Luigi Amerio

Abstract almost-periodic functions and functional equations.

Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **20** (1965), n.3, p. 287–334.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1965_3_20_3_287_0>

Bollettino dell'Unione Matematica Italiana, Zanichelli, 1965.

ABSTRACT ALMOST-PERIODIC FUNCTIONS AND FUNCTIONAL EQUATIONS

LUIGI AMERIO

CHAPTER I

ALMOST-PERIODIC FUNCTIONS IN BANACH SPACES

1. - Definition of almost-periodic function. Elementary properties.

The general theory of almost-periodic functions with complex values, created by HARALD BOHR [1] in his two classical papers published on Acta Mathematica in 1925 and 1926, has been greatly developed by WEYL, DE LA VALLÉE-POUSSIN, BOCHNER, STEPANOV, WIENER, BOGOLIUBOV, LEVITAN. An important class of these functions had already been studied, at the beginning of the century, by BOHL and by ESCLANGON.

Bohr's theory was then, in a particular case, extended by MUCKENHAUPT [2] and, subsequently, by BOCHNER [3] and by BOCHNER and von NEUMANN [4] to very general abstract spaces. The extension to BANACH spaces has, in particular, revealed itself of great interest, in view of the fundamental importance of these spaces in theory and applications.

To this extension will be devoted the first chapter of the present paper. In chapter 2 we shall deal with the applications to a p. partial, or, more generally, abstract differential equations, linear or non linear. This means, essentially, the extension of the classical theorems of BOHR-NEUGEBAUER and of FAVARD [5] on ordinary linear differential equations. Such extension can be made, as will be seen, following a procedure whose nature will be quite clear already in the problem of the integration of a.p. functions.

Let X be a BANACH space; if $x \in X$, we shall indicate by $\| x \|$. or by $\| x \|_X$, the corresponding norm.

Let J be the interval $-\infty < t < +\infty$ and

$$(1.1) \qquad x = f(t)$$

a *continuous* function, defined on J and with values in X: an application, in other words, $t \rightarrow f(t)$, from J to X. Continuity will obviously be intended in the *strong sense* (i. e. $\lim\limits_{\tau \rightarrow 0} f(t+\tau) = f(t)$ means that $\| f(t+\tau) - f(t) \| \rightarrow 0$).

When t varies on J the point $x = f(t)$ describes, in the X space, a set which is called the range of the function $f(t)$, indicating it by $\mathcal{R}_{f(t)}$.

A set $E \subseteq J$ is said to be *relatively dense* (r.d.) if there exists a number $l > 0$ (*inclusion length*) such that every interval $a \vdash\!\dashv a+l$ contains at least one point of E.

We shall now say that the function $f(t)$ is almost-periodic (a p.) if to every $\varepsilon > 0$ there corresponds a r.d. set $\{\tau\}_\varepsilon$, such that

$$(1.2) \qquad \operatorname*{Sup}_t \| f(t+\tau) - f(t) \| \leq \varepsilon, \qquad \forall \tau \in \{\tau\}_\varepsilon .$$

Each $\tau \in \{\tau\}_\varepsilon$ is called an *ε-almost period of $f(t)$; to the set* $\{\tau\}_\varepsilon$ therefore corresponds an *inclusion length* l_ε and it is clear that, when $\varepsilon \rightarrow 0$, the set $\{\tau\}_\varepsilon$ becomes rarified, while (in general) $l_\varepsilon \rightarrow +\infty$.

The above definition was given by BOCHNER and is an obvious extension of the definition adopted by BOHR for his theory of a.p. functions. It is, undoubtedly, in itself a very significant definition: its real depth can actually be understood only «a posteriori», from the beauty of the theory constructed on it and the importance of its applications.

The theory of a.p. functions with values in a BANACH space is, in the way it is treated by BOCHNER [3], similar to Bohr's theory of numerical a.p. functions: new developements arise, as is natural, in connection with questions on compactness and boundedness. These questions (which have been recently studied especially in Italy) are of interest particularly in the integration of a.p. functions and, more generally, in the integration of abstract a.p. differential equations.

It is obvious that a continuous periodic function is also a.p. The almost-periodicity condition is however much less restrictive than that of periodicity: for instance, *all trigonometric polynomials*

$$P(t) = \sum_{1}^{n}{}_k a_k e^{i\lambda_k t} \qquad\qquad (a_k \in X, \quad \lambda_k \in J)$$

are a.p. functions; not only, but, as we shall see later, *the class of a.p. functions coincides with the closure, with respect to the uniform convergence on J, of the set of such polynomials.*

Let us now indicate the first properties of a.p. functions, which can be easily desumed from their definition. In what follows we shall omit the proofs, except at some fundamental or typical point. We add that when we say that $f(t)$ is uniformly continuous, or bounded, or that the sequence $\{f_n(t)\}$ converges uniformly etc.. we *always* mean that this occurs on the whole interval J.

When, for sake of clarity, it may be necessary to state in which space $f(t)$ takes its values, we shall say, for instance, that $f(t)$ is X — continuous, or X — a.p., instead of continuous, or a.p. etc.

I - *$f(t)$ a.p. $\Rightarrow$ $f(t)$ uniformly continuous (u.c).*

II - *$f(t)$ a.p. $\Rightarrow$ $\mathfrak{R}_{f(t)}$ relatively compact (r.c.).*

This means that the closure $\overline{\mathfrak{R}}_{f(t)}$ is compact. It may be noted that, in the numerical case (or, equivalently, when X is Euclidean) property II reduces to Bohr's (*$f(t)$ a.p. $\Rightarrow \mathfrak{R}_{f(t)}$ bounded*). In a general BANACH space however, the r.c. sets are bounded sets of a very particular nature. The fact that the range $\mathfrak{R}_{f(t)}$ is r.c. is equivalent to the following: $\forall \varepsilon > 0$, there exists a finite number of points $f(t_1), \ldots, f(t_\nu)$ such that

$$\mathfrak{R}_{f(t)} \subset \bigcup_{k=1}^{\nu} (f(t_k), \varepsilon)$$

(where (x, ε) denotes the open sphere with centre x and radius ε); equivalently, every sequence $\{f(t_n)\}$ contains a convergent subsequence (in other words, for $\mathfrak{R}_{f(t)}$, the principle of Bolzano–Weierstrass holds).

III - *$f_n(t)$ a.p.$(n = 1, 2, \ldots)$, $f_n(t) \rightarrow f(t)$ uniformly $\Rightarrow f(t)$ a.p.*

The class of a.p. functions is therefore *closed* with respect to the topology of uniform convergence.

IV - $f(t)$ *a.p.*, $f'(t)$ *uniformly continuous* $\Rightarrow f'(t)$ *a.p.*

V - $x = f(t) X -$ *a.p.*, $y = g(x)$ *with values in* Y *(Banach) and continuous on* $\overline{\mathfrak{R}}_{f(t)} \Rightarrow g(f(t))$ $Y-$ *a.p.*

In particular:

$$f(t) \ a.p., \ k > 0 \Rightarrow \| f(t) \|^k \ a.p.$$

2. - Bochner's criterion.

The class of a.p. functions has been characterized by BOCHNER by means of a compactness criterion, which plays an essential rôle in the theory and in applications. The starting point consists in considering, together with a given function $f(t)$, the set of its translates $\{f(t+s)\}$ and its closure $\overline{\{f(t+s)\}}$ with respect to uniform convergence.

We shall prove Bochner's criterion by the following analysis.

Let G *be the Banach space of continuous and bounded functions* $f(t)$, *from* J *to* X $(G = C(J;\, X) \cap L^\infty(J;\, X))$, *with norm corresponding to uniform convergence:* if $\tilde{f}$ is the point of G which corresponds to the function $f(t)$, it will therefore be

$$\tilde{f} = \{f(t);\ t \in J\}, \qquad \|\tilde{f}\| = \operatorname*{Sup}_t \|f(t)\| .$$

Let us now consider, together with $f(t)$, the set of the translates $f(t+s)$, $\forall s \in J$. If

$$\tilde{f}(s) = \{f(t+s);\ t \in J\},$$

we have defined an application, $s \to \tilde{f}(s)$, from J to G; furthermore, $\tilde{f}(0) = \tilde{f}$.

We shall call *transformation of Bochner* the operation by which we pass from $f(t)$ to $\tilde{f}(s)$: $\tilde{f}(s)$ will be called the *Bochner transform* of $f(t)$, using also the notation

$$\tilde{f}(s) = \mathfrak{B}(f(t)).$$

Bearing in mind the definition of G, it is clear that the transformation just defined is *linear*; moreover, the correspondance between $f(t)$ and $\tilde{f}(s)$ is *one-to-one*, (note that $\tilde{f}(0) = \{f(t);\ t \in J\}$, that is $f(t)$ is the function corresponding to the value $\tilde{f}(0)$).

The range $\mathfrak{R}_{\tilde{f}(s)}$, in G, of the transform $\tilde{f}(s)$ has the following important properties.

α) $\mathfrak{R}_{\tilde{f}(s)}$ *is a spherical line:* more precisely

$$(2.1) \qquad \| \tilde{f}(s) \| = \operatorname*{Sup}_t \| f(t+s) \| = \operatorname*{Sup}_t \| f(t) \| = \| \tilde{f}(0) \| ;$$

β) $\mathfrak{R}_{\tilde{f}(s)}$ *is described in such a way that the «principle of conservation of distances» holds: in fact*

$$(2.2) \qquad \begin{aligned} \| \tilde{f}(s+\tau) - \tilde{f}(s) \| &= \operatorname*{Sup}_t \| f(t+s+\tau) - f(t+s) \| = \\ &= \operatorname*{Sup}_t \| f(t+\tau) - f(t) \| = \| \tilde{f}(\tau) - \tilde{f}(0) \| ; \end{aligned}$$

γ) $\mathfrak{R}_{\tilde{f}(s)}$ *r.c.* $\Longleftrightarrow \tilde{f}(s)$ *a.p.*

This property, that identifies almost-periodicity with relative compactness, is of the greatest importance.

From theorem II it is obvious that $\tilde{f}(s)$ a.p. $\Rightarrow \mathfrak{R}_{\tilde{f}(s)}$ r.c.

Let us now prove that $\mathfrak{R}_{\tilde{f}(s)}$ r.c. $\Rightarrow \tilde{f}(s)$ a.p.

Firstly, we shall prove that $\mathfrak{R}_{\tilde{f}(s)}$ r.c. $\Rightarrow \tilde{f}(s)$ uniformly continuous. Suppose this is not true; there exist then a constant $\rho > 0$ and two sequences $\{s'_n\}$, $\{s''_n\}$ such that

$$(2.3) \qquad \lim_{n \to \infty} (s_n'' - s'_n) = 0, \quad \| \tilde{f}(s''_n) - \tilde{f}(s'_n) \| \geq \rho .$$

From the property β) and the second of (2.3) it follows

$$(2.4) \qquad \| \tilde{f}(s''_n - s'_n) - \tilde{f}(0) \| \geq \rho .$$

As the range $\mathfrak{R}_{\tilde{f}(s)}$ is r.c. it is possible to extract from $\{s''_n - s'_n\}$ a subsequence (again indicated by $\{s''_n - s'_n\}$) such that

$$(2.5) \qquad \lim_{n \to \infty} \tilde{f}(s''_n - s'_n) = \tilde{g}$$

and, by (2.4),

$$(2.6) \qquad \| \tilde{g} - \tilde{f}(0) \| \geq \rho .$$

On the other hand

$$\tilde{g} = \{ g(t) ;\ t \in J \} \in G$$

and from (2.5) we get

$$\lim_{n \to \infty} \| \tilde{f}(s''_n - s'_n) - \tilde{g} \| = \lim_{n \to \infty} \operatorname*{Sup}_t \| f(t + s''_n - s'_n) - g(t) \| = 0 .$$

It results then, $\forall t \in J$,

$$\lim_{n \to \infty} f(t + s''_n - s'_n) = g(t)$$

and therefore, for the first of (2 3), $f(t)$ being continuous,

$$f(t) = g(t),$$

that is $\| \tilde{f} - \tilde{g} \| = 0$, against (2.6).

We now prove that $\tilde{f}(s)$ is a. p. For this, it is sufficient to show that, $\forall \varepsilon > 0$, the set $\{\tau\}_\varepsilon$ of the ε — almost-periods is r.d.

As the range $\mathcal{R}_{\tilde{f}(s)}$ is r.c., there exist, in correspondance of the given ε, ν values $\tilde{f}(s_1), \dots \tilde{f}(s_\nu)$ such that

$$\mathcal{R}_{\tilde{f}(s)} \subset \bigcup_k^{1 \dots \nu} (\tilde{f}(s_k), \varepsilon).$$

Let us put $l = 2 \max | s_k |$ and fix $a \in J$ arbitrarily; we consider now the interval $\left(a - \frac{l}{2}\right) \vdash\!\dashv \left(a + \frac{l}{2}\right)$ of length l. The point $\tilde{f}(a)$ belongs to one of the ν spheres $(\tilde{f}(s_k), \varepsilon)$; let us suppose that

$$\tilde{f}(a) \in (\tilde{f}(s_{\bar{k}}), \varepsilon) \tag{2.7}$$

and consider the value

$$\tau = a - s_{\bar{k}}. \tag{2.8}$$

It will be

$$a - \frac{l}{2} \leq \tau \leq a + \frac{l}{2} \tag{2.9}$$

and, furthermore, by the property β) and (2.8),

$$\| \tilde{f}(s + \tau) - \tilde{f}(s) \| = \| \tilde{f}(s_{\bar{k}} + \tau) - \tilde{f}(s_k) \| = \| \tilde{f}(a) - \tilde{f}(s_{\bar{k}}) \| . \tag{2.10}$$

From (2.7), (2.10) it follows that, $\forall s \in J$,

$$\| \tilde{f}(s + \tau) - \tilde{f}(s) \| < \varepsilon,$$

that is τ is an ε-almost-period for $\tilde{f}(s)$. For (2.9), the set $\{\tau\}_\varepsilon$ is also r.d., a being arbitrary and l independent of a.

The almost-periodicity of $\tilde{f}(s)$ is therefore proved.

δ) $f(t)$ *a.p.* $\Longleftrightarrow$ $\tilde{f}(s)$ *a.p.*

This property follows immediately from (2.2), as

$$\operatorname*{Sup}_t \| f(t + \tau) - f(t) \| = \operatorname*{Sup}_s \| \tilde{f}(s + \tau) - \tilde{f}(s) \| .$$

Properties δ) and γ) express *Bochner's criterion:*

VI - $f(t)$ *a.p.* $\Longleftrightarrow$ $\mathcal{R}_{\tilde{f}(s)}$ *r.c.*

In other words:

Let $f(t)$ be continuous, from J to X. A necessary and sufficient condition for $f(t)$ to be a.p. is that from every sequence $\{s_n\}$ it may be possible to extract a subsequence $\{l_n\}$ such that the sequence $\{f(t+l_n)\}$ be uniformly convergent.

A very important consequence of Bochner's criterion is that *the sum $f(t)+g(t)$ of two $X-a.p.$ functions is $X-a.p.$; the product $\varphi(t)f(t)$ of $f(t)$, $X-a.p.$, by a numerical a.p. function $\varphi(t)$, is a.p.*

Therefore: *The set of $X-a.p.$ functions $f(t)$ is a vector space. The corresponding set, constituted by the Bochner transforms $\tilde{f}(s)$, forms a subspace (closed, by theorem III) of the space G.*

OBSERVATION I. - As we have already pointed out, Bochner's transformation sets up a correspondance between a function $f(t)$ and its transform $\tilde{f}(s)=\mathfrak{B}(f(t))$, for which the «*principle of conservation of distance*» holds (property β)). If we assume that $f(t)$ itself satisfies a condition more general than β), but essentially of the same nature, we obtain a sufficient condition for almost-periodicity due to BOCHNER [6], and utilized in the study of the homogeneous wave equation. It is well worth noting that such a condition is suggested by the «principle of conservation of energy», which holds for the solutions of that equation.

VII. - *Let us assume that $f(t)$, from J to X, satisfies the following conditions:*

1) *$f(t)$ is continuous and its range $\mathfrak{R}_{f(t)}$ is r.c.;*

2) $\forall \tau \in J$

$$\operatorname*{Inf}_t \| f(t+\tau)-f(t) \| \geq \sigma \operatorname*{Sup}_t \| f(t+\tau)-f(t) \| , \tag{2.11}$$

σ being a constant >0 independent of τ.

$f(t)$ is then a.p.

Let us observe that, if $\| f(t+\tau)-f(t) \|$ is independent of t (that is if the principle of «conservation of distance» holds) 2) is satisfied with $\sigma=1$. The proof of theorem VII was obtained by BOCHNER in the same way as the proof (given in property γ)) that the set $\{\tau\}_\varepsilon$ of ε-almost-periods is r.d.

OBSERVATION II. - *Let $f(t)$ be $X-a.p.$* We shall say that *a sequence $\{s_n\}$ is regular* (with respect to $f(t)$) *if $\{f(t+s_n)\}$ is uniformly convergent*: in other words, *if the sequence $\{\tilde{f}(s_n)\}$ is convergent*

Let us call S_f the set of all regular sequences with respect to $f(t)$. Obviously $S_f = S_{\tilde{f}}$.

Given a second function $g(t)$, *Y—a.p.*, *it is important,* (as will also appear from what follows) *to recognise when* $S_f \subseteq S_g$.

To solve this problem we shall make use of the *translation function* (defined by BOCHNER):

$$v_f(\tau) = \operatorname*{Sup}_t \| f(t+\tau) - f(t) \| = \| \tilde{f}(\tau) - \tilde{f}(0) \| \qquad (\tau \in J). \tag{2.12}$$

In an analogous way, consider the function

$$v_g(\tau) = \operatorname*{Sup}_t \| g(t+\tau) - g(t) \| = \| \tilde{g}(\tau) - \tilde{g}(0) \| .$$

We now construct, in the following way, a function $\omega_{f,g}(\varepsilon)$, which will be called *comparison function* of f with g: $\forall \varepsilon$, with $0 < \varepsilon < \operatorname*{Sup}_{\tau \in J} v_f(\tau)$, we set

$$\omega_{f,g}(\varepsilon) = \operatorname*{Sup}_{v_f(\tau) \leq \varepsilon} v_g(\tau).$$

$\omega_{f,g}(\varepsilon)$ is therefore the supremum of the translation function $v_g(\tau)$ when τ varies in the set of the ε-almost-periods of $f(t)$: each ε-almost-period of $f(t)$ is also a $\omega_{f,g}(\varepsilon)$-almost-period of $g(t)$. It follows that $\omega_{f,g}(\varepsilon)$ is a monotonic, non decreasing function of ε: the limit

$$\omega_{f,g}(0+) \geq 0 \tag{2.13}$$

therefore exists and the following proposition can be proved:

VIII. - $S_f \subseteq S_g \Longleftrightarrow \omega_{f,g}(0+) = 0$.

It follows that, *in order that all sequences* $\{s_n\}$, *regular with respect to* $f(t)$, *be regular with respect to* $g(t)$, *it is necessary and sufficient that the comparison function* $\omega_{f,g}(\varepsilon)$ *be infinitesimal with* ε: it is, in other words, required that, *taken an arbitrary sequence* $\{\tau_n\}$ *of* ε_n*-almost-periods of* $f(t)$, *with* $\varepsilon_n \to 0$, τ_n *be,* $\forall n$, *a* σ_n*-almost-period for* $g(t)$ $(\operatorname*{Sup}_t \| g(t+\tau_n) - g(t) \| = \sigma_n)$ *with* $\sigma_n \to 0$.

3. - Harmonic analysis of a.p. functions.

The harmonic analysis of a.p. functions extends to these the theory of FOURIER expansions of periodic functions.

In the numerical case, this analysis was made by BOHR, WEYL, DE LA VALLÉE POUSSIN, BOCHNER, who constructed at first the

FOURIER series associated to a given $f(t)$: subsequently, it was proved (approximation theorem) that this series is, in some convenient sense, summable to the value $f(t)$. Starting point of the theory is the *theorem of the mean.*

BOGOLIUBOV [7] has, instead, followed an apposite procedure, proving at first directly the *approximation theorem* and later deducting the theorem of the mean and the FOURIER expansion.

For a general BANACH space, the first procedure has been generalised by BOCHNER [3], BOCHNER and von NEUMANN [4], KOPEC [8], ZAIDMAN [9], the second by AMERIO [10].

In what follows we shall keep to this second point of view, as the direct proof of the approximation theorem can be easily obtained, even for BANACH spaces.

First of all, we observe that, $\forall a \in X$, $\lambda \in J$, the function

$$ae^{i\lambda t}$$

is periodic. From this follows the almost-periodicity of all trigonometric polynomials

$$P(t) = \sum_{1}^{m}{}_k a_k e^{i\lambda_k t} \qquad (a_k \in X,\ \lambda_k \in J) \tag{3.1}$$

and, consequently, of every $f(t)$ which is the limit of a uniformly convergent sequence of trigonometric polynomials. The approximation theorem enables us to prove that, in this way, it is actually possible to obtain *all* a.p. functions.

IX. - *If* $f(t)$ *is a.p. there exists,* $\forall \varepsilon > 0$, *a trigonometric polynomial* $P_\varepsilon(t)$ *such that*

$$\operatorname{Sup}_t \| f(t) - P_\varepsilon(t) \| \leq \varepsilon. \tag{3.2}$$

The *theorem of the mean* can be easily deduced from the approximation theorem.

X. - *If* $f(t)$ *is a.p. there exists the mean value*

$$\mathfrak{M}(f(t)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt. \tag{3.3}$$

We observe, first of all, that if

$$\Delta(\alpha) = \begin{cases} 1 & \text{for } \alpha = 0 \\ 0 & \text{for } \alpha \neq 0, \end{cases} \tag{3.4}$$

it results

$$\mathfrak{M}(e^{i\alpha t}) = \Delta(\alpha). \tag{3.5}$$

From (3.1), (3.4), (3.5) follows that the mean value exists for all trigonometric polynomials $P(t)$:

$$\mathfrak{M}(P(t)) = \sum_{1}^{m}{}_k\, a_k \Delta(\lambda_k). \tag{3.6}$$

From this relation and (3.2), (3.3) follows immediately.

Let us now observe that, *if* $f(t)$ *is a.p., also* $f(t)\, e^{-i\lambda t}$ *is a.p.* $\forall\, \lambda \in J$. The function

$$a(\lambda;\, f) = \mathfrak{M}(f(t)e^{-i\lambda t})$$

is therefore defined on J. $a(\lambda;\, f)$ takes its values in X, as does $f(t)$: we shall call this function the *Bohr transform* of the a. p. function $f(t)$.

It can be seen at once that $a(\lambda;\, f) = 0$ *on the whole of* J, *with the exclusion, at most, of a sequence* $\{\lambda_n\}$.

It is, in fact, sufficient to construct a sequence of approximation polynomials $\{P_l(t)\}$ such that

$$\operatorname*{Sup}_t \| f(t) - P_l(t) \| \leq \frac{1}{l} \qquad (l = 1,\ 2,\ \ldots).$$

If then

$$P_l(t) = \sum_{1}^{n_l}{}_k\, a_{lk}\, e^{i\lambda_{lk} t},$$

$$\{\mu_n\} = \bigcup_{l,\, k} \lambda_{lk},$$

we have, for $\lambda \notin \{\mu_n\}$, observing that $a(\lambda;\, P_l) = 0\ \forall\, l$,

$$\| a(\lambda;\, f) \| = \| a(\lambda;\, P_l) + a(\lambda;\, f - P_l) \| = \| a(\lambda;\, f - P_l) \| \leq \frac{1}{l}$$

and, consequently, $a(\lambda;\, f) = 0$.

The values λ_n (and obviously $\lambda_n \in \{\mu_k\}$) for which $a(\lambda_n;\, f) \neq 0$ are called the *characteristic exponents* of $f(t)$. If we put

$$a(\lambda_n;\, f) = a_n,$$

we can associate to $f(t)$ the FOURIER series

$$f(t) \sim \sum_{1}^{\infty}{}_n a_n e^{i\lambda_n t}. \tag{3.7}$$

The a_n are called the *Fourier coefficients* of $f(t)$.

It can be seen that *it is possible to choose the approximating polynomials $P_l(t)$ in such a way that the characteristic exponents λ_{lk} belong to the sequence $\{\lambda_n\}$; hence*

$$P_l(t) = \sum_{1}^{n_l}{}_k \, a_{lk} e^{i\lambda_k t}. \tag{3.8}$$

Moreover one proves the fundamental *uniqueness theorem*:

XI. - *$f(t)$ and $g(t)$ X—a.p., $a(\lambda;\, f) \equiv a(\lambda;\, g) \Rightarrow f(t) \equiv g(t)$.*

The correspondance between almost-periodic functions and their Bohr transforms is therefore one-to-one. A property of the transform $a(\lambda;\, f)$ is given by the following proposition.

XII - $a(\lambda';\, f) = 0 \Rightarrow \lim_{\lambda \to \lambda'} a(\lambda;\, f) = 0$, *that is the Bohr transform is continuous at all points in which it vanishes.*

Furthermore,

$$\lim_{\lambda \to \infty} a(\lambda;\, f) = 0.$$

OBSERVATION I. - It results

$$\lim_{n \to \infty} a_n = 0 \tag{3.9}$$

that is *the sequence of Fourier coefficients of an a.p. function is infinitesimal as n diverges.*

If

$$\operatorname{Sup}_t \| f(t) \| = M \tag{3.10}$$

we also have, by (3.7),

$$\| a_n \| = \| a(\lambda_n;\, f) \| \leq M. \tag{3.11}$$

If X is a Hilbert space, property (3.9) *can be made much more precise. It can in fact be deduced without difficulty* (from the approximation theorem) *that Parseval's equality holds:*

$$\mathfrak{M}(\| f(t) \|^2) = \sum_{1}^{\infty}{}_n \| a_n \|^2. \tag{3.12}$$

OBSERVATION II. - To BOCHNER we owe the construction of a notable approximation polynomial, suggested by Fejer's theorem on the CESARO-summability of the FOURIER series of periodic functions.

Let, first of all, $L = (\beta_1, \beta_2, ...)$ be a *base* for the sequence $\{\lambda_n\}$ of characteristic exponents of the function $f(t)$; this means that *the real numbers* β_n *are linearly independent* (*i. e.* $\sum_{1}^{n}{}_k c_k \beta_k = 0$, c_k *integers* $\Rightarrow c_k = 0$) *and that each* λ_n *is a linear combination with rational coefficients of a finite number of the* β_k. It is clear that a base always exists: if the λ_n are linearly independent, we can take $L = \{\lambda_n\}$; if not, L can be obtained by eliminating those λ_n which are linear combinations of the preceding ones.

If the base L is infinite, Bochner's polynomial $\sigma_m(t)$ is defined by the formula

$$\sigma_m(t) = \sum_{\nu_1, ..., \nu_m}^{-(m!)^2 ... (m!)^2} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \\ \cdots \left(1 - \frac{|\nu_m|}{(m!)^2}\right) a\left(\sum_{1}^{m}{}_k \frac{\nu_k \beta_k}{m!}; f\right) e^{i\left(\sum_{1}^{m}{}_k \frac{\nu_k \beta_k}{m!}\right)t}, \tag{3.13}$$

while, if the base is finite ($L = (\beta_1, ..., \beta_q)$), it is, for $m \geq q$,

$$\sigma_m(t) = \sum_{\nu_1, ..., \nu_q}^{-(m!)^2 ... (m!)} \left(1 - \frac{|\nu_1|}{(m!)^2}\right) \cdots \\ \cdots \left(1 - \frac{|\nu_q|}{(m!)^2}\right) a\left(\sum_{1}^{q}{}_k \frac{\nu_k \beta_k}{m!}; f\right) e^{i\left(\sum_{1}^{q}{}_k \frac{\nu_k \beta_k}{m!}\right)t}. \tag{3.14}$$

It can be proved (and the proof is easily deduced from theorem IX) that

$$\lim_{m \to \infty} \sigma_m(t) = f(t) \tag{3.15}$$

uniformly.

We observe that in (3.13), (3.14)

$$a\left(\sum_{1}^{m}{}_k \frac{\nu_k\beta_k}{m!}\,;\,f\right) \neq 0$$

only if $\sum_{1}^{m}{}_k \frac{\nu_k\beta_k}{m!} = \lambda_j$, characteristic exponent.

Relations (3.13) can therefore be written in the form

$$\sigma_m(t) = \sum_{1}^{N_m}{}_k \gamma_{mk} a_k e^{i\lambda_k t} \tag{3.16}$$

where the convergence factors γ_{mk} depend only from m and the exponents λ_k (but not from the coefficients a_h).

OBSERVATION III. – In observation II of § 2 we found that the facts that $S_f \subseteq S_g$ and that the comparison function $\omega_{f,g}(\varepsilon)$ was infinitesimal, as $\varepsilon \to 0$, were connected.

The harmonic analysis of a.p. functions expresses the condition $S_f \subseteq S_g$ by means of a relation between the characteristic exponents $\{\lambda_n\}$ of $f(t)$ and those, $\{\mu_n\}$, of $g(t)$.

XIII. – *$S_f \subseteq S_g$ if, and only if, each exponent μ_n is a linear combination with integer coefficients of a finite number of exponents λ_k, that is*

$$\mu_n = \sum_{1}^{q_n}{}_k c_{nk}\lambda_k \qquad (c_{nk} \text{ integers}). \tag{3.17}$$

Moreover the almost-periods and the characteristic exponents of an a.p. function are explicitly connected by the following propositions.

XIV. – *Let $\{\lambda_n\}$ be the sequence of characteristic exponents. To every fixed arbitrary $\varepsilon > 0$, there correspond a positive integer N_ε and a number $\delta_\varepsilon > 0$ such that every solution τ of the system of inequalities*

$$| e^{i\lambda_n\tau} - 1 | \leq \delta_\varepsilon \qquad (n = 1, \ldots, N_\varepsilon) \tag{3.18}$$

is an ε-almost-period of $f(t)$.

(It may be noted that, by a theorem of Kronecker, system (3.18) is compatible for every $\delta_\varepsilon > 0$).

For (3.8) it is, in fact,

$$\| f(t+\tau) - f(t) \| \leq \| f(t+\tau) - P_l(t+\tau) \| + \| P_l(t+\tau) - P_l(t) \| +$$
$$+ \| P_l(t) - f(t) \| \leq \frac{2}{l} + \sum_{1}^{n_l}{}_k \| a_{lk} \| \; | e^{i\lambda_k\tau} - 1 | \,.$$

Chosen l in such a way that $\frac{2}{l} \leq \frac{\varepsilon}{2}$, let us impose on τ to satisfy the system of inequalities

$$| e^{i\lambda_k \tau} - 1 | \leq \frac{\varepsilon}{2} \left(\sum_{1}^{n_l} k \, \| a_{lk} \| \right)^{-1} = \delta_\varepsilon \qquad (k = 1, \ldots, n_l).$$

It will then be $\| f(t + \tau) - f(t) \| \leq \varepsilon \quad \forall t \in J$, which proves our thesis.

XV. - *Let* $\{ \lambda_n \}$ *be the sequence of characteristic exponents. Then every* ε*-almost-period,* τ*, satisfies the system of inequalities*

$$| e^{i\lambda_n \tau} - 1 | \leq \varepsilon \, \| a_n \|^{-1}. \tag{3.19}$$

In fact

$$\| a_n (e^{i\lambda_n \tau} - 1) \| = \| \mathfrak{M}((f(t + \tau) - f(t)) e^{-i\lambda_n \tau}) \| \leq \varepsilon.$$

4. - Weakly almost-periodic functions.

Given the BANACH space X, we shall call X^* its dual space (constituted by the linear functionals continuous on X). If $x \in X$, $x^* \in X^*$, we shall indicate by $< x^*, x >$ the complex value that, through the functional x^*, corresponds to x, and by $\| x^* \|$ the norm of x^*.

We shall say that $f(t)$*, with values in* X*, is weakly almost periodic (w.a.p.) if,* $\forall x^* \in X^*$*, the numerical function*

$$< x^*, f(t) >$$

is a.p. [11].

As may be seen, the definition given here has, with respect to that of an a.p. function, the same relation as the definition of weakly continuous function has with respect to that of continuous function.

It is clear (as $| < x^*, x > | < \| x^* \| x \|$) that $f(t)$ *a.p.* $\Rightarrow f(t)$ *w.a.p.* In order to indicate that $\{ x_n \}$ is a sequence converging weakly to x (i.e. if $< x^*, x_n > \to < x^*, x >$, $\forall x^* \in X^*$) we shall make use of the following notations

$$x_n \xrightarrow{*} x, \quad \text{or} \quad \lim_{n \to \infty}{}^* x_n = x.$$

x is called the *weak limit* (which, if it exists, is also unique) of the sequence $\{x_n\}$. Let us remember that, in an arbitrary BANACH space, a sequence $\{x_n\}$ can be *scalarly convergent* (i. e. $\lim\limits_{n\to\infty} < x^*, x_n >$ exists and is finite $\forall x^* \in X^*$) without necessarily being weakly convergent, that is without there being an x which is its weak limit. If this circumstance is not present (i. e. if scalar convergence implies weak convergence) the space X is said to be *semicomplete* (reflexive, and, in particular, HILBERT spaces are semicomplete).

Let us now indicate some properties of w. a. p. functions.

XVI. - *$f(t)$ w.a.p. $\Rightarrow \mathfrak{R}_{f(t)}$ bounded and separable.*

Where necessary, we can therefore assume that X is separable.

XVII - *$f_n(t)$ w.a.p. $(n = 1, 2, \ldots)$, $f_n(t) \xrightarrow{*} f(t)$ uniformly $\Rightarrow f(t)$ w. a. p. ($f_n(t) \xrightarrow{*} f(t)$ uniformly means that, $\forall x^* \in X^*$, $< x^*, f_n(t) > \to < x^*, f(t) >$ uniformly).*

XVIII - *$f(t)$ w.a.p., $f(t + s_n) \xrightarrow{*} f_s(t)$ $\forall t \in J \Rightarrow$ that the convergence is uniform.*

XIX - *Let X be semicomplete and $f(t)$ weakly continuous. Then $f(t)$ w.a.p. $\Longleftrightarrow \forall \{s_n\}$ there exsists $\{l_n\} \subseteq \{s_n\}$ such that $\{f(t + l_n)\}$ is uniformly weakly convergent.*

This proposition extends Bochner's criterion to w.a.p. functions (with, however, a restrictive hypothesis on the nature of the space X).

As we have already observed, $f(t)$ a. p. $\Rightarrow f(t)$ w. a. p. It is interesting to note that the property that has to be added to weak almost-periodicity to obtain almost-periodicity is one of compactness (by theorem XVI, a w a. p. function is bounded).

The following theorem can, in fact, be proved.

XX - *$f(t)$ w.a.p. and $\mathfrak{R}_{f(t)}$ r.c. $\Longleftrightarrow f(t)$ a.p.*

OBSERVATION. - *Let us assume that X is a separable Hilbert space* and let $\{z_n\}$ be a complete orthonormal sequence. Then, if $f(t)$ takes its values in X, it results, $\forall t \in J$,

$$f(t) = \sum_{1}^{\infty}{}_n\, \varphi_n(t) z_n \tag{4.1}$$

where $\varphi_n(t) = (f(t), z_n)$ is the scalar product of $f(t)$ by z_n.

By (4.1) we have

$$\| f(t) \|^2 = \sum_{1}^{\infty}{}_n\, | \varphi_n(t) |^2. \tag{4.2}$$

Let us now prove the following propositions.

XXI - $f(t)$ *w.a.p.* $\Longleftrightarrow \varphi_n(t)$ *a.p.*, $\sum_n^{\infty} |\varphi_n(t)|^2 \leq M^2 < +\infty$.

XXII - $f(t)$ *a.p.* $\Longleftrightarrow \varphi_n(t)$ *a.p.*, $\sum_{1}^{\infty}{}_n |\varphi_n(t)|^2$ *uniformly convergent.*

The necessity of the condition expressed by XXI is evident.

For its sufficiency, observe that, chosen $y \in X$ arbitrarily, it results

$$y = \sum_1^{\infty}{}_n \eta_n z_n \quad (\eta_n = (y, z_n)), \quad \sum_1^{\infty}{}_n |\eta_n|^2 = \|y\|^2).$$

Consider the scalar product $(f(t), y)$; to prove our thesis it is sufficient to show that it is a. p. Now we have

$$(f(t), y) = \sum_1^{\infty}{}_n \varphi_n(t)\bar{\eta}_n \tag{4.3}$$

and the series on the right hand side (constituted by a. p. functions) converges uniformly, since it is, by Schwarz's inequality,

$$\left|\sum_p^q{}_n \bar{\eta}_n \varphi_n(t)\right| \leq \Big\{\sum_p^q{}_n |\eta_n|^2\Big\}^{1/2} \Big\{\sum_p^q{}_n |\varphi_n(t)|^2\Big\}^{1/2} \leq M \Big\{\sum_p^q{}_n |\eta_n|^2\Big\}^{1/2}.$$

To prove that the condition expressed by XXII is necessary, we observe that, $f(t)$ being a. p., the range $\mathfrak{R}_{f(t)}$ is r.c. This implies that, $\forall\, \varepsilon > 0$, there exist a finite number of points $f(t_1), \dots f(t_\nu)$ such that

$$\mathfrak{R}_{f(t)} \subset \bigcup_k^{1 \dots \nu} (f(t_k), \varepsilon).$$

Let us now fix an index m such that

$$\| f_m(t_k) \| = \Big\{\sum_m^{\infty}{}_n |\varphi_n(t_k)|^2\Big\}^{1/2} \leq \varepsilon \qquad (k = 1, \dots \nu)$$

Chosen arbitrarily $\bar{t} \in J$, we have, for a certain t_j, $\|f(\bar{t}) - f(t_j)\| < \varepsilon$, and it results

$$\Big\{\sum_m^{\infty}{}_n |\varphi_n(\bar{t})|^2\Big\}^{1/2} = \| f_m(\bar{t}) \| \leq \| f_m(t_j) \| + \| f_m(\bar{t}) - f_m(t_j) \| \leq$$

$$\leq \| f_m(t_j) \| + \| f(\bar{t}) - f(t_j) \| < 2\varepsilon.$$

As $\bar{t}$ is arbitrary, the thesis is proved.

It is obvious that the condition is sufficient because the uniform convergence of the series $\sum_{1}^{\infty}{}_n \mid \varphi_n(t) \mid^2$ is equivalent to the uniform convergence of the series (of a. p. functions) on the right hand side of (4.1).

It may be noted that, by (4.2), *if the series of a.p. functions*

$$\sum_{1}^{\infty}{}_n \mid \varphi_n(t) \mid^2 \tag{4.4}$$

converges, the norm $\| f(t) \|$ *is a. p.*

We may now ask if the converse is true, that is if, *X being a* Hilbert *space,* $f(t)$ *w.a.p* , $\| f(t) \|$ *a.p.* $\Rightarrow$ $f(t)$ *a.p.* (on this subject it may be noted that, in a Hilbert space, the following proposition holds: $f(t)$ *weakly continuous,* $\| f(t) \|$ *continuous* $\Rightarrow f(t)$ *continuous*).

The answer to this question is however negative, as may be shown by examples. It is necessary to extend the hypothesis of almost-periodicity of the norm to a whole family of functions; precisely to the family associated to $f(t)$ by Bochner's criterion.

Let $f(t)$ be w.a.p. and S_f indicate the family of sequences $\{s_k\}$ *regular* with respect to $f(t)$: in other words, such that

$$\lim_{n \to \infty}{}^{*} f(t + s_n) = f_s(t) \tag{4.5}$$

uniformly.

We obtain in this way a family $\Phi_f = \{f_s(t)\}$ of w.a.p. functions and the following theorem can be proved

XXIII - $f(t)$ *w.a.p.*, $\| f_s(t) \|$ *a.p.* $\forall f_s \in \Phi_f \Rightarrow f(t)$ *a.p.*

5. - Integration of a. p. functions.

If $f(t)$ is an a. p. function with values in a Banach space X, we will write, in what follows,

$$F(t) = \int_0^t f(\eta)\, d\eta. \tag{5.1}$$

The problem of the integration of a. p. functions in Banach spaces is of notable interest also because it serves, so to say, as a *model* for classifying Banach spaces in relation to the theory of abstract a. p. equations.

If X is Euclidean, then Bohr's theorem holds: $F(t)$ bounded $\Rightarrow F(t)$ a. p.

For the general case (X arbitrary BANACH space), *the almost-periodicity of* $F(t)$ *has been proved by Bochner* [3] *under the hypothesis that* $\mathfrak{R}_{F(t)}$ *is r.c.*

This condition is obviously much more restrictive than that of boundedness; it can not however be substituted in the general case by latter, as can be shown on examples (AMERIO [12]). Nevertheless AMERIO [12] has proved that *Bohr's enunciation remains unaltered if the space* X *is uniformly convex* (it holds therefore in HILBERT spaces, in l^p and L^p, with $1 < p < +\infty$).

Let us prove the following theorems.

XXIV - (*Bochner*) — X *arbitrary,* $f(t)$ *a. p.,* $\mathfrak{R}_{F(t)}$ *r. c.* $\Rightarrow$ $\Rightarrow F(t)$ *a. p.*

XXV - (*Amerio*) — X *uniformly convex,* $f(t)$ *a. p.,* $F(t)$ *bounded* $\Rightarrow F(t)$ *a. p.*

a) Proof of theorem XXIV. As $\mathfrak{R}_{F(t)}$ is r. c., $F(t)$ is bounded:

$$\operatorname*{Sup}_t \| F(t) \| = M < +\infty. \tag{5.2}$$

Furthermore, $\forall x^* \in X^*$,

$$| < x^*, F(t) > | = | < x^*, \int_0^t f(\eta)\, d\eta > | =$$

$$= | \int_0^t < x^*, f(\eta) > d\eta \, | \leq \| x^* \| \, M.$$

As $< x^*, f(t) >$ is a.p., from Bohr's theorem it follows that $< x^*, F(t) >$ is a.p.; $F(t)$ is therefore w.a.p.

$\mathfrak{R}_{F(t)}$ has been supposed r.c.; our thesis follows then from theorem XX.

b) Proof of theorem XXV. We have already proved in *a*) (utilizing only the boundedness of $F(t)$) that $F(t)$ is w.a.p. It is therefore sufficient, making use of the properties of uniformly convex spaces, to prove that $\mathfrak{R}_{F(t)}$ is r. c.

We first of all remember that a space X is called a uniformly convex (or CLARKSON) space if in the interval $0 < \sigma \leq 2$ there exists a function $\omega(\sigma)$, with $0 < \omega(\sigma) \leq 1$, such that

$$\| x \|, \; \| y \| \leq 1 \;\; and \;\; \| x - y \| \geq \sigma \Rightarrow \left\| \frac{x+y}{2} \right\| \leq 1 - \omega(\sigma). \tag{5.3}$$

A HILBERT space is uniformly convex. From the parallelogram theorem it follows, in fact,

$$\left\| \frac{x-y}{2} \right\|^2 + \left\| \frac{x+y}{2} \right\|^2 = \frac{1}{2} \left(\| x \|^2 + \| y \|^2 \right)$$

and consequently, if $\| x \| \leq 1$, $\| y \| \leq 1$, $\| x - y \| \geq \sigma$,

$$\left\| \frac{x+y}{2} \right\|^2 \leq 1 - \frac{\sigma^2}{4}.$$

It can be shown that the l^p and L^p $(1 < p < +\infty)$ spaces are uniformly convex; in addition, that uniformly convex spaces are also reflexive.

We now observe that from (5.3) it follows, for any x and y,

$$\text{(5.4)} \qquad \| x - y \| \geq \sigma \max \{ \| x \|, \| y \| \} \Rightarrow \left\| \frac{x+y}{2} \right\| \leq$$
$$\leq (1 - \omega(\sigma)) \max \{ \| x \|, \| y \| \}.$$

Let us assume that the range $\mathfrak{R}_{F(t)}$ is not r.c. There exist then a constant $\rho > 0$ and a sequence $\{ s_n \}$ such that

$$\text{(5.5)} \qquad \| F(s_j) - F(s_k) \| \geq \rho \qquad (j \neq k).$$

We can suppose that $\{ s_n \}$ is regular with respect to $f(t)$ and $F(t)$, that is

$$\text{(5 6)} \qquad \begin{aligned} &\lim_{n \to \infty} f(t + s_n) = f_s(t) \\ &\lim_{n \to \infty}{}^{*} F(t + s_n) = F_s(t) \end{aligned}$$

uniformly. The last relation follows from Bochner's criterion (theorem XIX), noting that the space X is semicomplete (being reflexive).

It is also

$$F(t + s_j) = F(s_j) + \int_0^t f(\eta + s_j) \, d\eta$$

and, consequently, for $j \neq k$,

$$\| F(t + s_j) - F(t + s_k) \| \geq \| F(s_j) - F(s_k) \| -$$
$$- \left\| \int_0^t (f(\eta + s_j) - f(\eta + s_k)) \, d\eta \right\|.$$

If we fix $t \in J$, we will have, by (5.5) and the first of (5.6),

$$\| F(t+s_j) - F(t+s_k) \| \geq \frac{\rho}{2} \quad \text{for } j > k \geq n_t .$$

Therefore, by (5.2),

$$\| F(t+s_j) - F(t+s_k) \| \geq \frac{\rho}{2M} \max \{ \| F(t+s_j) \| , \ \| F(t+s_k) \| \}$$

and, by (5.4),

$$\left\| \frac{F(t+s_j) + F(t+s_k)}{2} \right\| \leq \left(1 - \omega\left(\frac{\rho}{2M}\right)\right) \max \{ \| F(t+s_j) \| ,$$
$$\| F(t+s_k) \| \} \leq \left(1 - \omega\left(\frac{\rho}{2M}\right)\right) M.$$

From the second of (5.6) it then follows

$$\| F_s(t) \| \leq \left(1 - \omega\left(\frac{\rho}{2M}\right)\right) M$$

and, consequently,

$$\sup_t \| F_s(t) \| \leq \left(1 - \omega\left(\frac{\rho}{2M}\right)\right) M. \tag{5.7}$$

Relation (5.7) is absurd; from the second of (5.6) follows in fact, the weak convergence being uniform,

$$\lim_{n \to \infty}{}^{*} F_s(t - s_n) = F(t)$$

and therefore

$$\| F(t) \| \leq \min \lim_{n \to \infty} \| F_s(t - s_n) \| \leq \left(1 - \omega\left(\frac{\rho}{2M}\right)\right) M,$$

which contradicts (5.2).

OBSERVATION. – The problem if Bohr's theorem holds or not in the case of X reflexive is still open.

It must be noted however that there exist non reflexive spaces in which Bohr's theorem holds; such is, for instance, the space l^1 [13]. More generally, let us consider the space $X = l^p \{ X_n \}$, with $1 \leq p < +\infty$; $\{ X_n \}$ is a sequence of BANACH spaces and $x \in X$ means

$$x = \{ x_n \}, \text{ with } x_n \in X_n , \quad \| x \| = \{ \sum_{1}^{\infty}{}_n \| x_n \|^p \}^{1/p} < +\infty.$$

Let $f(t) = \{f_n(t)\}$ be a.p. with values in X. It is then

$$F(t) = \int_0^t f(\eta)\,d\eta = \left\{ \int_0^t f_n(\eta)\,d\eta \right\} = \{F_n(t)\}.$$

and the following theorem can be proved [14].

XXVI - *If*, $\forall X_n$, *the property*

$$f_n(t) \ a.p.,\ F_n(t) \ bounded \Rightarrow F_n(t) \ a.p.$$

holds, then the same property holds for X*:*

$$f(t) \ a.p.,\ F(t) \ bounded \Rightarrow F(t) \ a.p.$$

In the proof of this theorem an extension to real a. p. functions of Dini's classical theorem on monotonic sequences of continuous functions is used.

Precisely, let $\{\varphi_n(t)\}$ be a bounded, monotonic sequence of real a.p. functions, with

$$\varphi_1(t) \leq \varphi_2(t) \leq \dots \leq \varphi_n(t) \leq \dots \leq M < +\infty. \tag{5.8}$$

There exists therefore, $\forall t \in J$, the finite limit

$$\Phi(t) = \lim_{n \to \infty} \varphi_n(t). \tag{5.9}$$

While it is not possible to say that, if $\Phi(t)$ is a.p., convergence is uniform, Dini's theorem can however be extended in the same order of ideas as theorem XXIII. Let S be the set of sequences $s = \{s_n\}$ regular with respect to *all* $\varphi_n(t)$; $\forall s \in S$, it is therefore

$$\lim_{k \to \infty} \varphi_n(t + s_k) = \varphi_{sn}(t) \qquad (n = 1, 2, \dots) \tag{5.10}$$

uniformly, where $\varphi_{sn}(t)$ is, like $\varphi_n(t)$, a.p.

From Bochner's criterion and applying Cantor's diagonal process, it follows immediately that every sequence $r = \{r_n\}$ contains a subsequence $s \in S$.

We observe that from (5.8), (5.10) it follows that

$$\varphi_{s1}(t) \leq \varphi_{s2}(t) \leq \dots \leq \varphi_{sn}(t) \leq \dots \leq M$$

and consequently

$$\lim_{n \to \infty} \varphi_{sn}(t) = \Phi_s(t), \qquad \Phi_0(t) = \Phi(t).$$

We can prove the following proposition.

XXVII - *If* $\Phi_s(t)$ *is a.p.* $\forall s \in S$, *the sequence* $\varphi_n(t)$ *converges uniformly.*

This theorem (proved by AMERIO [15]) has been generalized by BOCHNER [16] to *almost-automorphic* functions. For another proof see DOLCHER [17].

CHAPTER II

FUNCTIONAL ALMOST-PERIODIC EQUATIONS

1. - Almost-periodic solutions of the wave equation.

α) In the present §, we shall deal with the mixed problem, according to HADAMARD, for the *wave equation* (*or equation of the vibrating membrane*)

$$\frac{\partial^2 y}{\partial t^2} = \sum_{j,k}^{1\ldots m} \left(a_{jk}(x) \frac{\partial y}{\partial x_k}\right) - a(x)y + f(t, x), \tag{1.1}$$

and consider, more precisely, the first mixed problem.

Let Ω be an open, bounded and connected set of the EUCLIDEAN space $\Re^m$, $\partial\Omega$ the boundary of Ω, $x = \{x_1, \ldots, x_m\}$ an arbitrary point of R^m.

The problem considered consists in finding a solution $y = y(t, x)$ satisfying the *initial conditions*

$$y(0, x) = y_0(x), \qquad y_t(0, x) = y_1(x) \qquad (x \in \Omega) \tag{1.2}$$

and the *boundary condition*

$$y(t, x)\,|_{x \in \partial\Omega} = 0 \qquad (t \in J). \tag{1.3}$$

It corresponds therefore to the study of the motion of a *vibrating membrane, with fixed edge.* The functions considered in (1.1) are assumed to be *real.*

It is classical, in the theory of hyperbolic equations, to look for so-called *weak* or *generalized* solutions. As we shall see, these solutions are associated to the *variational* theory of the vibrating membrane, and, whenever they satisfy convenient regularity conditions, are solutions of (1.1).

The variational form of equation (1.1) can be obtained applying to it Green's formula (and bearing in mind the boundary condition (1.3)).

First of all, it is well to define the functional spaces in which the problem is correctly posed. As we shall see, they will be HILBERT spaces. These spaces play, as is well known, a very special rôle in the mathematical description of physical problems. This is not surprising if one observes that such phenomena are essentially bound to the concept of *energy*, hence to that of *scalar product:* now HILBERT spaces are defined precisely as those BANACH spaces in which the scalar product is defined, with the *same formal properties* which this functional has in ordinary EUCLIDEAN spaces.

We assume that the coefficients $a_{jk}(x)$, $a(x)$ are measurable and bounded functions on Ω and that

$$(1{,}4) \qquad a_{jk}(x) = a_{kj}(x), \quad \sum_{j,\,k}^{1\ldots m} a_{jk}(x)\,\eta_j\eta_k \geq \nu\,\{\sum_1^m{}_j \eta_j^2\}^{1/2} \quad (\nu > 0), \quad a(x) \geq 0.$$

The second of (1.4) is valid for all real values of $\eta_1, \ldots, \eta_m$.

Let us consider the following HILBERT spaces.

1) The space L^2 of real functions $y = \{y(x);\ x \in \Omega\}$ which are square integrable in Ω, with the usual definition of scalar product (and, consequently, of norm):

$$(1.5) \qquad (y,\ z)_{L^2} = \int_\Omega y(x)\,z(x)\,d\Omega, \quad \|\, y \,\|_{L^2} = (y,\, y)_{L^2}^{1/2}.$$

2) The space H_0^1 of the functions $y = \{y(x);\ x \in \Omega\}$ which are square integrable in Ω together with their first partial derivatives. These derivatives must be intended in *Sobolev's generalized sense, or in that of the theory of distributions* (in other words: $g(x) = \dfrac{\partial y}{\partial x_k}$ means that $g(x) \in L^2$ and is such that

$$\int_\Omega y(x)\,\frac{\partial\varphi(x)}{\partial x_k}\,d\Omega = -\int_\Omega g(x)\,\varphi(x)\,d\Omega,$$

$\forall\,\varphi(x)$ continuous on Ω together with all its partial derivatives and with compact support on Ω). Therefore the vanishing on $\partial\Omega$ must be intended in the sense of the variational theory of elliptic equations (as will be pointed out later).

The scalar product and the norm in H_0^1 can be defined in the following way:

$$(1.6) \qquad (y, z)_{H_0^1} = \int_\Omega \left(\sum_{j,k}^{1 \dots m} a_{jk}(x) \frac{\partial y(x)}{\partial x_j} \frac{\partial z(x)}{\partial x_k} + a(x)\, y(x) z(x) \right) d\Omega,$$

$$\| y \|_{H_0^1} = (y, y)_{H_0^1}^{1/2}.$$

It is worth while recalling that the space H_0^1 can be obtained as the *closure* in the norm defined above, of the space of functions continuous on Ω together with their first derivatives and with compact support on Ω.

As is well known, $H_0^1 \subset L^2$; the *embedding* of H_0^1 in L^2 is moreover not only *continuous* ($\| y \|_{L^2} \leq \rho \| y \|_{H_0^1} \ \forall y \in H_0^1$, with ρ positive constant, independent of y) but even *compact* (or *completely continuous*): this means that every sequence $\{ y_n \}$ bounded in H_0^1 contains a subsequence $\{ z_n \}$ which converges in L^2.

3) The space $E = H_0^1 \times L^2$, Cartesian product of H_0^1 by L^2.

Each element $Y \in E$ is therefore constituted by a couple $\{ y_0, y_1 \}$, with $y_0 \in H_0^1$, $y_1 \in L^2$ and

$$(1.7) \qquad (Y, Z)_E = (y_0, z_0)_{H_0^1} + (y_1, z_1)_{L^2},$$

$$\| Y \|_E = \{ \| y_0 \|^2_{H_0^1} + \| y_1 \|^2_{L^2} \}^{1/2}.$$

We shall call E the *energy space* and the metric defined by the second of (1.7) the *energy metric*.

Let us assume that the known term $f(t, x)$ and the unknown function $y(t, x)$ satisfy the following conditions.

1) Put $f(t) = \{ f(t, x);\ x \in \Omega \}$, $f(t)$ takes its values in L^2, for almost all $t \in J$ and has a summable norm in every bounded interval Δ; in other words

$$\int_\Delta \| f(t) \|_{L^2} \, dt = \int_\Delta dt \left\{ \int_\Omega f^2(t, x) d\Omega \right\}^{1/2} < +\infty.$$

Hence

$$(1.8) \qquad f(t) \in L^1_{\text{loc}}(J;\ L^2).$$

2) Put $y(t) = \{ y(t, x);\ x \in \Omega \}$, $y(t)$ takes its values in H_0^1 and is *continuous on J*; in other words

$$(1.9) \qquad \lim_{\tau \to 0} \| y(t + \tau) - y(t) \|_{H_0^1} =$$

$$= \lim_{\tau \to 0} \left\{ \int_\Omega \sum_{j,k}^{1 \dots m} a_{jk}(x) \frac{\partial (y(t+\tau, x) - y(t, x))}{\partial x_j} \frac{\partial (y(t+\tau, x) - y(t, x))}{\partial x_k} + \right.$$

$$\left. + a(x)(y(t+\tau, x) - y(t, x))^2 \right) d\Omega = 0.$$

We shall, further, assume that $y(t)$ is *L^2-differentiable* and that *its derivative* $y'(t) = \left\{ \frac{\partial y(t,x)}{\partial t};\ x \in \Omega \right\}$ (in the *strong* sense, i.e.

$$\lim_{\tau \to 0} \left\| \frac{y(t+\tau) - y(t)}{\tau} - y'(t) \right\|_{L^2} = 0\Big)$$

is continuous on J.

Hence

$$y(t) \in C(J;\ H_0^1), \qquad y'(t) \in C(J;\ L^2).$$

We may note that, as $y(t, x)$ represents the displacement at the time t of the point x of the membrane, the quantities

$$\frac{1}{2}\ \| y(t) \|^2_{H_0^1}, \qquad \frac{1}{2}\ \| y'(t) \|^2_{L^2}$$

measure the *potential energy* and the *kinetic energy* respectively, at the given time, of the membrane.

If $Y(t) = \{ y(t),\ y'(t) \}$ $\left(\text{i.e. } Y(t, x) = \left\{ y(t, x),\ \frac{\partial y(t, x)}{\partial t} \right\}\right)$, the function $Y(t)$ will be continuous, with values in E; furthermore the quantity

$$(1.10) \qquad \frac{1}{2}\ \| Y(t) \|^2_E = \frac{1}{2}\ \| y(t) \|^2_{H_0^1} + \frac{1}{2}\ \| y'(t) \|^2_{L^2}$$

measures the *total energy*, at the time t, of the membrane. The denomination, given to E, of energy space is therefore justified.

Let us now recall that, *by Hamilton's principle, the functions* $y(t)$, $t \in J$, *which describe the possible motions of the membrane are those for which, in whatever way a bounded interval* Δ *is taken, the integral*

$$\int_\Delta \left\{ \frac{1}{2}\ \| y'(t) \|^2_{L^2} - \frac{1}{2}\ \| y(t) \|^2_{H_0^1} + (f(t),\ y(t))_{L^2} \right\} dt$$

(*Hamiltonian action*) *is stationary with respect to all the variations* $l(t)$ (*of the same functional class as* $y(t)$) *and with support* $X \subseteq \Delta$.

By imposing that this variation must vanish, we obtain the *variational wave equation*

$$(1.11) \qquad \int_J \{ (y'(t),\ l'(t))_{L^2} - (y(t),\ l(t))_{H_0^1} + (f(t)),\ l(t))_{L^2} \}\, dt = 0$$

that must be verified $\forall\ l(t)$ *with compact support.*

If $y(t)$ is a solution of (1.11) such that the corresponding $y(t, x)$ satisfies further regularity conditions and if $\partial\Omega$ is sufficiently smooth, it can be proved, by a well known procedure, that (1.1) and (1.3) are satisfied. The solutions of the variational equation are called, for this reason, *weak* or *generalized* solutions of the differential wave equation.

Let us now consider, for (1.11), the *initial value problem.* If we take $y_0 \in H_0^1$, $y_1 \in L^2$ arbitrarily, we want to find a solution $y(t)$, $t \in J$, which satisfies the initial conditions

$$y(0) = y_0, \quad y'(0) = y_1, \tag{1.12}$$

corresponding to (1.2).

It can be proved that such a solution exists and is unique [18]. It can be obtained by the method of elementary solutions.

Let us consider the sequence $\{u_n\}$ of *eigensolutions* of the equation (that must be satisfied $\forall v \in H_0^1$)

$$(u, v)_{H_0^1} = \lambda^2 (u, v)_{L^2}. \tag{1.13}$$

(1.13) admits a sequence $\{\lambda_n\}$ of *eigenvalues* such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots, \lim_{n \to \infty} \lambda_n = +\infty, \tag{1.14}$$

to which the *eigenfunctions* u_n correspond, which satisfy the orthogonality conditions

$$\left(\frac{u_n}{\lambda_n}, \frac{u_m}{\lambda_m}\right)_{H_0^1} = (u_n, u_m)_{L^2} = \delta_{nm}. \tag{1.15}$$

The sequence $\{u_n\}$ is also *complete* both in L^2 and in H_0^1. If we put

$$\begin{aligned} y(t) &= \sum_1^\infty {}_n\, \omega_n(t) \frac{u_n}{\lambda_n} \quad \left(\omega_n(t) = \left(y(t), \frac{u_n}{\lambda_n}\right)_{H_0^1}\right), \\ y'(t) &= \sum_1^\infty {}_n \frac{\omega'_n(t)}{\lambda_n} u_n \quad \left(\frac{\omega'_n(t)}{\lambda_n} = (y(t), u_n)_{L^2}\right) \end{aligned} \tag{1.16}$$

(which is correct, as $y(t)$ and $y'(t)$ take their values in H_0^1 and L^2 respectively) and

$$f(t) = \sum_1^\infty {}_n \varphi_n(t) u_n \quad (\varphi_n(t) = (f(t), u_n)_{L^2}), \tag{1.17}$$

we find that $\omega_n(t)$ satisfies the equation

$$\omega_n''(t) + \lambda_n^2 \omega_n(t) = \lambda_n \varphi_n(t).$$

Consequently

$$(1.18) \qquad \omega_n(t) = \alpha_n \cos \lambda_n t + \beta_n \sin \lambda_n t + \int_0^t \varphi_n(\eta) \sin \lambda_n (t - \eta)\, d\eta,$$

where the constants α_n and β_n are determined by the initial conditions (1.12) and, precisely,

$$\alpha_n = \left(y_0, \frac{u_n}{\lambda_n}\right)_{H_0^1}, \quad \beta_n = (y_1, u_n)_{L^2}.$$

Let us now consider the function $Y(t) = \{y(t), y'(t)\}$, which we shall again call *solution* of (1.11); it is clear that the range $\mathcal{R}_{Y(t)}$ is, in E, a continuous line.

If $Z(t) = \{z(t), z'(t)\}$ is a second solution, corresponding to the known term $g(t)$, we can prove the following *fundamental relation*

$$(1.20) \qquad \frac{d}{dt}(Y(t), Z(t))_E = (f(t), z'(t))_{L^2} + (g(t), y'(t))_{L^2},$$

from which, setting $Z(t) = Y(t)$, follows

$$(1.21) \qquad \frac{d}{dt}\frac{1}{2} \; \| Y(t) \|_E^2 = (f(t), y'(t))_{L^2}$$

and, integrating between t_1 and t_2,

$$(1.22) \qquad \frac{1}{2} \| Y(t_2) \|^2_E - \frac{1}{2} \| Y(t_1) \|^2_E = \int_{t_1}^{t_2} (f(t), y'(t))_{L^2}\, dt.$$

The right hand term represents the work in the time interval $t_1 \vdash\!\dashv t_2$ of the force $f(t) = \{f(t, x); x \in \Omega\}$; *this work equals the variation of the energy of the membrane.*

Let, in particular, $f(t) \equiv 0$ and consider the *homogeneous wave equation.* If $U(t) = \{u(t), u'(t)\}$ is a solution, it follows from (1.21)

$$(1.23) \qquad \| U(t) \|_E = \text{const.}$$

The ranges are therefore *spherical lines*, with their centers in the origin: *the solutions of the homogeneous wave equation satisfy the principle of conservation of energy.*

We now go back to the general case and consider an arbitrary $f(t)$.

Let us assume that (1.11) admits *one* solution $Y_0(t)$ which is

E–bounded (i.e. $\sup_{t\in J} \| Y_0(t) \|_E < +\infty$).

From (1.23) it follows that *all* the solutions $Y(t)$ are bounded.

The following *minimax theorem* [19] (which is interesting also from the point of view of mathematical physics) then holds.

I - *If*

$$\mu(Y) = \operatorname*{Sup}_t \| Y(t) \|_E, \quad \tilde{\mu} = \operatorname*{Inf}_{\{Y(t)\}} \mu(Y),$$

then there exists one, and only one, solution $\tilde{Y}(t)$ *such that*

$$\mu(\tilde{Y}) = \tilde{\mu},$$

that is, such that the supremum of the energy has the smallest possible value.

As an application, it can easily be seen that, *if* $f(t)$ *is periodic, with period* T, *the minimal solution* $\tilde{Y}(t)$ *is also periodic, with period* T.

β) - The almost-periodicity of the solutions $U(t)$ of the *homogenous* wave equation has been proved by various Authors under more and more general hypotheses: MUCKENHAUPT [2], BOCHNER [6], BOCHNER and VON NEUMANN [20], SOBOLEV [21], LADYZENSKAJA [22].

Very significant is Bochner's *deduction of the almost-periodicity of* $U(t)$, *under the hypothesis that* $\mathfrak{R}_{U(t)}$ *is r. c., from the energy conservation principle.* In fact $U(t+\tau) - U(t)$ being a solution, $\forall \tau \in J$, of the homogeneous equation), it results

$$\| U(t+\tau) - U(t) \| = \| U(\tau) - U(0) \| .$$

The «principle of conservation of distances» is then satisfied and our thesis follows from theorem VII.

Subsequently, SOBOLEV succeeded in eliminating the compactness hypothesis and assumed only that the boundary $\partial\Omega$ has continuous curvatures: finally, LADYZENSKAJA has abolished also this last hypothesis. Therefore *all the solutions* $U(t)$ *are a.p.*

Observe in fact that (by (1.16) and (1.18) with $f(t) \equiv 0$)

$$U(t) = \left\{ \sum_{1}^{\infty}{}_n (\alpha_n \cos \lambda_n t + \beta_n \sin \lambda_n) \frac{u_n}{\lambda_n} , \right. \qquad (1.24)$$

$$\left. \sum_{1}^{\infty}{}_n (-\alpha_n \sin \lambda_n t + \beta_n \cos \lambda_n t) u_n \right\}$$

and the series of a.p. functions on the right hand side of (1.24) converge uniformly. It is, in fact, for $1 \leq p \leq q$,

$$\| \sum_{p}^{q}{}_n (\alpha_n \cos \lambda_n t + \beta_n \sin \lambda_n t) \frac{u_n}{\lambda_n} \|^2_{H_0^1} +$$

$$+ \| \sum_{p}^{q}{}_n (- \alpha_n \sin \lambda_n t + \beta_n \cos \lambda_n) u_n \|^2_{L^2} = \sum_{p}^{q}{}_n (\alpha_n^2 + \beta_n^2)$$

where

$$\sum_{1}^{\infty}{}_n (\alpha_n^2 + \beta_n^2) = \| U(0) \|^2_E < + \infty.$$

γ) *Let us now consider the non-homogeneous wave equation, assuming that $f(t)$ is a.p. as a function with values in L^2.* In this case it is possible that no bounded solutions exist, as the so-called « resonance phenomenon » may take place; however, as already observed in α), if a bounded solution exists, all solutions are bounded.

Regarding the almost-periodicity of $Y(t)$, ZAIDMAN [23] has proved that *if the range of $Y(t)$ is r.c., then $Y(t)$ is a.p.* Subsequently AMERIO [24] has eliminated the compactness hypothesis, substituting it with a *boundedness* hypothesis, which is strictly necessary and has an evident physical interpretation.

The following theorem therefore holds.

II. - *$f(t)$ a.p., $Y(t)$ bounded $\Rightarrow$ $Y(t)$ a.p.*

The proof was obtained by AMERIO by two different methods (those, substantially, of theorems XXVI and XXV).

Observe, first of all, that if $\sup_t \| Y(t) \| = M < + \infty$, the functions $\omega_n(t)$ defined by (1.18) are a.p. together with their first derivatives $\omega'_n(t)$. By the first procedure [24] we prove the uniform convergence of the series defining $Y(t)$:

$$Y(t) = \left\{ \sum_{1}^{\infty}{}_n \omega_n(t) \frac{u_n}{\lambda_n}, \quad \sum_{1}^{\infty}{}_n \frac{\omega'_n(t)}{\lambda_n} u_n \right\},$$

that is the uniform convergence of the series of a.p. non-negative functions

$$\sum_{1}^{\infty}{}_n \left(\omega_n^2 (t) + \frac{\omega_n'^2 (t)}{\lambda_n^2} \right) = \| Y(t) \|^2_E ,$$

making use of theorem XXVII.

The second method [25] consists in *recognizing at first* (and this is immediate) *that $Y(t)$ is w a.p.* (i.e. $(Y(t), G)_E$ is a.p. $\forall\, G \in E$).

Subsequently, *one proves*, as in theorem XXV, *that the range* $\mathfrak{R}_{Y(t)}$ *is r.c.*

Contributions and generalizations to the problem treated in [24] and [25] have subsequently been given by BOCHNER [26], ZAIDMAN [27], PROUSE [28]. For C-almost-periodicity of the solutions, see VAGHI [29]. VAGHI [30] has also generalized *theorem* II *to the weak solutions* (relative to the problem (1.3)) *of the equation*

$$\frac{\partial^2 y}{\partial t} + \alpha(t)\frac{\partial y}{\partial t} + \beta(t)y = \gamma(t)\sum_{j,k}^{1\dots m}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial y}{\partial x_k}\right) + f(t, x), \tag{1.25}$$

$\alpha(t)$, $\beta(t)$, $\gamma(t)$ *being real periodic functions, of period* T, $\gamma(t) > 0$ *and* $f(t) = \{f(t, x);\ x \in \Omega\}$ $L^2 - a.p.$

In particular: *the E-bounded eigensolutions of* (1.25) *are a.p.*

It may be noted that, although (1.25) is an equation with coefficients depending on t of a very particular type, it is actually the *variation equation* of an interesting equation.

Let

$$z''(t) + g(z, z') = 0 \tag{1.26}$$

be a non-linear second order equation, with $g(z, z')$ continuous function together with its derivatives g_z, $g_{z'}$, for $-\infty < z$, $z' < +\infty$. Assume that (1.26) has a periodic solution $z = z_0(t)$ with period T. This function is also a solution independent of x of the non-linear partial differential equation

$$\frac{\partial^2 z}{\partial t^2} + g\left(z, \frac{\partial z}{\partial t}\right) = \sum_{j,k}^{1\dots m}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial z}{\partial x_k}\right). \tag{1.27}$$

The variation equation corresponding to (1.27) is then (if $y(t, x)$ is the variation given to $z_0(t)$)

$$\frac{\partial^2 y}{\partial t^2} + g_z(z_0(t), z'_0(t))y + g_{z'}(z_0(t), z'_0(t))\frac{\partial y}{\partial t} = \sum_{j,k}^{1\dots m}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial y}{\partial x_j}\right).$$

We obtain therefore a particular case of (1.25): by (1.3), we consider variations which vanish, $\forall\, t$, on $\partial\Omega$.

Lastly, we recall that ZAIDMAN [31] has studied the *elliptic* equation with coefficients independent of time and has proved *the almost-periodicity of L^2-bounded solutions, even when* $\Omega = R^m$.

OBSERVATION. - *Theorem* II *extends to the wave equation the classical theorem of Bohr-Neugebauer* on linear ordinary differential equations, with constant coefficients and a. p. known term.

This theorem, in fact, states that *every bounded integral of the equation*

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots a_n y = f(t)$$

with $a_1, \dots, a_n$ *constants,* $f(t)$ *a.p., is a.p.*

2. – A. p. solutions of linear functional a. p. equations.

a) In theorem II the BOHR - NEUGEBAUER theorem was generalized to the wave equation. We shall now deal with the extension of the important results obtained by FAVARD [32] on ordinary linear differential systems, with a. p. coefficents and known term.

Let us consider such a system, in vector form

$$x'(t) = A(t)\, x(t) + f(t) \tag{2.1}$$

($x(t)$, $f(t)$ vectors of the complex Euclidean space $\mathfrak{R}^m$, $A(t)$ $[m, m]$ matrix, $f(t)$ and $A(t)$ a. p functions).

Indicate with S_A the set of sequences $s = \{s_n\}$ which are regular with respect to $A(t)$; therefore, $\forall s \in S_A$,

$$\lim_{n \to \infty} A(t + s_n) = A_s(t)$$

uniformly, and $A_s(t)$ is, like $A(t) = A_0(t)$, an a. p. matrix.

Favard's theory considers the *family of homogeneous equations*

$$u'(t) = A_s(t)\, u(t) \tag{2.2}$$

and assumes that, $\forall s \in S_A$, *the bounded eigensolutions of* (2.2) *satisfy the condition*

$$\operatorname*{Inf}_t \ \| u(t) \| > 0. \tag{2.3}$$

Favard then proves that, *if equation* (2.1) *has a bounded solution* $x_0(t)$, *it has also one,* $\tilde{x}(t)$, *which is a. p.*

More precisely, $\tilde{x}(t)$ *is that solution, which exists and is unique, for which the functional*

$$\mu(x) = \operatorname*{Sup}_t \ \| x(t) \|$$

takes its smallest value when $x(t)$ varies in the class of the bounded solutions of (2.1).

It is clear from what precedes that, *if* (2.2) *admits,* $\forall s \in S$, *as its only bounded solution, the solution which is identically zero, then a bounded solution* $\tilde{x}(t)$ *of* (2.1) (if it exists) *is necessarily a. p.*

This is *Favard's first theorem*, which gives a very interesting connection between the *uniqueness of a bounded solution and its almost-periodicity.*

We can see that, in the case of systems with a. p. coefficents, we do not prove the almost-periodicity of all the bounded solutions: it can, in fact, be seen on examples that the bounded eigen-solutions of the homogeneous equation $u'(t) = A(t)u(t)$ are not necessarily a. p. It has however been possible to prove the *almost-periodicity of the minimal solution* $\tilde{x}(t)$, *if condition* (2.3) *holds.*

β) Let V and H be two Hilbert spaces; we assume $V \subseteq H$, dense in H and with a *continuous embedding* ($\| v \|_H \leq k \| v \|_V$, k embedding constant that can be assumed $= 1$).

Set

$$(2.4) \qquad Q(x; l) = \int_J \{(x'(\eta), l'(\eta))_H - (A(\eta)x(\eta), l(\eta))_V + (B(\eta)x'(\eta), l(\eta))_V\} \, d\eta,$$

we consider the linear second order functional abstract equation [33]

$$(2.5) \qquad Q(x;\ l) = \int_J (f(\eta),\ l(\eta))_H \, d\eta.$$

In (2.4) $A(\eta)$ and $B(\eta)$ are bounded linear operators $\forall \eta \in J$, from V to V and from H to V respectively; therefore

$$A(\eta) \in \mathfrak{L}(V,\ V) = \mathfrak{A}, \quad B(\eta) \in \mathfrak{L}(H,\ V) = \mathfrak{B}.$$

$\mathfrak{A}$ and $\mathfrak{B}$ are two Banach spaces and we shall assume that $A(\eta)$ and $B(\eta)$ are continuous functions in their respective uniform topologies, that is as functions with values in $\mathfrak{A}$ and $\mathfrak{B}$ (precisely: $\| A \|_{\mathfrak{A}} = \operatorname{Sup}_{\| x \|_V = 1} \| Ax \|_V$, $\| B \|_{\mathfrak{B}} = \operatorname{Sup}_{\| y \|_H = 1} \| By \|_V$).

In (2.4), (2.5) $x(t)$ is the *unknown function*, $l(\eta)$ the *test function*, $f(\eta)$ the *known term*. We shall assume that they belong to the following functional spaces:

$$(2.6) \qquad \begin{gathered} x(\eta),\ l(\eta) \in L^2_{\mathrm{loc}}(J;\ V) \\ x'(\eta),\ l'(\eta),\ f(\eta) \in L^2_{\mathrm{loc}}(J;\ H). \end{gathered}$$

$l(\eta)$ has, in addition, compact support and (2.5) must be true for *all* test functions $l(\eta)$.

The derivatives $x'(\eta)$, $l'(\eta)$ are intended in the sense of distributions (*i. e.* $\int_J (x(\eta), u'(\eta))_H \, d\eta = - \int_J (x'(\eta), u(\eta))_H \, d\eta \ \forall\, u(\eta)$ from J to H whith compact support and indefinitely differentiable).

It may be noted that in this way we do not impose that $x(\eta)$ and $x'(\eta)$ be continuous; the reason for this is that in the present theory of the initial value problem for (2.5) (particulary in the hyperbolical case) the spaces (2.6) are considered.

Equation (2,5) (or a more general one, that however can be treated in the same way) corresponds to the weak formulation of many classical problems on partial differential equations. This, for example, holds for the second order hyperbolic equation with coefficents depending on η (in addition to the spacial variables $\xi_1, \ldots, \xi_m$).

Favard's results have been generalized to (2,5) by AMERIO [34] in the way we shall see in the present §.

Let us observe that, the functions not being continuous, we connot speak of almost-periodicity in the sense of BOHR-BOCHNER: it is however possible to introduce in the folloving way almost-periodicity in the sense of STEPANOV.

Indicating with Δ the interval $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$, consider the Hilbert space $L^2(\Delta; H)$ of the functions $g(\eta)$ with values in H, almost everywere on Δ, and with square summable norm on $\Delta : g \in L^2(\Delta; H)$ means therefore that $g = \{g(\eta); \eta \in \Delta\}$ and

$$\| g \|_{L^2(\Delta;\, H)} = \{ \int_\Delta \| g(\eta) \|_H^2 \, d\eta \}^{1/2}.$$

It is then possible to associate to $f(\eta)$ a function $f_\Delta(t)$ from J to $L^2(\Delta; H)$, defined by

$$f_\Delta(t) = \{f(t+\eta);\ \eta \in \Delta\}.$$

Therefore

$$\| f_\Delta(t) \|_{L_2(\Delta;\, H)} = \{ \int_\Delta \| f(t+\eta) \|_H^2 \, d\eta \}^{1/2}$$

and, consequently

$$\lim_{\tau \to 0} \| f_\Delta(t+\tau) - f_\Delta(t) \|_{L^2(\Delta,\, H)} = 0, \tag{2.7}$$

which shows that $f_\Delta(t)$ is continuous.

In what follows, we shall, for simplicity's sake, write $f(t)$ instead of $f_\Delta(t)$, adding the indication of the space in which $f(t)$ is to be considered; we shall therefore write $f(t) = \{f(t+\eta); \eta \in \Delta\}$ and, by (2.7), $f(t)$ is $L^2(\Delta; H)$ — *continuous*. In other words:

$$f(\eta) \in L^2_{\mathrm{loc}}(J;\, H) \Rightarrow f(t)\, L^2(\Delta;\, H) \text{ — continuous.}$$

Analogously, $x(t)$ and $l(t)$ are $L^2(\Delta;\ V)$ — continuous, while $x'(t) = \{x'(t+\eta);\ \eta \in \Delta\}$ and $l'(t)$ are $L^2(\Delta;\ H)$ — continouous.

Regarding the operation of differentiation, it must be noted that $x'(t)$ *defines* the $L^2(\Delta;\ H)$ — *derivative* of $x(t)$; in other words

$$\lim_{\tau \to 0} \left\| \frac{x(t+\tau) - x(t)}{\tau} - x'(t) \right\|_{L^2(\Delta;\ H)} = 0.$$

If $E = V \times H$, we shall indicate with W the space of functions $G = \{g(\eta);\ \eta \in \Delta\}$ with $g(\eta) \in L^2(\Delta;\ V)$, $g'(\eta) \in L^2(\Delta; H)$. It follows that the functions $X(\eta) = \{x(\eta), x'(\eta)\}$, $X(t) = \{X(t+\eta);\ \eta \in \Delta\}$ take their values in E and W respectively. We have also

$$\| X(\eta) \|_E = \{ \| x(\eta) \|_V^2 + \| x'(\eta) \|_H^2 \}^{1/2},$$

$$\| X(t) \|_W = \{ \int_\Delta \| X(t+\eta) \|_E^2 d\eta \}^{1/2} = \{ \int_\Delta (\| x(t+\eta) \|_V^2 + \| x'(t+\eta) \|_H^2) d\eta \}^{1/2},$$

and $X(t)$ is W-continuous.

We shall say that the function $f(\eta)$ is a. p. according to Stepanov (H — a. p. S^2) if to every $\varepsilon > 0$ it is possible to associate a relatively dense set $\{\tau\}_\varepsilon$ such that

$$\sup_t \{ \int_\Delta \| f(t+\tau+\eta) - f(t+\eta) \|_H^2 d\eta \}^{1/2} \leq \varepsilon.$$

As BOCHNER [3] observed, this definition can be reduced to the classical one of a. p. function (in the sense of BOHR - BOCHNER). The condition given is, in fact, equivalent to the following

$$\sup_t \| f(t+\tau) - f(t) \|_{L^2(\Delta;\ H)} \leq \varepsilon,$$

The notations

$$f(\eta)\ H - a.\ p.\ S^2, \qquad f(t)\ L^2(\Delta;\ H) - a.\ p.$$

are therefore equivalent.

We shall say that $f(\eta)$ is H — w. a. p. S^2 if the corresponding $f(t)$ is $L^2(\Delta;\ H)$ w. a. p., that is if the scalar product

$$(f(t),\ g)_{L^2(\Delta;\ H)} = \int_\Delta (f(t+\eta),\ g(\eta))_H\, d\eta$$

is a. p. $\forall g \in L^2(\Delta;\ H)$.

Analogously, the conditions

$x(\eta)\ V$ — *a. p.* S^2 and $x'(\eta)\ H$ — a. p. S^2 (i. e. $X(\eta)\ E$ — *a. p.* S^2)

are equivalent to the condition

$$X(t)\ W - a.\ p.$$

By the definition, $X(\eta)$ E — w. a. p. S^2 means thath $X(t)$ is W — w. a. p., that is, $\forall G \in W$, the scalar product

$$(X(t),\ G)_W = \int_\Delta (X(t+\eta),\ G(\eta))_E\, d\eta =$$

$$+ \int_\Delta ((x(t+\eta),\ g(\eta))_V + (x'(t+\eta),\ g'(\eta))_H)\, d\eta$$

is an a. p. function.

We shall again call $X(t)$, $L(t) = \{l(t+\eta);\ \eta \in \Delta\}$ (with $L(\eta) = \{l(\eta),\ l'(\eta)\}$), $f(t)$, *solution, test function and known term* of equation (2.5).

γ) Let $Z(t) = \{Z(t+\eta);\ \eta \in \Delta\}$ be a W — bounded function (i. e. $\operatorname{Sup}_t \| Z(t) \|_W < +\infty$).

We shall put

$$\varphi(Z,\ \tau) = \operatorname{Sup}_t \| Z(t+\tau) - Z(t) \|_W \qquad (\forall \tau \in J) \tag{2.8}$$

$$\mu(Z) = \operatorname{Sup} \| Z(t) \|_W . \tag{2.9}$$

Let Λ_Z be the set of W — bounded functions $X(t)$ such that

$$\varphi(X,\ \tau) \leq \varphi(Z,\ \tau) \qquad \forall \tau \in J. \tag{2.10}$$

Furthermore, let $\Lambda_{Z,Q,f}$ be the set of the solutions of (2.5) $\in \Lambda_Z$ and $\Lambda_{Z,Q}$ the set of eingensolutions $U(t)$ of the homogeneous equation

$$Q(u;\ l) = 0$$

which are differences between functions $\in \Lambda_{Z,Q,f}$.

It is clear that the sets Λ_Z, $\Lambda_{Z,Q,f}$, $\Lambda_{Z,Q}$ (to which must be added the identically null solution) are *convex*; *if*, in addition, *a W — bounded solution $X_o(t)$ exists, the set $\Lambda_{X_0,Q,f}$, is not empty.*

The following *minimax theorem* can now be proved.

I - *Let us assume that*:

1) *There exists a W — bounded $Z(t)$ such that $\Lambda_{Z,Q,f}$ is not empty;*

2) $\forall\, U(t) \in \Lambda_{Z,Q}$, *it results*

$$\operatorname{Inf}_t \| U(t) \|_W > 0. \tag{2.11}$$

Then, if

$$\tilde{\mu} = \operatorname{Inf}_{\Lambda_{Z,Q,f}} \mu(X),$$

there exists in $\Lambda_{Z,\,Q,\,f}$ *one, and only one, solution* $\tilde{X}(t)$ *such that*

$$\mu(\tilde{X}) = \tilde{\mu}.$$

It can be observed, comparing this theorem with the minimax theorem regarding the wave equation and with the one of FAVARD for ordinary systems, that the class of solutious $\Lambda_{Z,\,Q,\,f}$ is now more restricted, owing to condition (2.10). The reason for this will appear in theorem III. On the other hand, the rather restrictive condition (2.11) (which will be called "Favard's condition„) is here broadened because we impose that it be satisfied only by the $U(t) \in \Lambda_{Z,\,Q}$, and not by all the W — bounded eigensolutions.

Lastly, we observe that *condition* (2.11) *implies that, in* $\Lambda_{X_0,\,Q,\,f}$, *the uniqueness theorem for the imitial value problem must hold,* that is

$$X_1(t),\ X_2(t) \in \Lambda_{X_0,\,Q,\,f},\ X_1(0) = X_2(0) \Rightarrow X_1(t) \equiv X_2(t).$$

In fact, if $X_1(t_o) \neq X_2(t_o)$, then $X_1(t) - X_2(t) = U(t) \in \Lambda_{X_0,\,Q}$ and from (2.11) follows $\| U(0) \| > 0$, which is absurd.

δ) In what follows, we shall assume that *the operators* $A(\eta)$, $B(\eta)$ *are* $\mathfrak{A} - a.\ p.$ *and* $\mathfrak{B} - a.\ p.$ *respectively and that* $f(t)$ *is* $L^2(\Delta;\ H) - w.\,a.\,p.$

If $s = \{s_n\}$ is a *regular* sequence with respect to $A(\eta)$, $B(\eta)$, $f(\eta)$ simultaneausly and if S is the family of such sequences, it results, uniformly,

$$
\begin{aligned}
&\lim_{n \to \infty} A(\eta + s_n) = A_s(\eta) \\
(2.12)\qquad &\lim_{n \to \infty} B(\eta + s_n) = B_s(\eta) \\
&\lim_{n \to \infty}{}^{*} f(t + s_n) = f_s(t)
\end{aligned}
$$

and $A_s(\eta)$, $B_s(\eta)$ are a. p., while $f_s(t)$ is w. a. p.

Let us now consider, $\forall\, s \in S$, the equation

$$Q_s(x;\ l) = \int_{\Delta} (f_s(\eta),\ l(\eta))_H\, d\eta, \tag{2.13}$$

where

$$Q_s(x,\ l) = \int_J \{(x'(\eta),\ l'(\eta))_H - (A_s(\eta)\, x(\eta),\ l(\eta))_V + (B_s(\eta)\, x'(\eta),\ l(\eta))_V\}\, d\eta .$$

It can be shown that, *if* (2.5), *considered in an interval* $a \vdash +\infty$

(i. e. if $l(\eta) = 0$ for $\eta \leq a$) *admits in the interval* $a + \frac{1}{2} \overline{\rightarrow} + \infty$ *a* W — *bounded solution* $\bar{X}(t)$, *then there exists also a* W — *bounded solution* $X_o(t)$ ($\sup_t \| X_o(t) \|_W < + \infty$).

It is then easy to prove that *each equation* (2.12) *has one* W—*bounded solution* $X_s(t)$; more precisely

$$\varphi(X_s;\ \tau) \leq \varphi(X_o;\ \tau), \quad \mu(X_s) < \mu(X_s).$$

The set Λ_{X_0, Q_s, f_s} is therefore not empty, $\forall s \in S$.

The following *theorem of* W — *weak almost — periodicity* then holds.

II - *Let us assume that*:

1) *Equation* (2.5) *has one* W — *bounded solution* $X_o(t)$;

2) *The operators* $A(\eta)$, $B(\eta)$ *and the known term* $f(t)$ *are respectively* $\mathfrak{A}$ — *a. p.*, $\mathfrak{B}$ — *a. p.* $L^2(\Delta;\ H)$ — *w. a. p.*;

3) $\forall s \in S$ *and* $U(t) \in \Lambda_{X_0, Q_s}$ *it results*

$$\inf_t \| U(t) \|_W > 0. \tag{2.14}$$

Then the minimal solution $\tilde{X}(t)$ (*in the set* $\Lambda_{X_0, Q, f}$) *is* W—*w.a.p.* *Furthermore*, $\forall s \in S$,

$$\lim_{n \to \infty}{}^* \tilde{X}(t + s_n) = \tilde{X}_s(t)$$

uniformly and $\tilde{X}_s(t)$ *is a w. a. p. solution of* (2.13), *minimal in the set* Λ_{X_0, Q_s, f_s}.

The proof (for which the *uniqueness* of the minimal solution is essential) can be obtained by extending a procedure based on BOCHNER's criterion, given by FAVARD for ordinary systems.

ε) *Let us assume that equation* (2.5) *has a* W — *bounded and* W — *uniformly continuous solution* $X_o(t)$. In this case, *also the minimal solution* $\tilde{X}(t)$ *is* W - *uniformly continuous*.

By (2.1), we have, in fact,

$$\varphi(\tilde{X};\ \tau) \leq \varphi(X_o;\ \tau) \qquad \forall \tau \in J,$$

and, therefore, $X_o(t)$ being uniformly continuous,

$$\lim_{\tau \to 0} \varphi(\tilde{X};\ \tau) \leq \lim_{\tau \to 0} \varphi(X_o;\ \tau) = 0.$$

The following W - *almost - periodicity theorem* can now be proved.

III - *Let us assume that:*

1) *Equation* (2.5) *has a* W — *bounded and* W — *uniformly continuous solution* $X_0(t)$;

2) *The operators* $A(\eta)$, $B(\eta)$ *and the known term* $f(t)$ *are respectively* $\mathfrak{A}$ — *a. p.*, $\mathfrak{B}$ — *a. p.*, $L^2(\Delta; H)$ *w. a. p.; $A(\eta)$ satisfies, in addition, the ellipticity condition*

$$R(A(\eta)x, x)_V \geq \nu \|x\|_V^2 \qquad (\nu > 0);$$

3) $\forall s \in S$ *and* $\forall U(t) \in \Lambda_{X_0, Q_s}$, *it results*

$$\operatorname*{Inf}_t \|U(t)\|_W > 0;$$

4) *The embedding of* V *in* H *is compact.*

Then the minimal solution, $\tilde{X}(t)$, *is* W — *a. p.*

The theorem is proved by showing that the range of $\tilde{X}(t)$ is r. c., using a compactness theorem which holds for the solutions of (2.5).

It may be observed that, for the problem of the vibrating membrane, treated in § 1, conditions 2) and 4) are obviously verified and so is condition 3) (of FAVARD) because, by the energy conservation principle, $\|U(t)\|_W =$ const.

The problem of proving theorem III eliminating the hypotesis that $X_0(t)$ is W — uniformly continouous (and assuming eventually that $f(t)$ is $L^2(\Delta; H)$ — a. p. and not only $L^2(\Delta; H)$ w. a. p.) is still open. See, on this subject, a typical example, examined by PROUSE [35].

Finally, it is evident that theorems II and III extend also to equation (2.5) the *first theorem of Favard.*

OBSERVATION I - It is posssible to effect the *harmonic analysis of the minimal solution* $\tilde{X}(t)$ (and consequently of the $\tilde{X}_s(t)$). If, for simplicity's sake, we assume $f(t)$ $L^2(\Delta; V)$ — a. p., every sequence $s \in S$ is regular also for $\tilde{X}(t)$. From theorem XIII of CHAPTER I it follows that *every characteristic exponent of* $\tilde{X}(t)$ *is a linear combination, with integer coefficients, of a finite number of characteristic exponents of* $A(\eta)$, $B(\eta)$, $f(t)$.

OBSERVATION II - Theorems I, II and III hold also for more general equations [36], assuming the space V to be *uniformly convex* and H *semicomplete.* In this case, the proof of Theorem III can be obtained by assuming that a continuous dependance theorem from the operator and the known term holds; such a theorem however has been proved only in particular cases. ZAIDMAN [37] has shown that this is true for the *wave equation*; more precisely, for the *Cauchy problem* under the assumption that the initial values (1.2) are given on the *whole* of R^m (i. e $\Omega = R^m$) and $a(x) \geq \rho > 0$. In this case the *E — bounded eigensolutions are not, in general, a. p. Such is, however, the minimal solution $\tilde{X}(t)$ if $f(t)$ is L^2 — a. p.*

OBSERVATION III – Theorem III concerns the hypperbolic case of equation (2.5), with variable operators. Problems related to the *parabolic* and *elliptic* cases have been examined respectively by PROUSE [38] and by RICCI and VAGHI [39].

3. – A. p. solutions of the Navier - Stokes equation.

α) Let Ω be an open, bounded and connected set of the plane $\xi = \{\xi_1, \xi_2\}$ with boundary $\partial\Omega$. The "classical" problem we shall examine consists in determining, in the interval J, the vectors $x(\eta, \xi) = \{x_1(\eta, \xi), x_2(\eta, \xi)\}$ and the scalars $p = p(\eta, \xi)$ satisfying the NAVIER - STOKES system

$$
(3.1) \qquad \begin{aligned} & \frac{\partial x_i}{\partial \eta} + \sum_1^2{}_i\, x_i \frac{\partial x_i}{\partial \xi} - \mu \Delta x_i = - \frac{\partial p}{\partial \xi_j} + f_i(\eta, \xi) \\ & \frac{\partial x_1}{\partial \xi_1} + \frac{\partial x_2}{\partial \xi_2} = 0 \qquad (\xi \in \Omega;\ j = 1, 2) \end{aligned}
$$

and the boundary conditions

$$
(3.2) \qquad x_i(\eta, \xi)|_{\xi \in \partial\Omega} = 0 \qquad (\eta \in J;\ j = 1, 2)
$$

In (3.1), x_i are the components of the fluid velocity, p the pressure, $\mu > 0$ the viscosity coefficient, Δ the LAPLACE operator end $f_i(\eta, \xi)$ are the given components of the force of mass. All functions considered are real.

To this classical problem can be associated, as is well known, a weak problem, and it is to this problem that we shall refer.

Let $\mathfrak{M}$ be the manifold of indefinitely differentiable vectors, with null divergence (i. e. satisfing the second of (3.1)) and compact support in Ω. We shall indicate by N and N^1 the closures of $\mathfrak{M}$ in L^2 and $H_0{}^1$ respectively.

As N and N^1 are subspaces respectively of L^2, and H_o^1, it will be

$$(3.3)\qquad \begin{aligned} (u, v)_N &= (u, v)_{L^2} = \int_\Omega u(\xi)\, v(\xi)\, d\Omega \\ (u, v)_{N^1} &= (u, v)_{H_0^1} = \int_\Omega \sum_1^2{}_i \frac{\partial u}{\partial \xi_i} \frac{\partial v}{\partial \xi_i}\, d\Omega . \end{aligned}$$

Furthermore, we shall put

$$(3.4)\qquad b(u, v, w) = \int \sum_{i,j}^{1,2} u_j \frac{\partial v_j}{\partial \xi_\iota} w_\iota\, d\Omega .$$

The *weak* form which can be deducted for the NAVIER - STOKES equation is then the following

$$(3.5)\quad \int_J \{ u(x(\eta), h(\eta))_{H_0^1} - (x(\eta), h'(\eta))_{L^2} + b(x(\eta), x(\eta), h(\eta)) \}\, d\eta = \int_J (f(\eta), h(\eta))_{L^2}\, d\eta$$

In equation (3.5)

$$x(\eta) = \{ x(\eta, \xi); \xi \in \Omega \},\ h(\eta) = \{ h(\eta, \xi); \xi \in \Omega \},\ f(\eta) = \{ f(\eta, \xi); \xi \in \Omega \}$$

and

$$h'(\eta) = \{ \frac{\partial h(\eta, \xi)}{\partial \eta};\ \xi \in \Omega \}.$$

$x(\eta)$ is the *unknown function*, $h(\eta)$ the *test function*, $f(\eta)$ the *known term*. (3.5) must hold for all $h(\eta)$ with *compact support* on J.

The functions considered must belong to the following functional spaces:

$$\begin{aligned} & x(\eta) \in L^2{}_{\mathrm{loc}}(J;\ N^1) \cap L^\infty_{\mathrm{loc}}(J;\ N) \\ & h(\eta) \in C(J;\ N^1),\ h'(\eta) \in L^2{}_{\mathrm{loc}}(J;\ N) \\ & f(\eta) \in L^2{}_{\mathrm{loc}}(J;\ L^2). \end{aligned}$$

It has been proved by PRODI [40] that *the solutions* $x(\eta)$, *modified eventually on a set of measure zero, are* L^2 — *continuous*: consequently $x(\eta) \in L^2{}_{\mathrm{loc}}(J;\ N^1) \cap C(J;\ N)$.

The theory of the NAVIER - STOKES equation in an interval $\alpha \vdash\!\dashv \beta$ can be made assuming in (3.5) the interval $\alpha \vdash\!\dashv \beta$ in place of J as interval of integration; $h(\eta)$ is then assumed with compact support on $\alpha \frown \beta$. *The initial value problem* $x(\alpha) = x_o$ *has then one and only one solution* $\forall\, x_o \in N$; this has been proved, with different techniques, by HOPF [41], LADYZENSKAJA [42], LIONS and PRODI [43], PRODI [40]. The case of $f(t)$ periodic (in two or more dimensions) has been considered by PRODI [40], YUDOVIC [44], PROUSE [45].

β) The study of the solutions in an unbounded interval and the proof of the existence of an a. p. solution has been made by PROUSE [47] who has proved, among others, the following theorems.

I - *Let* $x(\eta)$ *be the solution, in* $\eta_o \vdash +\infty$, *of the Navier-Stokes equation, satisfying the initial condition* $x(\eta_o) = x_o$. *Then*

$$\text{(3.6)}\quad \sup_{\eta \geq \eta_0} \| f(\eta) \|_{L^2} = K < +\infty \quad \Rightarrow \quad \sup_{\eta \geq \eta_0} \| x(\eta) \|_{L^2} = M < +\infty,$$

where M *depends only from* $K, \| x_o \|_{L^2}, \mu, \Omega$.

II - *Let*

$$\sup_{\eta} \| f(\eta) \|_{L^2} = K < \infty .$$

There exists then at least one L^2*-bounded solution* $\bar{x}(\eta)$*: precisely,*

$$\sup_{\eta} \| \bar{x}(\eta) \|_{L^2} = K < \infty ,$$

where M *depends only from* K, μ, Ω.

If, in addition, K *is sufficiently small* ($K \leq K_o$, *depending only from* μ, Ω) *the bounded solution* $\bar{x}(\eta)$ *is unique.*

We introduce now, as in § 2, the functions

$$x(t) = \{ x(t+\eta);\ \eta \in \Delta \}, \quad f(t) = \{ f(t+\eta);\ \eta \in \Delta \},$$

Δ being the interval $-\frac{1}{2} \leq \eta \leq \frac{1}{2}$.

The following propositions then hold.

III - $\sup_{\eta} \| f(\eta) \|_{L^2} \leq K_o$, $f(t) L^2(\Delta;\ L^2) - w.\ a.\ p. \Rightarrow \bar{x}(\eta)\, L^2 - w.\ a.\ p.$ *and* $\bar{x}(t)\, L^2(\Delta;\ L^2) - a.\ p.$

As we can see, *the almost periodicity of the bounded solution follows from its uniqueness;* we are in the same order of ideas as FAVARD's first theorem (generalised by AMERIO [47] to *non linear* ordinary systems).

We now give the final result.

IV - $f(t)\, L^2(\Delta;\, L^2) - a.\ p.$, $\operatorname{Sup}_\eta \| f(\eta) \|_{L^2} = K_1 \leq K_o$ *(K_1 depending only from μ, Ω)* $\Rightarrow \tilde{x}(\eta)\, L^2 - a.\ p.$, $\tilde{x}(t)\, L^2(\Delta;\, H_o^1) - a.\ p.$

PROUSE's theory holds if Ω is a two — demensional set because the following inequality of LADYZENSKAJA

$$\| x \|_{L^4} \leq \sqrt{2}\, \| x \|_{L^2} \| x \|_{H_0^1}$$

is used, which is true only for two — dimensional sets.

γ) A study of the NAVIER - STOKES equation in more than two dimensions, in view of proving that "sufficiently small„ solutions are a. p., has been made by FOIAS [48]. This Author has proved the following theorem.

V - *Let $f(\eta)$ be $L^2 - a.\ p.$ If there exists, in J, a solution $\tilde{x}(\eta)$ such that, for a certain p, with $3 < p \leq +\infty$, it is*

$$\operatorname*{Sup}_{\eta} \| \tilde{x}(\eta) \|_{L^p} < K$$

(where K depends only from p and Ω), then $\tilde{x}(\eta)$ is $L^2 - a.\ p.$

4 - A. p. solutions of the wave equation with non - linear disspative term.

α) Let Ω be an open, bounded and connected set $\in R^m$, with $m \leq 5$, satisfying the cone property.

We consider, for $\xi \in \Omega$, the *wave equation with dissipative term*

$$(4.1) \quad \frac{\partial^2 x}{\partial \eta^2} + \beta\left(\frac{\partial x}{\partial \eta}\right) = \sum_{i,k}^{1 \dots m} \frac{\partial}{\partial \xi_i}\left(a_{ik}(\xi) \frac{\partial x}{\partial \xi_k}\right) - a(\xi)\, x + f(\eta, \xi) \quad (\eta \in J),$$

where the functions $a_{ik}(\xi)$, $a(\xi)$ satisfy the conditions given in § 1.

We shall assume that the boundary condition

$$(4.2) \qquad x(\eta, \xi)|_{\xi \in \partial\Omega} = 0 \quad (\eta \in J)$$

holds.

We shall, furthermore, assume that $\beta(\zeta)$ *is a coutinuous increasing function of $\zeta \in J$*, with $\beta(0) = 0$. In (4.1), the term $\beta\left(\frac{\partial x}{\partial \eta}\right)$ represents therefore a *passive resistance*, opposite to the velocity $\frac{\partial x}{\partial \eta}$; $f(\eta, \xi)$ is the *force of mass.*

Setting, as usual,

$$x(\eta) = \{ x(\eta, \xi);\ \xi \in \Omega \}; \quad f(\eta) = \{ f(\eta, \xi);\ \xi \in \Omega \},$$

and

$$\beta(x'(\eta)) = \{\beta\left(\frac{\partial x(\eta, \xi)}{\partial \eta}\right); \ \xi \in \Omega\}$$

we shall say that $x(\eta)$ is a weak solution of (4.1), satisfying the boundary conditions (4.2) if it satisfies the equation

$$(4.3) \quad \int_J \{(x''(\eta), h(\eta))_{L^2} + (x(\eta), h(\eta))_{H_0^1} + (\beta(x'(\eta)), h(\eta))_{L^2}\} d\eta = \\ = \int_J (f(\eta), h(\eta))_{L^2} d\eta .$$

In (4.3) we assume that the *unknown function* $x(\eta)$, the *test function* $h(\eta)$ and the *known term* $f(\eta)$ satisfy the conditions:

1) $x(\eta), x'(\eta) \in L^{\infty}_{\mathrm{loc}}(J; H_o^1), \quad x''(\eta) \in L^{\infty}_{\mathrm{loc}}(J; L^2)$;

2) $h(\eta) \in L^2_{\mathrm{loc}}(J; H_o^1)$ and has compact support;

3) $f(\eta), f'(\eta) \in L^2_{\mathrm{loc}}(J; L^2)$.

Furthermore, we shall assume that $\beta(\zeta)$ has, on J, a continuous derivative, verifying the conditions

$$(4.4) \qquad k_1(1 + |\zeta|^{\rho-1}) \leq \beta'(\zeta) \leq k_2(1 + |\zeta|^{\rho-1}),$$

with

$$(4.5) \qquad o < k_1 \leq k_2, \ \ 1 \leq \rho \leq 1 + \frac{4}{m-1} \quad (m \leq 5).$$

Let us observe that (4.4) *is satisfied if we take*

$$(4.6) \qquad \beta(\zeta) = \mu\zeta + \nu\zeta|\zeta| \qquad (\mu > o, \ \nu > o),$$

that is if we consider a passive resistance of "viscous,, type for small velocities and of "hydraulic,, type for large velocities: it is therefore a physically well acceptable law.

Cauchy's problem for (4.3) as been studied, for arbitrary ρ and m, by LIONS and STRAUSS [49] who have proved the following existence and uniqueness theorem. *If* $f(\eta), f'(\eta) \in L^2_{\mathrm{loc}}(J_o; L^2)$, *with* $J_o = 0 \vdash + \infty$, *there exists in* J_o *one and only one solution* $x(\eta)$ *such that*

$$x(0) = x_o, \qquad x'(0) = x_1,$$

$\forall x_o \in H_0^1 \cap H^2, \qquad x_1 \in H_0^1.$

The asymptotic behaviour of $x(\eta)$ *as* $\eta \to +\infty$ *and the existence of bounded or a. p. solutions* has been studied by PROUSE [50]: hypoteses $m \leq 5$ and the second of (4.5) have been introduced by PROUSE in order that, if $Q = \Omega \times \Delta$, the embedding of $H^1(Q)$ in $L^{\rho+1}(Q)$ be continuous; it would be interesting to see if these conditions, whose phisical meaning is not clear, can be eliminated. We may add that, in the case of *periodic* $f(\eta)$, which was treated by PROUSE [51] under the same assumptions, these were subsequently eliminated by PRODI [52].

PROUSE has proved, among others, the following theorems (we have again called solution the function $X(\eta) = \{x(\eta), x'(\eta)\}$).

I - *If*

$$\max_{t \to +\infty} \lim \| f(t) \|_{L^2(\Delta; L^2)} < +\infty$$

then all solutions are, among themselves, E — asymptotic, when $\eta \to +\infty$. *In other words, if* $X_1(\eta)$, $X_2(\eta)$ *are any two solutions*

$$\lim_{\eta \to +\infty} \| X_1(\eta) - X_2(\eta) \|_E = \lim_{\eta \to +\infty} \{ \| x_1(\eta) - x_2(\eta) \|^2_{H_0^1} + \\ + \| x_1'(\eta) - x_2'(\eta) \|^2_{L^2} \}^{1/2} = 0.$$

II - *Let* $f(t)$, $f'(t)$ *be* $L^2(\Delta; L^2)$ *— bounded. There exists then one, and only one,* E *— bounded solution* $\tilde{X}(\eta)$.

By theorem I we have then, $\forall$ solution $X(\eta)$,

$$\lim_{\eta \to +\infty} \| \tilde{X}(\eta) - X(\eta) \|_E = 0.$$

III - *If* $f(t)$ *is* $L^2(\Delta; L^2)$ *— w. a. p. and if* $f'(t)$ *is* $L^2(\Delta; L^2)$ *— bounded, then* $\tilde{X}(t)$ *is* $L^2(\Delta; E)$ *— w. a. p.*

IV - *If* $f(t)$ *is* $L^2(\Delta; L^2)$ *— a. p. and* $f'(t)$ *is* $L^2(\Delta; L^2)$ *— bounded, then* $\tilde{X}(\eta)$ *is* E *— a. p.*

As a conclusion of PROUSE's analysis, it can be said that *if* $f(\eta)$ *is a. p. according to Stepanov* (L^2 *— a. p.* $S^2 \Longleftrightarrow L^2(\Delta; L^2)$ *— a. p.*) *and if* $\sup_t \{ \int_\Delta \| f'(t + \eta) \|^2_{L^2} d\eta \}^{1/2} < +\infty$, *there exists one, and only one, solution* $\tilde{X}(\eta)$ *which is a. p. as a function taking its values in the energy space, that no other solutions with bounded energy exist and that, when* $\eta \to +\infty$, *all solutions* $X(\eta)$ *are asymptotic, in the energy space, to the solution* $\tilde{X}(\eta)$.

BIBLIOGRAPHY

[1] H. BOHR, *Zur Theorie der fastperiodischeF unktionen*, Acta Math., 46 (1925), 47 (1926), 237 - 281.

[2] C. F. MUCKENHAUPT, *Almost - periodic functions and vibrating systems*, Journ of Math. and Phys. MIT, 8 (1929), 163 - 198.

[3] S. BOCHNER, *Abstrakte fastperiodische Funktionen*, Acta Math, 61 (1933), 149 - 184.

[4] S. BOCHNER, J. VON NEUMANN, *Almost-periodic functions of groups, II*, Trans. Am. Math Soc., 37 (1935). In the present paper we have considered only the theory of a. p. functions from the *interval* J to X, this being the case that is of greatest interest in the applications to functional equations.

[5] J. FAVARD, *Leçons sur les fonctions presque - périodiques*, « Gauthier Villars » Paris, 1933, 85 - 99.

[6] S. BOCHNER, *Fastperiodischen Lösungen der Wellengleichung*, « Math. Ann. », 62 (1934), 227 - 237.

[7] Cfr. C. CORDUNEANU, *Functii aproape-periodice*, « Ac. Rep. Popul. Romîne », 1961.

[8] J. KOPEC, *On vector-valued almost-periodic functions*, « Ann. Soc. Pol. Math. », 25 (1962), 100 - 105.

[9] S. ZAIDMAN, *Solutions presque - périodiques des équations hyperboliques*, Ann. Ec. Norm, Sup., 79 (1962). 151 - 198.

[10] L. AMERIO, *Sul teorema di approssimazione delle funzioni quasi-periodiche, Ancora sul teorema di approssimazione delle funzioni quasi periodiche*, Rend. Acc. Naz. dei Lincei, 34 (1963), 97 - 104, 36 (1964), 101 - 106.

[11] J. KOPEC, *On linear differential equations in Banach spaces*, Zes. Nauk. Uniw. Mickiewicza Mat. Chem., 1 (1957) 3-7 (Mat. Rev. 21 (1960), 4282); L. AMERIO, *Funzioni debolmente quasi-periodiche*, Rend. Sem. Mat. Univ. Padova, 30 (1960), 288-301 (in the latter paper are proved the results recalled in the text; the definition of w.a.p. function and theorem XX are also found in the paper of KOPEC). For another definition of weak almost periodic functions cfr. W. E. EBERLEIN, *Abstract ergodic theorems and weak almost periodic functions* Trans. Am. Math. Soc. 67 (1949); *The spectrum of weakly almost-periodic functions*, « Mich. J. Math. », 3 (1956).

12] L. AMERIO, *Sull'integrazione delle funzioui quasi-periodiche astratte*, Ann. di Mat., 53 (1961), 371 - 382; *Sull'integrazione delle funzioni quasi-periodiche a valori in uno spazio hilbertiano*, Rend. Acc. Naz. dei Lincei, 28 (1960), 1 - 4

[13] M. L. RICCI, P. RIZZONELLI, *Sulle funzioni l^1-quasi-periodiche*. Rend. Ist. Lomb. (A) 95 (1961), 941 - 946.

[14] L. AMERIO, *Sull'integrazione delle funzioni $l^p \mid X_n \mid$-quasi-periodiche, con $1 \leq p < +\infty$*, Ric. di Mat. 12, (1963), 4 - 12.

[15] L. AMERIO, *Quasi-periodicità degli integrali ad energia limitata dell'equazione delle onde con termine noto quasi-periodico, I, II, III*, Rend. Acc. Naz. dei Lincei, 28 (1960), 147-152, 1-6, 1-6.

[16] S. BOCHNER, *Uniform convergence of monotonic sequences of functions*, Proc. Nat. Acad. Sc. U.S.A., 47 (1961).

[17] M. DOLCHER, *Su un criterio di convergenza uniforme per le successioni monotone di funzioni quasi-periodiche*, Rend. Sem. Mat. Univ. di Padova, 34 (1964).

[18] O. A. LADYZENSKAJA, *The mixed problem for hyperbolic equations* (in russian) - Moscov, Leningrad (1953); S. NAGY, *Vibrations d'une corde non homogène*, Bull. Soc. Math. de France, 75 (1947), 193-208; L. AMERIO, *Problema misto e quasi periodicità per l'equazione delle onde non omogenea*, Ann. di Mat., 29 (1960), 393-416.

[19] L. AMERIO, loc. cit. in ([18]).

[20] S. BOCHNER, J. VON NEUMANN, *On compact solutions of operational differential equations*, Ann. of Math., 36 (1935), 255-290.

[21] S. L. SOBOLEV, *Sur la presque périodicité des solutions de l'equation des ondes*, Compt Rend. Acc. Sc. U.R.S.S. (1945).

[22] O. A. LADYZENSKAJA, loc. cit. in ([18]).

[23] S ZAIDMAN, *Sur la presque périodicité des solutions de l'equation des ondes non homogéne*, Journ. of Math. and Mech., 8 (1959), 369-382.

[24] L. AMERIO, loc cit in ([15]).

[25] L. AMERIO, *Sull'equazione delle onde con termine noto quasi-periodico*, Rend. di Mat., 19 (1960), 333-346.

[26] S. BOCHNER, *Almost-periodic solutions of the inhomogeneous wave equation*, Proc Nat. Acad. of Sc , 46 (1960).

[27] S. ZAIDMAN, loc. cit. in ([9]).

[28] G. PROUSE, *Analisi di alcuni classici problemi di propagazione*, Rend. Sem. Mat. Univ. Padova, 32 (1962), 338-373.

[29] C. VAGHI, *Soluzioni C-quasi-periodiche dell'equazione non omogenea delle onde*, «Ric. di Mat.», 12 (1963), 196-215; this paper utilizes some results on the regularization of the wave equation obtained by Il'in (*On solvability of the mixed problem for hyperbolic and parabololic equations*, Usp. Mat. Nauk, 15 (1960)).

[30] C. VAGHI, *Su un'equazione iperbolica con coefficenti periodici e termine noto quasi-periodico*, Rend. Ist. Lomb. (A) (1965).

[31] S. ZAIDMAN, *Quasi-periodicità per l'equazione di Poisson*, Rend. Acc. Naz. Lincei, 34 (1963) 241-245; *Teoremi di quasi-periodicità per alcune equazioni differenziali operazionali*, Rend. Sem. Mat. e Fis. di Milano, 33 (1963), 3-18.

[32] J. FAVARD, loc. cit. in ([5]).

[33] For the theory of these equations cfr. J. L. LIONS, *Equations différentielles opérationelles et problèmes aux limites*, «Springer», Berlin, 1961; *Equations différentielles opérationelles dans les espaces de Hilbert*, «course CIME», Varenna (Lake of Como), 1963.

[34] L. AMERIO, *Sulle equazioni lineari quasi-periodiche negli spazi hibertiani, I, II*, Rend. Acc. Naz. dei Lincei 31 (1961), 110 117, 197-205; *Soluzioni quasi periodiche delle equazioni lineari iperboliche quasi-periodiche*, Rend. Acc. Naz. dei Lincei, 33 (1962), 179-186; *Soluzioni quasi-periodiche di equazioni quasi.periodiche negli spazi hilbertiani*, Ann. di Mat., 61 (1963), 259-278; *Su un teorerema di minimax per le equazioni differenziali astratte*, Rend. Acc. Naz dei Lincei 34 (1963), 409-416; *Solutions presque-périodiques d'équations fonctionelles dans les espaces de Hilbert*, Coll. sur l'Analyse fontionnelle, Liège (4-6 May 1964), 11-35.

[35] G. PROUSE, *Su un esempio concernente l'equazione delle onde*, Rend. Acc. Naz. dei Lincei, 37 (1964), 40-47.

[36] L. AMERIO, *Sulle equazioni differenziali quasi-periodiche astratte*, Ric. di Mat., 9 (1960), 256-274.

[37] S. ZAIDMAN, *Solutions presque-périodiques dans le problème de Cauchy pour l'équation non homogene des ondes, I, II*, Rend. Acc. Naz. Lincei, 30 (1961), 678-681, 823-827.

[38] G. PROUSE, *Soluzioni quasi-periodiche delle equazioni lineari di tipo parabolico*, Rend. Ist. Lomb. (A) 96 (1962), 847-860.

[39] M. L. RICCI, C. VAGHI, *Soluzioni quasi-periodiche di un'equazione funzionale*, Ric. di Mat, 13 (1964, 242-260.

[40] G. PRODI, *Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale*, Rend. Sem. Mat. Univ. Padova, 30 (1960), 1-15.

[41] E. HOPF, *Ueber die Anfangswertausfgabe für die hydrodinamischen Grundgleichungen*, Math Nach., 4 (1951), 126-132.

[42] O. A. LADYZENSKAJA, *Solutions in the large of the boundary value problem for the Navier-Stokes equation in two spatial variables*, Comm Pure and Appl Math. 12 (1958), 427-433.

[43] J. L. LIONS, G. PRODI, *Un théorème d'existence et d'unicité dans les équations de Navier-Strokes en dimension 2*, C. R. Acad. Sc., 248 (1959), 3519-3521.

[44] V. I. IUDOVIC, *Periodical solutions of a viscous incompressible fluid*, Dokl. Akad. Nauk SSSR, 130 (1960), 1214-1217.

[45] G. PROUSE. *Soluzioni periodiche dell'equazione di Navier-Stokes*, Rend. Acc. Naz. dei Lincei, 35 (1963), 443-447.

[46] G. PROUSE, *Soluzioni quasi-periodiche dell'equazione differenziale di Navier-Stokes in due dimensioni*, Rend Sem. Mat. Univ. Padova, 33 (1963), 186-212.

[47] L. AMERIO, *Soluzioni quasi-periodiche, o limitate, di sistemi differenziali quasi-periodici, o limitati*, Ann. di Mat., 39 (1955).

[48] C. FOIAS, *Essais dans l'étude des solutions des équations de Navier-Stokes dans l'espace. L'unicité et la presque-periodicité des solutions petites*, « Rend. Sem. Mat. Univ. » Padova, 32 (1962), 261-294.

[49] J. L. LIONS, W. A. STRAUSS, *Some non-linear evolution equations*, « Bull. Soc. Math. Fr. » (1965); *Sur certains problèmes hyperboliques non linéaires*, C.R. Ac. Sc. Paris, 257 (1963).

[50] G. PROUSE, *Soluzioni quasi-periodiche dell'equazione non omogenea delle onde con termine dissipativo non lineare I, II, III*, « Rend. Acc. Naz. dei Lincei » (1965).

[51] G. PROUSE, *Soluzioni periodiche dell'equazione delle onde non omogenea con termine dissipativo quadratico*, Ric. di Mat, 13 (1964).

[52] G. PRODI, *Soluzioni periodiche dell'equazione delle onde con termine dissipativo non lineare*, « Rend. Sem. Univ. Padova » (1965).

Pervenuta alla Segreteria dell' U.M.I.
l'11 dicembre 1964