t < x^*, f(\eta) > d\eta | \leq || x^* || M.
\]

As \( < x^*, f(t) > \) is a.p., from Bohr's theorem it follows that \( < x^*, F(t) > \) is a.p.; \( F(t) \) is therefore w.a.p.

\( R_{\mathcal{F}(t)} \) has been supposed r.c.; our thesis follows then from theorem XX.

b) Proof of theorem XXV. We have already proved in a) (utilizing only the boundedness of \( F(t) \)) that \( F(t) \) is w.a.p. It is therefore sufficient, making use of the properties of uniformly convex spaces, to prove that \( R_{\mathcal{F}(t)} \) is r.c.

We first of all remember that a space X is called a uniformly convex (or Clarkson) space if in the interval \( 0 < \sigma \leq 2 \) there exists a function \( \omega(\sigma) \), with \( 0 < \omega(\sigma) \leq 1 \), such that

\[
(5.3) \quad || x ||, || y || \leq 1 \text{ and } || x - y || \geq \sigma \Rightarrow \left| \left| \frac{x + y}{2} \right| \right| \leq 1 - \omega(\sigma).
\]
A HILBERT space is uniformly convex. From the parallelogram theorem it follows, in fact,

$$\left\| \frac{x - y}{2} \right\|^2 + \left\| \frac{x + y}{2} \right\|^2 = \frac{1}{2} \left( \| x \|^2 + \| y \|^2 \right)$$

and consequently, if $\| x \| \leq 1$, $\| y \| \leq 1$, $\| x - y \| \geq \sigma$,

$$\left\| \frac{x + y}{2} \right\|^2 \leq 1 - \frac{\sigma^2}{4}.$$ 

It can be shown that the $L^p$ and $L^\infty$ ($1 < p < +\infty$) spaces are uniformly convex; in addition, that uniformly convex spaces are also reflexive.

We now observe that from (5.3) it follows, for any $x$ and $y$,

(5.4) $\| x - y \| \geq \sigma \max \{ \| x \|, \| y \| \} \Rightarrow \left\| \frac{x + y}{2} \right\| \leq (1 - \omega(\sigma)) \max \{ \| x \|, \| y \| \}$.

Let us assume that the range $R_F(t)$ is not r.c. There exist then a constant $\rho > 0$ and a sequence $|s_n|$ such that

(5.5) $\| F(s_j) - F(s_k) \| \geq \rho \quad (j \neq k)$.

We can suppose that $|s_n|$ is regular with respect to $f(t)$ and $F(t)$, that is

$$\lim_{n \to \infty} f(t + s_n) = f(t)$$

(5.6) $\lim_{n \to \infty} F(t + s_n) = F(t)$

uniformly. The last relation follows from Bochner's criterion (theorem XIX), noting that the space $X$ is semicomplete (being reflexive).

It is also

$$F(t + s_j) = F(s_j) + \int_0^t f(\eta + s_j) d\eta$$

and, consequently, for $j \neq k$,

$$\| F(t + s_j) - F(t + s_k) \| \geq \| F(s_j) - F(s_k) \| - \\int_0^t (f(\eta + s_j) - f(\eta + s_k)) d\eta \|.$$
If we fix $t \in J$, we will have, by (5.5) and the first of (5.6),
\[ \| F(t + s_j) - F(t + s_k) \| \geq \frac{\rho}{2} \quad \text{for } j > k > n_i. \]

Therefore, by (5.2),
\[ \| F(t + s_j) - F(t + s_k) \| \geq \frac{\rho}{2M} \max \| F(t + s_j) \|, \| F(t + s_k) \| \]
and, by (5.4),
\[ \left\| \frac{F(t + s_j) + F(t + s_k)}{2} \right\| \leq \left(1 - \omega \left(\frac{\rho}{2M}\right)\right) \max \| F(t + s_j) \|, \| F(t + s_k) \| \]
\[ \| F(t + s_j) \| \leq \left(1 - \omega \left(\frac{\rho}{2M}\right)\right) M. \]

From the second of (5.6) it then follows
\[ \| F_s(t) \| \leq \left(1 - \omega \left(\frac{\rho}{2M}\right)\right) M \]
and, consequently,
\[ \text{Sup}_t \| F_s(t) \| \leq \left(1 - \omega \left(\frac{\rho}{2M}\right)\right) M. \] (5.7)

Relation (5.7) is absurd; from the second of (5.6) follows in fact, the weak convergence being uniform,
\[ \lim_{n \to \infty} F_s(t - s_n) = F(t) \]
and therefore
\[ \| F(t) \| \leq \min_{n \to \infty} \| F_s(t - s_n) \| \leq \left(1 - \omega \left(\frac{\rho}{2M}\right)\right) M, \]
which contradicts (5.2).

**Observation.** – The problem if Bohr's theorem holds or not in the case of $X$ reflexive is still open.

It must be noted however that there exist non reflexive spaces in which Bohr's theorem holds; such is, for instance, the space $l^1$ [13]. More generally, let us consider the space $X = \ell^p \mid X_n \mid$, with $1 \leq p < + \infty$; $\mid X_n \mid$ is a sequence of Banach spaces and $x \in X$ means
\[ x = |x_n|, \text{ with } x_n \in X_n, \| x \| = \left\| \sum_{1}^{\infty} \| x_n \|^p \right\|^{1/p} < + \infty. \]
Let $f(t) = |f_n(t)|$ be a.p. with values in $X$. It is then

$$F(t) = \int_0^t f(\eta) d\eta = \{ \int_0^t f_n(\eta) d\eta \} = |F_n(t)|.$$ 

and the following theorem can be proved [14].

**XXVI - If, $\forall X_n$, the property**

$f_n(t)$ a.p., $F_n(t)$ bounded $\Rightarrow$ $F_n(t)$ a.p.

**holds, then the same property holds for $X$:**

$f(t)$ a.p., $F(t)$ bounded $\Rightarrow$ $F(t)$ a.p.

In the proof of this theorem an extension to real a.p. functions of Dini's classical theorem on monotonic sequences of continuous functions is used.

Precisely, let $|\varphi_n(t)|$ be a bounded, monotonic sequence of real a.p. functions, with

$$\varphi_1(t) \leq \varphi_2(t) \leq ... \leq \varphi_n(t) \leq ... \leq M < +\infty.$$ 

There exists therefore, $\forall t \in J$, the finite limit

$$\Phi(t) = \lim_{n \to \infty} \varphi_n(t).$$ 

While it is not possible to say that, if $\Phi(t)$ is a.p., convergence is uniform, Dini's theorem can however be extended in the same order of ideas as theorem XXIII. Let $S$ be the set of sequences $s = |s_n|$ regular with respect to all $\varphi_n(t); \forall s \in S$, it is therefore

$$\lim_{k \to \infty} \varphi_n(t + s_n) = \varphi_n(t) \quad (n = 1, 2, ...)$$

uniformly, where $\varphi_n(t)$ is, like $\varphi_n(t)$, a.p.

From Bochner's criterion and applying Cantor's diagonal process, it follows immediately that every sequence $r = |r_n|$ contains a subsequence $\sigma \in S$.

We observe that from (5.8), (5.10) it follows that

$$\varphi_{r_1}(t) \leq \varphi_{r_2}(t) \leq ... \leq \varphi_{r_n}(t) \leq ... \leq M$$

and consequently

$$\lim_{n \to \infty} \varphi_{r_n}(t) = \Phi_0(t), \quad \Phi_0(t) = \Phi(t).$$

We can prove the following proposition.
XXVII - If $\Phi_s(t)$ is a.p. $\forall s \in S$, the sequence $\varphi_n(t)$ converges uniformly.

This theorem (proved by AMERIO [15]) has been generalized by BOCHNER [16] to almost-automorphic functions. For another proof see DOLCHER [17].

CHAPTER II

FUNCTIONAL ALMOST-PERIODIC EQUATIONS

1. - Almost-periodic solutions of the wave equation.

a) In the present §, we shall deal with the mixed problem, according to HADAMARD, for the wave equation (or equation of the vibrating membrane)

$$
\frac{\partial^2 y}{\partial t^2} = \sum_{j, k}^m \left( a_{jk}(x) \frac{\partial y}{\partial x_k} \right) - a(x)y + f(t, x),
$$

and consider, more precisely, the first mixed problem.

Let $\Omega$ be an open, bounded and connected set of the Euclidean space $\mathbb{R}^m$, $\partial \Omega$ the boundary of $\Omega$, $x = (x_1, ..., x_m)$ an arbitrary point of $\mathbb{R}^m$.

The problem considered consists in finding a solution $y = y(t, x)$ satisfying the initial conditions

$$
y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x) \quad (x \in \Omega)
$$

and the boundary condition

$$
y(t, x) |_{x \in \partial \Omega} = 0 \quad (t \in J).
$$

It corresponds therefore to the study of the motion of a vibrating membrane, with fixed edge. The functions considered in (1.1) are assumed to be real.

It is classical, in the theory of hyperbolic equations, to look for so-called weak or generalized solutions. As we shall see, these solutions are associated to the variational theory of the vibrating membrane, and, whenever they satisfy convenient regularity conditions, are solutions of (1.1).

The variational form of equation (1.1) can be obtained applying to it Green's formula (and bearing in mind the boundary condition (1.3)).
First of all, it is well to define the functional spaces in which the problem is correctly posed. As we shall see, they will be Hilbert spaces. These spaces play, as is well known, a very special rôle in the mathematical description of physical problems. This is not surprising if one observes that such phenomena are essentially bound to the concept of energy, hence to that of scalar product: now Hilbert spaces are defined precisely as those Banach spaces in which the scalar product is defined, with the same formal properties which this functional has in ordinary Euclidean spaces.

We assume that the coefficients \( a_{jk}(x), a(x) \) are measurable and bounded functions on \( \Omega \) and that

\[
\begin{align*}
(1.4) \quad a_{jk}(x) &= a_{kj}(x), \\
\sum_{j, k} a_{jk}(x) \eta_j \eta_k &\geq \nu \sum_{j} |\eta_j|^2 / 2 \quad (\nu > 0), \quad a(x) \geq 0.
\end{align*}
\]

The second of (1.4) is valid for all real values of \( \eta_1, \ldots, \eta_m \).

Let us consider the following Hilbert spaces.

1) The space \( L^2 \) of real functions \( y = y(x); x \in \Omega \) which are square integrable in \( \Omega \), with the usual definition of scalar product (and, consequently, of norm):

\[
(1.5) \quad (y, z)_{L^2} = \int_{\Omega} y(x) z(x) d\Omega, \quad ||y||_{L^2} = (y, y)_{L^2}^{1/2}.
\]

2) The space \( H^1 \) of the functions \( y = y(x); x \in \Omega \) which are square integrable in \( \Omega \) together with their first partial derivatives. These derivatives must be intended in Sobolev's generalized sense, or in that of the theory of distributions (in other words: \( g(x) = \frac{\partial y}{\partial x_k} \) means that \( g(x) \in L^1 \) and is such that

\[
\int_{\Omega} y(x) \frac{\partial \varphi(x)}{\partial x_k} d\Omega = - \int_{\Omega} g(x) \varphi(x) d\Omega,
\]

\( \forall \varphi(x) \) continuous on \( \Omega \) together with all its partial derivatives and with compact support on \( \Omega \). Therefore the vanishing on \( \partial \Omega \) must be intended in the sense of the variational theory of elliptic equations (as will be pointed out later).

The scalar product and the norm in \( H^1 \) can be defined in the
following way:

\[(y, z)_{H^1_0} = \int_{\Omega} \left( \sum_{j,k} a_{jk}(x) \frac{\partial y(x)}{\partial x_j} \frac{\partial z(x)}{\partial x_k} + a(x) y(x) z(x) \right) d\Omega, \]

\[||y||_{H^1_0} = \sqrt{(y, y)_{H^1_0}}, \]

It is worth while recalling that the space \(H^1_0\) can be obtained as the closure in the norm defined above, of the space of functions continuous on \(\Omega\) together with their first derivatives and with compact support on \(\Omega\).

As is well known, \(H^1_0 \subset L^2\); the embedding of \(H^1_0\) in \(L^2\) is moreover not only continuous (\(||y||_{L^p} \leq \rho ||y||_{H^1_0} \forall y \in H^1_0\), with \(\rho\) positive constant, independent of \(y\)) but even compact (or completely continuous): this means that every sequence \(\{y_n\}\) bounded in \(H^1_0\) contains a subsequence \(\{y_{n_k}\}\) which converges in \(L^2\).

3) The space \(E = H^1_0 \times L^2\), Cartesian product of \(H^1_0\) by \(L^2\).

Each element \(Y \in E\) is therefore constituted by a couple \(\{(y_0, y_1)\}\), with \(y_0 \in H^1_0\), \(y_1 \in L^2\) and

\[(Y, Z)_E = (y_0, z_0)_{H^1_0} + (y_1, z_1)_{L^2}, \]

\[||Y||_E = ||y_0||_{H^1_0} + ||y_1||_{L^2}.\]

We shall call \(E\) the energy space and the metric defined by the second of (1.7) the energy metric.

Let us assume that the known term \(f(t, x)\) and the unknown function \(y(t, x)\) satisfy the following conditions.

1) Put \(f(t) = \int f(t, x) ; x \in \Omega|, f(t)\) takes its values in \(L^2\), for almost all \(t \in J\) and has a summable norm in every bounded interval \(\Delta\); in other words

\[\int_\Delta ||f(t)||_{L^2} dt = \int_\Delta \left\{ \int_\Omega |f(t, x)| d\Omega \right\}^{1/2} < +\infty.\]

Hence

\[(f(t) \in L^1_{\text{loc}}(J; L^2)).\]

2) Put \(y(t) = \int y(t, x); x \in \Omega|, y(t)\) takes its values in \(H^1_0\) and is continuous on \(J\); in other words

\[\lim_{\tau \to 0} ||y(t + \tau) - y(t)||_{H^1_0} = \]

\[= \lim_{\tau \to 0} \left\{ \int_\Omega \sum_{j,k} a_{jk}(x) \frac{\partial (y(t + \tau, x) - y(t, x))}{\partial x_j} \frac{\partial (y(t + \tau, x) - y(t, x))}{\partial x_k} + a(x)(y(t + \tau, x) - y(t, x))^2 \right\} d\Omega = 0.\]
We shall, further, assume that $y(t)$ is $L^2$-differentiable and that its derivative $y'(t) = \left( \frac{\partial y(t, x)}{\partial t} ; x \in \Omega \right)$ (in the strong sense, i.e.

$$\lim_{\tau \to 0} \left\| \frac{y(t + \tau) - y(t)}{\tau} - y'(t) \right\|_{L^2} = 0$$

is continuous on $J$.

Hence

$$y(t) \in C(J; H^1_0), \quad y'(t) \in C(J; L^2).$$

We may note that, as $y(t, x)$ represents the displacement at the time $t$ of the point $x$ of the membrane, the quantities

$$\frac{1}{2} \left\| y(t) \right\|^2_{H^1_0}, \quad \frac{1}{2} \left\| y'(t) \right\|^2_{L^2}$$

measure the potential energy and the kinetic energy respectively, at the given time, of the membrane.

If $Y(t) = \left\| y(t) \right\|, y'(t) \left( \text{i.e. } Y(t, x) = \left\{ y(t, x), \frac{\partial y(t, x)}{\partial t} \right\} \right)$, the function $Y(t)$ will be continuous, with values in $E$; furthermore the quantity

$$\frac{1}{2} \left\| Y(t) \right\|^2_L = \frac{1}{2} \left\| y(t) \right\|^2_{H^1_0} + \frac{1}{2} \left\| y'(t) \right\|^2_{L^2}$$

measures the total energy, at the time $t$, of the membrane. The denomination, given to $E$, of energy space is therefore justified.

Let us now recall that, by Hamilton’s principle, the functions $y(t), t \in J$, which describe the possible motions of the membrane are those for which, in whatever way a bounded interval $\Delta$ is taken, the integral

$$\int_{\Delta} \left\{ \frac{1}{2} \left\| y'(t) \right\|^2_{L^2} - \frac{1}{2} \left\| y(t) \right\|^2_{H^1_0} + (f(t), y(t))_{L^2} \right\} dt$$

(Hamiltonian action) is stationary with respect to all the variations $l(t)$ of the same functional class as $y(t))$ and with support $X \subset \Delta$.

By imposing that this variation must vanish, we obtain the variational wave equation

$$\int_\Delta (y'(t), l'(t))_{L^2} - (y(t), l(t))_{H^1_0} + (f(t), l(t))_{L^2} \mid dt = 0$$

that must be verified $\forall l(t)$ with compact support.
If \( y(t) \) is a solution of (1.11) such that the corresponding \( y(t, x) \) satisfies further regularity conditions and if \( \Omega \) is sufficiently smooth, it can be proved, by a well known procedure, that (1.1) and (1.3) are satisfied. The solutions of the variational equation are called, for this reason, weak or generalized solutions of the differential wave equation.

Let us now consider, for (1.11), the initial value problem. If we take \( y_0 \in H_0^1, y_1 \in L^2 \) arbitrarily, we want to find a solution \( y(t), t \in J \), which satisfies the initial conditions
\[
(1.12) \quad y(0) = y_0, \quad y'(0) = y_1,
\]
corresponding to (1.2).

It can be proved that such a solution exists and is unique [18]. It can be obtained by the method of elementary solutions.

Let us consider the sequence \( |u_n| \) of eigenfunctions of the equation (that must be satisfied \( \forall v \in H_0^1 \))
\[
(1.13) \quad (u, v)_{H_0^1} = \lambda^2 (u, v)_{L^2}.
\]

(1.13) admits a sequence \( |\lambda_n| \) of eigenvalues such that
\[
(1.14) \quad 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots, \lim_{n \to \infty} \lambda_n = +\infty,
\]
to which the eigenfunctions \( u_n \) correspond, which satisfy the orthogonality conditions
\[
(1.15) \quad \left( \frac{u_n}{\lambda_n}, \frac{u_m}{\lambda_m} \right)_{H_0^1} = (u_n, u_m)_{L^2} = \delta_{nm}.
\]

The sequence \( |u_n| \) is also complete both in \( L^2 \) and in \( H_0^1 \). If we put
\[
(1.16) \quad y(t) = \sum_{n=1}^{\infty} \frac{\omega_n(t)}{\lambda_n} u_n \quad \left( \frac{\omega_n(t)}{\lambda_n} = \left( y(t), \frac{u_n}{\lambda_n} \right)_{H_0^1} \right),
\]
\[
(1.17) \quad y'(t) = \sum_{n=1}^{\infty} \frac{\omega'_n(t)}{\lambda_n} u_n \quad \left( \frac{\omega'_n(t)}{\lambda_n} = \left( y(t), \frac{u_n}{\lambda_n} \right)_{L^2} \right)
\]
(which is correct, as \( y(t) \) and \( y'(t) \) take their values in \( H_0^1 \) and \( L^2 \) respectively) and
\[
(1.17) \quad f(t) = \sum_{n=1}^{\infty} \varphi_n(t) u_n \quad \left( \varphi_n(t) = \left( f(t), u_n \right)_{L^2} \right),
\]
we find that $\omega_n(t)$ satisfies the equation

$$\omega_n(t) + \lambda_n^2 \omega_n(t) = \lambda_n \varphi_n(t).$$

Consequently,

$$\omega_n(t) = x_n \cos \lambda_n t + \beta_n \sin \lambda_n t + \int_0^t \varphi_n(\eta) \sin \lambda_n (t - \eta) \, d\eta,$$

where the constants $x_n$ and $\beta_n$ are determined by the initial conditions (1.12) and, precisely,

$$x_n = \left( y_n, \frac{u_n}{\lambda_n} \right)_{H^0}, \quad \beta_n = (y_n, u_n)_{L^2}.$$

Let us now consider the function $Y(t) = y(t), y'(t)|$, which we shall again call solution of (1.11); it is clear that the range $\mathcal{R} Y(t)$ is, in $E$, a continuous line.

If $Z(t) = |s(t), s'(t)|$ is a second solution, corresponding to the known term $g(t)$, we can prove the following fundamental relation

$$\frac{d}{dt}(Y(t), Z(t))_E = (f(t), s'(t))_{L^2} + (g(t), y'(t))_{L^2},$$

from which, setting $Z(t) = Y(t)$, follows

$$\frac{d}{dt} \frac{1}{2} \| Y(t) \|_E^2 = (f(t), y'(t))_{L^2}.$$

and. integrating between $t_i$ and $t_e$,

$$\frac{1}{2} \| Y(t_e) \|_E^2 - \frac{1}{2} \| Y(t_i) \|_E^2 = \int_{t_i}^{t_e} (f(t), y'(t))_{L^2} \, dt.$$

The right hand term represents the work in the time interval $t_i - t_e$ of the force $f(t) = |f(t, x); x \in \Omega|$; this work equals the variation of the energy of the membrane.

Let, in particular, $f(t) = 0$ and consider the homogeneous wave equation. If $U(t) = |u(t), u'(t)|$ is a solution, it follows from (1.21)

$$\| U(t) \|_E = \text{const.}$$

The ranges are therefore spherical lines, with their centers in the origin: the solutions of the homogeneous wave equation satisfy the principle of conservation of energy.

We now go back to the general case and consider an arbitrary $f(t)$.

Let us assume that (1.11) admits one solution $Y_0(t)$ which is
$E$-bounded (i.e. $\sup_{t \in J} ||Y(t)||_E < +\infty$).

From (1.23) it follows that all the solutions $Y(t)$ are bounded.

The following minimax theorem [19] (which is interesting also from the point of view of mathematical physics) then holds.

I - If

$$\nu(Y) = \sup_{t} ||Y(t)||_E, \quad \tilde{\nu} = \inf_{|Y(t)|} \nu(Y),$$

then there exists one, and only one, solution $\tilde{Y}(t)$ such that

$$\nu(\tilde{Y}) = \tilde{\nu},$$

that is, such that the supremum of the energy has the smallest possible value.

As an application, it can easily be seen that, if $f(t)$ is periodic, with period $T$, the minimal solution $\tilde{Y}(t)$ is also periodic, with period $T$.

β) - The almost-periodicity of the solutions $U(t)$ of the homogeneous wave equation has been proved by various Authors under more and more general hypotheses: MUCKENHAUPT [2], BOCHNER [6], BOCHNER and VON NEUMANN [20], SOBOLEV [21], LADYZENSKAJA [22].

Very significant is Bochner's deduction of the almost-periodicity of $U(t)$, under the hypothesis that $\mathbb{R}U(t)$ is $r.c.$, from the energy conservation principle. In fact $U(t + \tau) - U(t)$ being a solution, $\forall \tau \in J$, of the homogeneous equation, it results

$$||U(t + \tau) - U(t)|| = ||U(\tau) - U(0)||.$$

The «principle of conservation of distances» is then satisfied and our thesis follows from theorem VII.

Subsequently, SOBOLEV succeeded in eliminating the compactness hypothesis and assumed only that the boundary $\partial \Omega$ has continuous curvatures: finally, LADYZENSKAJA has abolished also this last hypothesis. Therefore all the solutions $U(t)$ are a.p.

Observe in fact that (by (1.16) and (1.18) with $f(t) = 0$)

$$U(t) = \left\{ \sum_{n=1}^{\infty} (\alpha_n \cos \lambda_n t + \beta_n \sin \lambda_n) \frac{u_n}{\lambda_n}, \right. \left. \sum_{n=1}^{\infty} (\alpha_n \sin \lambda_n t + \beta_n \cos \lambda_n t) u_n \right\}$$

(1.24)
and the series of a.p. functions on the right hand side of (1.24) converge uniformly. It is, in fact, for \(1 \leq p \leq q\),

\[
\| \sum_{n} (\alpha_n \cos \lambda_n t + \beta_n \sin \lambda_n t) \frac{u_n}{\lambda_n} \|_{H^2}^2 + \\
+ \| \sum_{n} (- \alpha_n \sin \lambda_n t + \beta_n \cos \lambda_n) u_n \|_L^2 = \sum_{n} (\alpha_n^2 + \beta_n^2)
\]

where

\[
\sum_{n} (\alpha_n^2 + \beta_n^2) = \| U(0) \|_E^2 < + \infty.
\]

γ) Let us now consider the non-homogeneous wave equation, assuming that \(f(t)\) is a.p. as a function with values in \(L^2\). In this case it is possible that no bounded solutions exist, as the so-called «resonance phenomenon» may take place; however, as already observed in γ), if a bounded solution exists, all solutions are bounded.

Regarding the almost-periodicity of \(Y(t)\), Zaidman [23] has proved that if the range of \(Y(t)\) is r.c., then \(Y(t)\) is a.p. Subsequently Amerio [24] has eliminated the compactness hypothesis, substituting it with a boundedness hypothesis, which is strictly necessary and has an evident physical interpretation.

The following theorem therefore holds.

II. \(f(t)\) a.p., \(Y(t)\) bounded \(\Rightarrow\) \(Y(t)\) a.p.

The proof was obtained by Amerio by two different methods (those, substantially, of theorems XXVI and XXV).

Observe, first of all, that if \(\sup_{t} \| Y(t) \| = M < + \infty\), the functions \(\omega_n(t)\) defined by (1.18) are a.p. together with their first derivatives \(\omega'_n(t)\). By the first procedure [24] we prove the uniform convergence of the series defining \(Y(t)\):

\[
Y(t) = \left\{ \sum_{1}^{\infty} \frac{\omega_n(t)}{\lambda_n} u_n, \sum_{1}^{\infty} \frac{\omega'_n(t)}{\lambda_n} u_n \right\},
\]

that is the uniform convergence of the series of a.p. non-negative functions

\[
\sum_{n=1}^{\infty} \left( \frac{\omega_n^2(t) + \omega'^2_n(t)}{\lambda_n^2} \right) = \| Y(t) \|_E^2,
\]

making use of theorem XXVII.

The second method [25] consists in recognizing at first (and this is immediate) that \(Y(t)\) is w.a.p. (i.e. \((Y(t), G)_E\) is a.p. \(\forall G \in E\)).
Subsequently, one proves, as in theorem XXV, that the range \( \mathcal{R}(\gamma) \) is r.c.

Contributions and generalizations to the problem treated in [24] and [25] have subsequently been given by Bochner [26], Zaidman [27], Provost [28]. For \( C \)-almost-periodicity of the solutions, see Vaghi [29]. Vaghi [30] has also generalized theorem II to the weak solutions (relative to the problem (1.3)) of the equation

\[
\frac{\partial^2 y}{\partial t^2} + \alpha(t) \frac{\partial y}{\partial t} + \beta(t) y = \gamma(t) \sum_{i,k} \frac{\partial}{\partial x_j} \left( a_{ik}(x) \frac{\partial y}{\partial x_k} \right) + f(t, x),
\]

\( \alpha(t), \beta(t), \gamma(t) \) being real periodic functions, of period \( T \), \( \gamma(t) > 0 \) and \( f(t) = f(t, x) \); \( x \in \Omega \setminus L^1 - a.p. \)

In particular: the \( E \)-bounded eigensolutions of (1.25) are a.p.

It may be noted that, although (1.25) is an equation with coefficients depending on \( t \) of a very particular type, it is actually the variation equation of an interesting equation.

Let

\[
z''(t) + g(z, z') = 0
\]

be a non-linear second order equation, with \( g(z, z') \) continuous function together with its derivatives \( g_z, g_{z'} \), for \( -\infty < z, z' < +\infty \). Assume that (1.26) has a periodic solution \( z = z_0(t) \) with period \( T \). This function is also a solution independent of \( x \) of the non-linear partial differential equation

\[
\frac{\partial^2 z}{\partial t^2} + g\left(z, \frac{\partial z}{\partial t}\right) = \sum_{i,k} \frac{\partial}{\partial x_j} \left( a_{ik}(x) \frac{\partial z}{\partial x_k} \right).
\]

The variation equation corresponding to (1.27) is then (if \( y(t, x) \) is the variation given to \( z_0(t) \))

\[
\frac{\partial^2 y}{\partial t^2} + g_z(z_0(t), z'_0(t)) y + g_{z'}(z_0(t), z'_0(t)) \frac{\partial y}{\partial t} = \sum_{i,k} \frac{\partial}{\partial x_j} \left( a_{ik}(x) \frac{\partial y}{\partial x_k} \right).
\]

We obtain therefore a particular case of (1.25): by (1.3), we consider variations which vanish, \( \forall t \), on \( \partial \Omega \).

Lastly, we recall that Zaidman [31] has studied the elliptic equation with coefficients independent of time and has proved the almost-periodicity of \( L^2 \)-bounded solutions, even when \( \Omega = \mathbb{R}^n \).

Observation. - Theorem II extends to the wave equation the classical theorem of Bohr–Neugebauer on linear ordinary differen-
tial equations, with constant coefficients and a.p. known term. This theorem, in fact, states that *every bounded integral of the equation*

\[ y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + a_n y = f(t) \]

*with* \( a_1, \ldots, a_n \) *constants,* \( f(t) \) *a.p.,* is a.p.


a) In theorem II the Bohr - Neugebauer theorem was generalized to the wave equation. We shall now deal with the extension of the important results obtained by Favard [32] on ordinary linear differential systems, with a. p. coefficients and known term.

Let us consider such a system, in vector form

\[ (2.1) \]

\[ x'(t) = A(t) x(t) + f(t) \]

\( (x(t), f(t)) \) vectors of the complex Euclidean space \( \mathbb{R}^m \), \( A(t) \) \([m, m]\) matrix, \( f(t) \) and \( A(t) \) a. p. functions.

Indicate with \( S_A \) the set of sequences \( s = \{s_n\} \) which are regular with respect to \( A(t) \); therefore, \( \forall s \in S_A \),

\[ \lim_{n \to \infty} A(t + s_n) = A_s(t) \]

uniformly, and \( A_s(t) \) is, like \( A(t) = A_0(t) \), an a. p. matrix.

Favard's theory considers the family of homogeneous equations

\[ (2.2) \]

\[ u'(t) = A_s(t) u(t) \]

and assumes that, \( \forall s \in S_A \), the bounded eigensolutions of \( (2.2) \) satisfy the condition

\[ \inf_{t} \| u(t) \| > 0. \]

Favard then proves that, *if equation (2.1) has a bounded solution* \( x_o(t) \), *it has also one, \( \widetilde{x}(t) \), which is a. p.*

More precisely, \( \widetilde{x}(t) \) *is that solution, which exists and is unique, for which the functional*

\[ \mu(x) = \sup_{t} \| x(t) \| \]

*takes its smallest value when* \( x(t) \) *varies in the class of the bounded solutions of (2.1).*

It is clear from what precedes that, *if (2.2) admits, \( \forall s \in S \), as its only bounded solution, the solution which is identically zero, then a bounded solution* \( x(t) \) *of (2.1) (if it exists) is necessarily a. p.*
This is Favard's first theorem, which gives a very interesting connection between the uniqueness of a bounded solution and its almost-periodicity.

We can see that, in the case of systems with a. p. coefficients, we do not prove the almost-periodicity of all the bounded solutions: it can, in fact, be seen on examples that the bounded eigen-solutions of the homogeneous equation \( u'(t) = A(t)u(t) \) are not necessarily a. p. It has however been possible to prove the almost-periodicity of the minimal solution \( x(t) \), if condition (2.3) holds.

\[ \text{2) Let } V \text{ and } H \text{ be two Hilbert spaces; we assume } V \subseteq H, \text{ dense in } H \text{ and with a continuous embedding (} \| v \|_H \leq k \| v \|_V, k \text{ embedding constant that can be assumed } = 1). \]

Set
\[ Q(x; l) = \int (A(\eta)x(\eta), l(\eta))_H - (A(\eta)x(\eta), l(\eta))_V + (B(\eta)x(\eta), l(\eta))_V \, d\eta, \]
we consider the linear second order functional abstract equation [33]
\[ Q(x; l) = \int (f(\eta), l(\eta))_H \, d\eta. \]

In (2.4) \( A(\eta) \) and \( B(\eta) \) are bounded linear operators \( \forall \eta \in J \), from \( V \) to \( V \) and from \( H \) to \( V \) respectively; therefore
\[ A(\eta) \in \mathcal{L}(V, V) = \mathcal{A}, \quad B(\eta) \in \mathcal{L}(H, V) = \mathcal{B}. \]

\( \mathcal{A} \) and \( \mathcal{B} \) are two Banach spaces and we shall assume that \( A(\eta) \) and \( B(\eta) \) are continuous functions in their respective uniform topologies, that is as functions with values in \( \mathcal{A} \) and \( \mathcal{B} \) (precisely: \( \| A \|_{\mathcal{A}} = \sup_{\| x \|_V = 1} \| Ax \|_V, \| B \|_{\mathcal{B}} = \sup_{\| y \|_H = 1} \| By \|_V \)).

In (2.4), (2.5) \( x(t) \) is the unknown function, \( l(\eta) \) the test function, \( f(\eta) \) the known term. We shall assume that they belong to the following functional spaces:
\[ x(\eta), l(\eta) \in L^1_{loc}(J; V) \]
\[ x'(\eta), l'(\eta), f(\eta) \in L^2_{loc}(J; H). \]

\( l(\eta) \) has, in addition, compact support and (2.5) must be true for all test functions \( l(\eta) \).

The derivatives \( x'(\eta), l'(\eta) \) are intended in the sense of distributions (i.e. \( \int (x(\eta), u'(\eta))_H \, d\eta = -\int (x'(\eta), u(\eta))_H \, d\eta + u(\eta) \)) from \( J \) to \( H \) with compact support and indefinitely differentiable).
It may be noted that in this way we do not impose that \( x(\eta) \) and \( x'(\eta) \) be continuous; the reason for this is that in the present theory of the initial value problem for (2.5) (particularly in the hyperbolical case) the spaces (2.6) are considered.

Equation (2.5) (or a more general one, that however can be treated in the same way) corresponds to the weak formulation of many classical problems on partial differential equations. This, for example, holds for the second order hyperbolic equation with coefficients depending on \( \eta \) (in addition to the spacial variables \( \xi_1, \ldots, \xi_m \)).

Favard's results have been generalized to (2.5) by Amerio [34] in the way we shall see in the present §.

Let us observe that, the functions not being continuous, we cannot speak of almost-periodicity in the sense of Bohr - Bochner: it is however possible to introduce in the following way almost-periodicity in the sense of Stepanov.

Indicating with \( \Delta \) the interval \(-\frac{1}{2} \leq \eta \leq \frac{1}{2}\), consider the Hilbert space \( L^2(\Delta; H) \) of the functions \( g(\eta) \) with values in \( H \), almost everywhere on \( \Delta \), and with square summable norm on \( \Delta : g \in L^2(\Delta; H) \) means therefore that \( g = [g(\eta); \gamma \in \Delta] \) and

\[
||g||_{L^2(\Delta; H)} = \int_{\Delta} ||g(\eta)||_H^2 d\eta^{1/2}.
\]

It is then possible to associate to \( f(\eta) \) a function \( f_\Delta(t) \) from \( J \) to \( L^2(\Delta; H) \), defined by

\[
f_\Delta(t) = f(t + \eta); \gamma \in \Delta].
\]

Therefore

\[
||f_\Delta(t)||_{L^2(\Delta; H)} = \int_{\Delta} ||f(t + \eta)||_H^2 d\eta^{1/2}
\]

and, consequently

\[
(2.7) \lim_{\tau \to 0} ||f_\Delta(t + \tau) - f_\Delta(t)||_{L^2(\Delta, H)} = 0,
\]

which shows that \( f_\Delta(t) \) is continuous.

In what follows, we shall, for simplicity's sake, write \( f(t) \) instead of \( f_\Delta(t) \), adding the indication of the space in which \( f(t) \) is to be considered; we shall therefore write \( f(t) = [f(t + \gamma); \gamma \in \Delta] \) and, by (2.7), \( f(t) \) is \( L^2(\Delta; H) \) — continuous. In other words:

\[
f(\eta) \in L^2_{\text{loc}}(J; H) \Rightarrow f(t) \in L^2(\Delta; H) \text{ — continuous}.
\]
Analogously, \( x(t) \) and \( l(t) \) are \( L^2(\Delta; V) \) — continuous, while \( x'(t) = |x'(t + \eta); \eta \in \Delta| \) and \( l'(t) \) are \( L^2(\Delta; H) \) — continuous.

Regarding the operation of differentiation, it must be noted that \( x'(t) \) defines the \( L^2(\Delta; H) \) — derivative of \( x(t) \); in other words

\[
\lim_{\tau \to 0} \frac{1}{\tau} \left( x(t + \tau) - x(t) - x'(t) \right)_{L^2(\Delta; H)} = 0.
\]

If \( E = V \times H \), we shall indicate with \( W \) the space of functions \( G = \{ g(\eta); \eta \in \Delta \} \) with \( g(\eta) \in L^2(\Delta; V) \), \( g'(\eta) \in L^2(\Delta; H) \). It follows that the functions \( X(\eta) = \{ x(\eta), x'(\eta) \}, X(t) = \{ x(t + \eta); \eta \in \Delta \} \) take their values in \( E \) and \( W \) respectively. We have also:

\[
\| X(\eta) \|_E = \| x(\eta) \|_V^2 + \| x'(\eta) \|_H \| \Delta \|^{1/2},
\]

\[
\| X(t) \|_W = \int_{\Delta} \| x(t + \eta) \|_V^2 + \| x'(t + \eta) \|_H \| \Delta \|^{1/2} \, d\eta \| \Delta \|^{1/2},
\]

and \( X(t) \) is \( W \) — continuous.

We shall say that the function \( f(\eta) \) is a. p. according to Stepanov \( (H - a. p. S^2) \) if to every \( \varepsilon > 0 \) it is possible to associate a relatively dense set \( \tau_i \) such that

\[
\sup_{\tau_i} \int_{\Delta} \| f(t + \tau + \eta) - f(t + \eta) \|_H \| \Delta \|^{1/2} \leq \varepsilon.
\]

As Bochner [3] observed, this definition can be reduced to the classical one of a. p. function (in the sense of Bohr — Bochner). The condition given is, in fact, equivalent to the following

\[
\sup_{\tau} \| f(t + \tau) - f(t) \|_{L^2(\Delta; H)} \leq \varepsilon,
\]

The notations

\[
f(\eta) \in H - a. p. S^2, \quad f(t) \in L^1(\Delta; H) - a. p.
\]

are therefore equivalent.

We shall say that \( f(\eta) \) is \( H - w. a. p. S^2 \) if the corresponding \( f(t) \) is \( L^1(\Delta; H) \) — w. a. p., that is if the scalar product

\[
(f(t), g)_{L^1(\Delta; H)} = \int_{\Delta} (f(t + \eta), g(\eta))_H \, d\eta
\]

is a. p. \( \forall g \in L^2(\Delta; H) \).

Analogously, the conditions

\[
x(\eta) \in V - a. p. S^1 \text{ and } x'(\eta) \in H - a. p. S^2 (\text{i. e. } X(\eta) \in E - a. p. S^2)
\]

are equivalent to the condition

\[
X(t) \in W - a. p.
\]
By the definition, \( X(\eta) \in W_{\text{a.p.}} \) means that \( X(t) \) is \( W \)-a.p., that is, \( \forall G \in W \), the scalar product

\[
(X(t), G)_W = \int_\Delta (X(t + \eta), G(\eta)) d\eta =
\]

\[
+ \int_\Delta (f(t + \eta), g(\eta)) + (f'(t + \eta), g'(\eta)) d\eta
\]

is an a. p. function.

We shall again call \( X(t), L(t) = \{l(t + \eta) ; \eta \in \Delta\} \) \( W \)-bounded function \( (\text{i.e.} \sup_t ||X(t)||w < \infty) \).

We shall put

\[
\varphi(Z, \tau) = \sup \| Z(t + \tau) - Z(t) \|w \quad (\forall \tau \in J)
\]

\[
\nu(Z) = \sup \| Z(t) \|w.
\]

Let \( \Lambda_Z \) be the set of \( W \)-bounded functions \( X(t) \) such that

\[
\varphi(X, \tau) \leq \varphi(Z, \tau) \quad \forall \tau \in J.
\]

Furthermore, let \( \Lambda_Z, \Lambda_Z, \Lambda, \Lambda_Z, \Lambda \) be the set of the solutions of (2.5) \( \in \Lambda_Z \) and \( \Lambda_Z, \Lambda \) the set of eingenvalues \( U(t) \) of the homogeneous equation

\[
Q(u; l) = 0
\]

which are differences between functions \( \in \Lambda_Z, \Lambda \).

It is clear that the sets \( \Lambda_Z, \Lambda, \Lambda, \Lambda, \Lambda \) (to which must be added the identically null solution) are convex; if, in addition, an \( W \)-bounded solution \( X_c(t) \) exists, the set \( \Lambda_Z, \Lambda, \Lambda, \Lambda \), is not empty.

The following minimax theorem can now be proved.

I - Let us assume that:

1) There exists a \( W \)-bounded \( Z(t) \) such that \( \Lambda_Z, \Lambda, \Lambda, \Lambda \) is not empty;

2) \( \forall U(t) \in \Lambda_Z, \Lambda \), it results

\[
\inf_t \| U(t) \|w > 0.
\]

Then, if

\[
\tilde{\nu} = \inf_{\Lambda_Z, \Lambda, \Lambda, \Lambda} \mu(X),
\]

...
there exists in $\Lambda Z, Q, f$ one, and only one, solution $\bar{X}(t)$ such that
\[ \mu(\bar{X}) = \bar{\mu}. \]

It can be observed, comparing this theorem with the minimax theorem regarding the wave equation and with the one of FAVARD for ordinary systems, that the class of solutions $\Lambda Z, Q, f$ is now more restricted, owing to condition (2.10). The reason for this will appear in theorem III. On the other hand, the rather restrictive condition (2.11) (which will be called "Favard's condition") is here broadened because we impose that it be satisfied only by the $U(t) \in \Lambda Z, Q,$ and not by all the $W -$ bounded eigensolutions.

Lastly, we observe that condition (2.11) implies that, in $\Lambda X_0, Q, f,$ the uniqueness theorem for the initial value problem must hold, that is

\[
X_1(t), \ X_2(t) \in \Lambda X_0, Q, f, \ X_1(0) = X_2(0) \Rightarrow X_1(t) = X_2(t).
\]

In fact, if $X_1(t_0) \neq X_2(t_0)$, then $X_1(t) - X_2(t) = U(t) \in \Lambda X_0, Q$ and from (2.11) follows $\|U(0)\| > 0$, which is absurd.

\(\delta\) In what follows, we shall assume that the operators $A(\eta), B(\eta)$ are $\mathcal{A} -$ a. p. and $\mathcal{B} -$ a. p. respectively and that $f(t)$ is $L^*(\Lambda; H) - w. a. p.$

If $s = |s_n|$ is a regular sequence with respect to $A(\eta), B(\eta), f(\eta)$ simultaneously and if $S$ is the family of such sequences, it results, uniformly,

\[
\lim_{n \to \infty} A(\eta + s_n) = A_\tau(\eta)
\]

\[
\lim_{n \to \infty} B(\eta + s_n) = B_\tau(\eta)
\]

\[
\lim_{n \to \infty} f(t + s_n) = f_\tau(t)
\]

and $A_\tau(\eta), \ B_\tau(\eta)$ are a. p., while $f_\tau(t)$ is w. a. p.

Let us now consider, $\forall s \in S$, the equation

\[ Q_s(x; l) = \int(f(\eta), \ l(\eta)) d\eta, \]

where

\[
Q_s(x, l) = \int|\lambda x'(\eta), \ l'(\eta) - (A_\eta x(\eta), \ l(\eta))v +
+ (B_\eta x'(\eta), \ l(\eta))v| d\eta.
\]

It can be shown that, if (2.5), considered in an interval $a \leftarrow +\infty$
(i.e. if \( l(\gamma) = 0 \) for \( \gamma \leq a \) admits in the interval \( a + \frac{1}{2} \rightarrow + \infty \) a \( W \)-bounded solution \( \tilde{X}(t) \), then there exists also a \( W \)-bounded solution \( X_0(t) \) (Sup \( || X_0(t) ||_W < + \infty \)).

It is then easy to prove that each equation (2.12) has one \( W \)-bounded solution \( X_s(t) \); more precisely

\[
\varphi(X_s; \tau) \leq \varphi(X_0; \tau), \quad \nu(X_s) < \nu(X_0).
\]

The set \( \Lambda_{X_0, q_s, t_s} \) is therefore not empty, \( \forall s \in S \).

The following theorem of \( W \)-weak almost-periodicity then holds.

\( II \) - Let us assume that:

1) Equation (2.5) has one \( W \)-bounded solution \( X_0(t) \);

2) The operators \( A(\eta), B(\eta) \) and the known term \( f(t) \) are respectively \( \mathcal{A} \)-a. p., \( \mathcal{B} \)-a. p. \( L^1(\Delta; H) \)-w. a. p.;

3) \( \forall s \in S \) and \( U(t) \in \Lambda_{X_0, q_s} \) it results

\[
\inf_{t} ||U(t)||_W > 0.
\]

Then the minimal solution \( \tilde{X}(t) \) (in the set \( \Lambda_{X_0, q_s} \)) is \( W \)-w.a.p. Furthermore, \( \forall s \in S \),

\[
\lim_{n \to \infty} \tilde{X}(t + s_n) = \tilde{X}_s(t)
\]

uniformly and \( \tilde{X}_s(t) \) is a w.a.p. solution of (2.13), minimal in the set \( \Lambda_{X_0, q_s, t_s} \).

The proof (for which the uniqueness of the minimal solution is essential) can be obtained by extending a procedure based on Bochner's criterion, given by Favard for ordinary systems.

4) Let us assume that equation (2.5) has a \( W \)-bounded and \( W \)-uniformly continuous solution \( X_0(t) \). In this case, also the minimal solution \( \tilde{X}(t) \) is \( W \)-uniformly continuous.

By (2.1), we have, in fact,

\[
\varphi(\tilde{X}; \tau) \leq \varphi(X_0; \tau), \quad \forall \tau \in J,
\]

and, therefore, \( X_0(t) \) being uniformly continuous,

\[
\lim_{\tau \to 0} \varphi(\tilde{X}; \tau) \leq \lim_{\tau \to 0} \varphi(X_0; \tau) = \varphi(0).
\]

The following \( W \)-almost-periodicity theorem can now be proved.
III - Let us assume that:

1) Equation (2.5) has a $W$-bounded and $W$-uniformly continuous solution $X_0(t)$;

2) The operators $A(\sigma)$, $B(\sigma)$ and the known term $f(t)$ are respectively $\mathcal{A}$ - a. p., $\mathcal{B}$ - a. p., $L^p(\Delta; H)$ w. a. p.; $A(\sigma)$ satisfies, in addition, the ellipticity condition

$$\langle R(A(\sigma)x, x) \rangle_V \geq \nu \|x\|^2_V \quad (\nu > 0);$$

3) $\forall s \in S$ and $\forall U(t) \in AX_0, q$, it results

$$\inf_{t} \|U(t)\|_W > 0;$$

4) The embedding of $Y$ in $B$ is compact.

Then the minimal solution, $\hat{X}(t)$, is $W$ - a. p.

The theorem is proved by showing that the range of $\hat{X}(t)$ is r. c., using a compactness theorem which holds for the solutions of (2.5).

It may be observed that, for the problem of the vibrating membrane, treated in § 1, conditions 2) and 4) are obviously verified and so is condition 3) (of FAVARD) because, by the energy conservation principle, $\|U(t)\|_W = \text{const}$.

The problem of proving theorem III eliminating the hypothesis that $X_0(t)$ is $W$ - uniformly continuous (and assuming eventually that $f(t)$ is $L^p(\Delta; H)$ - a. p. and not only $L^p(\Delta; H)$ w. a. p.) is still open. See, on this subject, a typical example, examined by PROUSE [35].

Finally, it is evident that theorems II and III extend also to equation (2.5) the first theorem of Favard.

Observation I - It is possible to effect the harmonic analysis of the minimal solution $\hat{X}(t)$ (and consequently of the $\hat{X}, (t)$). If, for simplicity's sake, we assume $f(t) \in L^p(\Delta; V)$ - a. p., every sequence $s \in S$ is regular also for $\hat{X}(t)$. From theorem XIII of Chapter I it follows that every characteristic exponent of $\hat{X}(t)$ is a linear combination, with integer coefficients, of a finite number of characteristic exponents of $A(\sigma)$, $B(\sigma)$, $f(t)$.

Observation II - Theorems I, II and III hold also for more general equations [36], assuming the space $V$ to be uniformly convex and $H$ semicomplete. In this case, the proof of Theorem III can be obtained by assuming that a continuous dependance
theorem from the operator and the known term holds; such a theorem however has been proved only in particular cases. Zaidman [37] has shown that this is true for the wave equation; more precisely, for the Cauchy problem under the assumption that the initial values (1.2) are given on the whole of $R^n$ (i.e. $\Omega = R^n$) and $a(x) \geq \rho > 0$. In this case the $E$-bounded eigensolutions are not, in general, a. p. Such is, however, the minimal solution $\tilde{X}(t)$ if $f(t)$ is $L^1 - a. p.$

Observation III - Theorem III concerns the hyperbolic case of equation (2.5), with variable operators. Problems related to the parabolic and elliptic cases have been examined respectively by Prouse [38] and by Ricci and Vaghi [39].


a) Let $\Omega$ be an open, bounded and connected set of the plane $\xi = |\xi_1, \xi_2|$ with boundary $\partial \Omega$. The "classical," problem we shall examine consists in determining, in the interval $J$, the vectors $x(\eta, \xi) = x_1(\eta, \xi), x_2(\eta, \xi)$ and the scalars $p = p(\eta, \xi)$ satisfying the Navier - Stokes system

\[
\frac{\partial x_1}{\partial \eta} + \sum_{i=1}^2 x_i \frac{\partial x_i}{\partial \xi} - \mu \Delta x_i = - \frac{\partial p}{\partial \xi_j} + f_i(\eta, \xi)
\]

(3.1)

\[
\frac{\partial x_1}{\partial \xi_1} + \frac{\partial x_2}{\partial \xi_2} = 0 \quad (\xi \in \Omega; j = 1, 2)
\]

and the boundary conditions

(3.2)

\[x_i(\eta, \xi)|_{\xi \in \partial \Omega} = 0 \quad (\eta \in J; j = 1, 2)\]

In (3.1), $x_i$ are the components of the fluid velocity, $p$ the pressure, $\mu > 0$ the viscosity coefficient, $\Delta$ the Laplace operator and $f_i(\eta, \xi)$ are the given components of the force of mass. All functions considered are real.

To this classical problem can be associated, as is well known, a weak problem, and it is to this problem that we shall refer.

Let $\mathcal{U}$ be the manifold of indefinitely differentiable vectors, with null divergence (i.e. satisfying the second of (3.1)) and compact support in $\Omega$. We shall indicate by $N$ and $N'$ the closures of $\mathcal{U}$ in $L^1$ and $H^1_0$ respectively.
As \( N \) and \( N^1 \) are subspaces respectively of \( L^2 \), and \( H^i \), it will be

\[
(u, v)_N = (u, v)_{L^2} = \int_{\Omega} u(\xi) v(\xi) \, d\Omega
\]

(3.3)

\[
(u, v)_{N^1} = (u, v)_{H^i} = \int_{\Omega} \sum_i \frac{\partial u}{\partial \xi_i} \frac{\partial v}{\partial \xi_i} \, d\Omega.
\]

Furthermore, we shall put

\[
b(u, v, w) = \int \sum_{i,j} u_j \frac{\partial v_j}{\partial \xi_i} w_i \, d\Omega.
\]

(3.4)

The weak form which can be deduced for the Navier-Stokes equation is then the following

\[
(3.5) \int_J \left[ u(x(\eta), h(\eta))_{H^i} - (x(\eta), h'(\eta))_{L^2} + b(x(\eta), x(\eta), h(\eta)) \right] \, d\eta = \int (f(\eta), h(\eta))_{L^2} \, d\eta
\]

In equation (3.5)

\[
x(\eta) = \left| x(\eta, \xi); \xi \in \Omega \right|, h(\eta) = \left| h(\eta, \xi); \xi \in \Omega \right|, f(\eta) = \left| f(\eta, \xi); \xi \in \Omega \right|
\]

and

\[
h'(\eta) = \left| \frac{\partial h(\eta, \xi)}{\partial \eta}; \xi \in \Omega \right|.
\]

\( x(\eta) \) is the unknown function, \( h(\eta) \) the test function, \( f(\eta) \) the known term. (3.5) must hold for all \( h(\eta) \) with compact support on \( J \).

The functions considered must belong to the following functional spaces:

\[
x(\eta) \in L^1_{1oc}(J; N^1) \cap L^\infty_{1oc}(J; N)
\]

\[
h(\eta) \in C(J; N^1), h'(\eta) \in L^1_{1oc}(J; N)
\]

\[
f(\eta) \in L^1_{1oc}(J; L^2).
\]

It has been proved by Prodi [40] that the solutions \( x(\eta) \), modified eventually on a set of measure zero, are \( L^1 \) — continuous: consequently \( x(\eta) \in L^1_{1oc}(J; N^1) \cap C(J; N) \).

The theory of the Navier-Stokes equation in an interval \( \alpha \leq \beta \) can be made assuming in (3.5) the interval \( \alpha \leq \beta \) in place of \( J \) as interval of integration; \( h(\eta) \) is then assumed with com-
pact support on $x^{-\beta}$. The initial value problem $x(t) = x_0$ has then one and only one solution $x(t) \in N$; this has been proved, with different techniques, by HOPF [41], LADYZENSKAJA [42], LIONS and PRODI [43], PRODI [40]. The case of $f(t)$ periodic (in two or more dimensions) has been considered by PRODI [40], YUDOVIC [44], PROUSE [45].

The study of the solutions in an unbounded interval and the proof of the existence of an a. p. solution has been made by PROUSE [47] who has proved, among others, the following theorems.

I - Let $x(t)$ be the solution, in $\tau \rightarrow + \infty$, of the Navier-Stokes equation, satisfying the initial condition $x(t_0) = x_0$. Then

$$\text{Sup } \|f(t)\|_{L^2} = K < + \infty \Rightarrow \text{Sup } \|x(t)\|_{L^2} = M < + \infty,$$

where $M$ depends only from $K, \|x_0\|_{L^2}, \nu, \Omega$.

II - Let

$$\text{Sup } \|x(t)\|_{L^2} = K < \infty.$$

There exists then at least one $L^2$-bounded solution $x(t)$: precisely,

$$\text{Sup } \|x(t)\|_{L^2} = K < \infty,$$

where $M$ depends only from $K, \nu, \Omega$.

If, in addition, $K$ is sufficiently small ($K \leq K_\nu$, depending only from $\nu, \Omega$) the bounded solution $x(t)$ is unique.

We introduce now, as in § 2, the functions

$$x(t) = x(t + \eta); \eta \in \Delta, \quad f(t) = f(t + \eta); \eta \in \Delta,$$

$\Delta$ being the interval $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$.

The following propositions then hold.

III - $\text{Sup } \|f(t)\|_{L^2} \leq K_\nu, f(t)L^2(\Delta; L^2) - w. a. p. \Rightarrow x(t) L^2 - w. a. p.$

and $x(t)L^2(\Delta; L^2) - a. p.$

As we can see, the almost periodicity of the bounded solution follows from its uniqueness; we are in the same order of ideas as FAVARD's first theorem (generalised by AMERIO [47] to non linear ordinary systems).

We now give the final result.
IV - \( f(t) L^p(\Delta; L^q) - a. p. \), \( \text{Sup} \| f(\eta) \|_{L^p} = K_1 \leq K_0 \) (\( K_i \) depending only from \( \mu \), \( \Omega \) implies \( \bar{x}(\eta) L^1 - a. p. \), \( \bar{x}(t) L^1(\Delta; H_0^1) - a. p. \).

Prouse's theory holds if \( \Omega \) is a two-dimensional set because the following inequality of Ladyzenskaja

\[
\| x \|_{L^p} \leq \sqrt{2} \| x \|_{L^1} \| x \|_{H^1_0}
\]

is used, which is true only for two-dimensional sets.

(\( K \) A study of the Navier-Stokes equation in more than two dimensions, in view of proving that "sufficiently small," solutions are a. p., has been made by Foias [48]. This Author has proved the following theorem.

Let \( f(\eta) \) be \( L^2 - a. p. \). If there exists, in \( J \), a solution \( \bar{x}(\eta) \) such that, for a certain \( p \), with \( 3 < p \leq +\infty \), it is

\[
\text{Sup} \| \bar{x}(\eta) \|_{L^p} \leq K
\]

(where \( K \) depends only from \( p \) and \( \Omega \), then \( \bar{x}(\eta) \) is \( L^2 - a. p. \))

4 - A. p. solutions of the wave equation with non-linear dissipative term.

\( a \) Let \( \Omega \) be an open, bounded and connected set \( \in \mathbb{R}^m \), with \( m \leq 5 \), satisfying the cone property.

We consider, for \( \xi \in \Omega \), the wave equation with dissipative term

\[
\frac{\partial^2 x}{\partial \eta^2} + \beta \left( \frac{\partial x}{\partial \eta} \right) = \sum_{i,k=1}^{m} \frac{\partial}{\partial \xi_i} \left( a_{ik}(\xi) \frac{\partial x}{\partial \xi_k} \right) - a(\xi) x + f(\eta, \xi) \quad (\eta \in J),
\]

where the functions \( a_{ik}(\xi) \), \( a(\xi) \) satisfy the conditions given in \( \S 1 \).

We shall assume that the boundary condition

\[
x(\eta, \xi) |_{\xi \in \partial \Omega} = 0 \quad (\eta \in J)
\]

holds.

We shall, furthermore, assume that \( \beta(\xi) \) is a continuous increasing function of \( \xi \in J \), with \( \beta(0) = 0 \). In (4.1), the term \( \beta \left( \frac{\partial x}{\partial \eta} \right) \) represents therefore a passive resistance, opposite to the velocity \( \frac{\partial x}{\partial \eta} \); \( f(\eta, \xi) \) is the force of mass.

Setting, as usual,

\[
x(\eta) = | x(\eta, \xi); \xi \in \Omega |; \quad f(\eta) = | f(\eta, \xi); \xi \in \Omega |
\]
and
\[ \beta(x'(\eta)) = \beta \left( \frac{\partial x(\eta, \xi)}{\partial \gamma} \right); \quad \xi \in \Omega. \]
we shall say that \( x(\eta) \) is a weak solution of (4.1), satisfying the boundary conditions (4.2) if it satisfies the equation
\[ \int_J (x''(\eta), h(\eta))_{L^2} + (x(\eta), h(\eta))_{H^1} + (\beta(x'(\eta)), h(\eta))_{L^2} \, d\eta = \int_J (f(\eta), h(\eta))_{L^2} \, d\eta. \]

In (4.3) we assume that the unknown function \( x(\eta) \), the test function \( h(\eta) \) and the known term \( f(\eta) \) satisfy the conditions:

1) \( x(\eta), x'(\eta) \in L^\infty_{loc}(J; H^\rho), \quad x''(\eta) \in L^\infty_{loc}(J; L^2) \);
2) \( h(\eta) \in L^2_{loc}(J; H^\rho) \) and has compact support;
3) \( f(\eta), f'(\eta) \in L^2_{loc}(J; L^\rho) \).

Furthermore, we shall assume that \( \beta(\xi) \) has, on \( J \), a continuous derivative, verifying the conditions
\[ k_1 (1 + |\xi|^{\rho-1}) \leq \beta'(\xi) \leq k_2 (1 + |\xi|^{\rho-1}), \]
with
\[ 0 < k_1 \leq k_2, \quad 1 \leq \rho \leq 1 + \frac{4}{m - 1} \quad (m \leq 5). \]

Let us observe that (4.4) is satisfied if we take
\[ \beta(\xi) = \mu \xi + \nu |\xi| \quad (\mu > 0, \nu > 0), \]
that is if we consider a passive resistance of "viscous," type for small velocities and of "hydraulic," type for large velocities: it is therefore a physically well acceptable law.

Cauchy's problem for (4.3) as been studied, for arbitrary \( \rho \) and \( m \), by Lions and Strauss [49] who have proved the following existence and uniqueness theorem. If \( f(\eta), f'(\eta) \in L^\infty_{loc}(J_0; L^\rho) \), with \( J_0 = 0 \leftarrow + \infty \), there exists in \( J_0 \) one and only one solution \( x(\eta) \) such that
\[ x(0) = x_0, \quad x'(0) = x_1, \]
\[ \forall x_0 \in H^\rho_0 \cap H^1, \quad x_1 \in H^1_0. \]
The asymptotic behaviour of \( x(\eta) \) as \( \eta \to +\infty \) and the existence of bounded or a. p. solutions has been studied by Prouse [50]: hypoteses \( m \leq 5 \) and the second of (4.5) have been introduced by Prouse in order that, if \( Q = \Omega \times \Delta \), the embedding of \( H^1(Q) \) in \( I_{P+1}(Q) \) be continuous; it would be interesting to see if these conditions, whose phisical meaning is not clear, can be eliminated. We may add that, in the case of periodic \( f(\eta) \), which was treated by Prouse [51] under the same assumptions, these were subsequently eliminated by Prodi [52].

Prouse has proved, among others, the following theorems (we have again called solution the function \( X(\eta) = (x(\eta), x'(\eta)) \).

I - If

\[
\max_{t \to +\infty} \lim_{\eta \to +\infty} \| f(t) \|_{L^1(\Delta; L^1)} < +\infty
\]

then all solutions are, among themselves, \( E \)—asymptotic, when \( \eta \to +\infty \). In other words, if \( X_1(\eta), X_2(\eta) \) are any two solutions

\[
\lim_{\eta \to +\infty} \| X_1(\eta) - X_2(\eta) \|_E = \lim_{\eta \to +\infty} \| x_1(\eta) - x_2(\eta) \|_{H^1} +
\]

\[
+ \| x_1'(\eta) - x_2'(\eta) \|_{L^2}^{1/2} = 0.
\]

II - Let \( f(t), f'(t) \) be \( L^2(\Delta; L^1) \) — bounded. There exists then one, and only one, \( E \) — bounded solution \( \bar{X}(\eta) \).

By theorem I we have then, \( \forall \) solution \( X(\eta) \),

\[
\lim_{\eta \to +\infty} \| \bar{X}(\eta) - X(\eta) \|_E = 0.
\]

III - If \( f(t) \) is \( L^2(\Delta; L^3) \) — w. a. p. and if \( f(t) \) is \( L^3(\Delta; L^3) \) — bounded, then \( \bar{X}(t) \) is \( L^3(\Delta; E) \) — w. a. p.

IV - If \( f(t) \) is \( L^2(\Delta; L^3) \) — a. p. and \( f'(t) \) is \( L^3(\Delta; L^3) \) — bounded, then \( \bar{X}(\eta) \) is \( E \) — a. p.

As a conclusion of Prouse's analysis, it can be said that if \( f(\eta) \) is a. p. according to Stepanov (\( L^2 \) — a. p. \( S' \Leftrightarrow L^2(\Delta; L^1) \) — a. p.) and if \( \sup_t \int_{\Delta} \| f'(t + \eta) \|_{L^2}^2 \, d\eta^{1/2} < +\infty \), there exists one, and only one, solution \( \bar{X}(\eta) \) which is a. p. as a function taking its values in the energy space, that no other solutions with bounded energy exist and that, when \( \eta \to +\infty \), all solutions \( X(\eta) \) are asymptotic, in the energy space, to the solution \( \bar{X}(\eta) \).
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