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On Kernel polynomials and related systems

by T. S. Chihara (Seattle University) (*) (**)

Summary. - *Systems of orthogonal polynomials involving kernel polynomials are constructed.*

1. Introduction. Let $\{P_n(x)\}$ denote a set of polynomials which are orthogonal with respect to a distribution $d\psi(x)$ on a subset of $[0, \infty)$. In [2, § 2], certain relations between the $P_n(x)$ and the special «kernel polynomials» which are orthogonal with respect to $xd\psi(x)$ were obtained. Since there is a more general concept of kernel polynomials [4, Th. 3.1.4], it is natural to extend the work of [2] to this more general case.

The generalization which we obtain enables us to give a rather complete discussion of a problem of Carlitz [1] which has been considered recently by Dickinson and Warsi [3].

2. Orthogonality relations. Let $\{P_n(x)\}$ be the monic polynomials which are orthogonal with respect to a distribution $d\psi(x)$ whose support is a subset of an interval $[a, b]$ with $a > -\infty$ (b may be infinite). Let a_0 be any real number such that $a_0 \leq a$ and let $\{Q_n(a_0, x)\}$ denote the monic kernel polynomials which are orthogonal with respect to the distribution $(x - a_0)d\psi(x)$. ($Q_n(a_0, x)$ can be expressed in terms of the $P_n(x)$ by means of Christoffel's formula [4, Th. 2.5] or the Christoffel-Darboux formula [4, Th. 3.2.2]).

Select any real number α such that $\sqrt{a - a_0} \leq \alpha$ and let

$$z = z(x) = x^2 + a - \alpha^2. \tag{2.1}$$

Let α_0 denote either of the two converse images of a_0 under the mapping z and let β denote the positive converse of b under z. We then have:

$$\begin{aligned} &z(\alpha) = a, \qquad z(\beta) = b, \qquad 0 \leq \alpha < \beta \\ &z(\alpha_0) = a_0, \qquad z - a_0 = x^2 - \alpha_0^2, \qquad |\alpha_0| \leq \alpha. \end{aligned} \tag{2.2}$$

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We now define polynomials $R_n(x) \equiv R_n(\alpha_0, x)$ by

$$R_{2n}(x) = P_n(z) \tag{2.3}$$
$$R_{2n+1}(x) = (x - \alpha_0)Q_n(a_0, z).$$

Next let

$$I_{MN} = \int_E R_M(x)R_N(x)d\varphi(x).$$

where $E = [-\beta, -\alpha] \cup [\alpha, \beta]$ and φ is a distribution function.

Writing $Q_n(x)$ in place of $Q_n(a_0, x)$, we have

$$I_{2m, N} = \begin{cases} \int_{x=\alpha}^{\beta} P_m(z)P_n(z)[d\varphi(x) - d\varphi(-x)], & N = 2n \\ \int_{x=\alpha}^{\beta} P_m(z)Q_n(z)[(x - \alpha_0)d\varphi(x) + (x + \alpha_0)d\varphi(-x)], & N = 2n + 1; \end{cases}$$

Thus $I_{2m, N} = 0$ for $N \neq 2m$ if we choose φ so that

$$d\varphi(x) - d\varphi(-x) = d\psi(z),$$
$$(x - \alpha_0)d\varphi(x) + (x + \alpha_0)d\varphi(-x) = 0. \qquad \alpha \leqq x \leqq \beta$$

We therefore obtain

$$d\varphi(x) = \frac{x + \alpha_0}{2x} d\psi(z)$$

$$d\varphi(-x) = -\frac{x - \alpha_0}{2x} d\psi(z)$$

and we have finally

$$I_{2m+1, 2n+1} = \int_{x=\alpha}^{\beta} Q_m(z)Q_n(z)[(x - \alpha_0)^2 d\varphi(x) - (x + \alpha_0)^2 d\varphi(-x)] =$$

$$= \int_{x=\alpha}^{\beta} Q_m(z)Q_n(z)(x^2 - a_0^2)d\psi(z) = 0$$

by (2.2) and the orthogonality of the $Q_n(z)$.

We therefore have

THEOREM 1. - The polynomials defined by (2.3) satisfy the orthogonality relation

$$\int_E R_m(x)R_n(x)\frac{x+a_0}{|x|}d\psi(x^2+a-\alpha^2)=2k_n\delta_{mn}$$

where $E=[-\beta, -\alpha]\bigcup[\alpha, \beta]$ and

$$k_{2m}=\int_a^b P_m^2(x)d\psi(x), \qquad k_{2m+1}=\int_a^b Q_m^2(x)(x-a_0)d\psi(x).$$

REMARKS. - (1) A more general quadratic transformation than (2.1) can be used but leads only to the equivalent of performing a linear transformation (in x) in our final results above.

(2) When b is finite, a similar result holds involving the kernel polynomials orthogonal with respect to $(b_0-x)d\psi(x)$, $b\leqq b_0$. In this case, (2.1) must be replaced by $z=b+\beta^2-x^2$, $\sqrt{b_0-b}\leqq\beta$. This is equivalent, essentially, to applying the preceding methods to the polynomials, $P_n(-x)$.

3. Recurrence formulas and the «symmetric case». Let the classical three term recurrence formulas satisfied by the three systems of orthogonal polynomials just considered be

(3.1) $$P_n(x)=(x-c_n)P_{n-1}(x)-\lambda_n P_{n-2}(x)$$

(3.2) $$Q_n(x)=(x-d_n)Q_{n-1}(x)-\nu_n Q_{n-2}(x), \quad Q_n(x)=Q_n(a_0, x),$$

(3.3) $$R_n(x)=(x-f_n)R_{n-1}(x)-\gamma_n R_{n-2}(x),$$

$$P_0(x)=Q_0(x)=R_0(x)=1, \quad P_{-1}(x)=Q_{-1}(x)=R_{-1}=0,$$

$$c_n, \ d_n, \ f_n \text{ real}, \quad \lambda_{n+1}>0, \quad \nu_{n+1}>0, \quad \gamma_{n+1}>0,$$

$$n=1, 2, 3, \ldots.$$

Proceeding in the same manner as in [2, § 2], we obtain from (3.3) with the aid of (2.3)

$$P_n(z) = (x - f_{2n})(x - \alpha_0)Q_{n-1}(z) - \gamma_{2n}P_{n-1}(z) \tag{3.4}$$

$$(x - \alpha_0)Q_n(z) = (x - f_{2n+1})P_n(z) - \gamma_{2n+1}(x - \alpha_0)Q_{n-1}(z). \tag{3.5}$$

Now since $z(-x) = z(x)$, it follows from (3.4) that

$$(x - f_{2n})(x - \alpha_0) = (x + f_{2n})(x + \alpha_0)$$

hence $f_{2n} = -\alpha_0$. Since $P_n(z(\alpha_0)) = P_n(a_0) \neq 0$ (since $a_0 \notin (a, b)$), it follows from (3.5) that $f_{2n+1} = \alpha_0$. Thus, with a reference to (2.2), we find

$$P_n(z) = (z - a_0)Q_{n-1} - \gamma_{2n}P_{n-1}(z), \tag{3.6}$$

$$n = 1, 2, 3, \dots$$

$$Q_n(z) = P_n(z) - \gamma_{2n+1}Q_{n-1}(z), \tag{3.7}$$

and it is easily verified that (3.7) holds for $n = 0$ also if we define $\gamma_1 = 0$. If we now eliminate $Q_{n-1}(z)$ and $Q_n(z)$ from (3.7) using (3.6), we obtain

$$P_{n+1}(z) = (z - a_0 - \gamma_{2n+1} - \gamma_{2n+2})P_n(z) - \gamma_{2n}\gamma_{2n+1}P_{n-1}(z) \qquad (n \geqq 0)$$

and similarly

$$Q_n(z) = (z - a_0 - \gamma_{2n} - \gamma_{2n+1})Q_{n-1}(z) - \gamma_{2n-1}\gamma_{2n}Q_{n-2}(z) \qquad (n \geqq 1).$$

Comparison of the latter with (3.1) and (3.2) thus yields the relations

$$f_{2n-1} = \alpha_0, \qquad f_{2n} = -\alpha_0 \tag{3.8}$$

$$\begin{aligned} &\gamma_{2n-1} + \gamma_{2n} = c_n - a_0, \qquad \gamma_{2n}\gamma_{2n+1} = \lambda_{n+1} \qquad (\gamma_1 = 0) \\ &\gamma_{2n} + \gamma_{2n+1} = d_n - a_0, \qquad \gamma_{2n+1}\gamma_{2n+2} = \nu_{n+1}, \end{aligned} \tag{3.9}$$

$$n = 1, 2, 3, \dots,$$

The relations in (3.9) are striking when compared with the corresponding equations obtained in [2, (2.6), (2.7)] (to which they, of course, reduce when $a_0 = 0$). Except for the presence of the « anti-symmetric » coefficients, f_n, they suggest the « symmetric » orthogonal polynomials satisfying

$$R_n^*(x) = xR_{n-1}^*(x) - \gamma_n R_{n-2}^*(x) \tag{3.10}$$

$$R_0^*(x) = 1, \qquad R_{-1}^*(x) = 0.$$

Indeed, because of (3.9), it follows from [2, § 2] that

$$R_{2n}^*(x) = P_n(x^2 + a_0), \qquad R_{2n+1}^*(x) = xQ_n(a_0, \ x^2 + a_0)$$

and the $R_n^*(x)$ are orthogonal with respect to the distribution, $d\varphi^*(x) = (\operatorname{sgn} x)d\psi(x^2 + a_0)$. From (2.3) and Theorem 1, we are thus led to the following theorem:

THEOREM 2. - Let the polynomials defined by (3.10) be orthogonal with respect to the distribution, $d\varphi^*(x)$, with support E^*. Then the polynomials defined by

$$R_n(x) = [x + (-1)^n\alpha_0]R_{n-1}(x) - \gamma_n R_{2-n}(x), \qquad n = 1, 2, 3, \dots \tag{3.11}$$

$$R_0(x) = 1, \qquad R_{-1}(x) = 0, \qquad \alpha_0 \text{ real},$$

satisfy

$$R_{2n}(x) = R_{2n}^*(\sqrt{x^2 - \alpha_0^2}), \tag{3.12}$$

$$R_{2n+1}(x) = (\operatorname{sgn} x)\left(\frac{x - \alpha_0}{x + \alpha_0}\right)^{1/2} R_{2n+1}^*(\sqrt{x^2 - \alpha_0^2})$$

$$\int_E R_m(x)R_n(x)\frac{x + \alpha_0}{|x|}\, d\varphi^*(\sqrt{x^2 - \alpha_0^2}) = k_n\delta_{mn} \tag{3.13}$$

where $E = \{x : x^2 = z^2 + \alpha_0^2, \ z \in E^*\}$

$$k_n = \int_{E^*} [R_n^*(x)]^2 d\varphi^*(x).$$

It may be observed that theorem 2 can be proven directely without reference to the preceding work. For if we consider the $R_n(x)$ as defined by (3 12), then substitution into (3.10) yields (3.11). Since (3.11) uniquely determines the $R_n(x)$, this verifies (3.12). The orthogonality relation (3.13) can now be verified directly using (3.12) and the fact that the sets E and E^* are symmetric with respect to the origin.

The preceding can be illustrated nicely using JACOBI polynomials:

$$P_n(x) = \binom{2n+\alpha+\beta}{n}^{-1} P_n^{(\alpha,\beta)}(2x-1), \qquad d\psi(x) = (1-x)^\alpha x^\beta dx,$$

$$Q_n(0,\ x) = \binom{2n+\alpha+\beta+1}{n}^{-1} P_n^{(\alpha,\beta+1)}(2x-1).$$

Here $a_0 = a = 0$, $b = 1$ while α and β are the usual parameters and not the same as in section 2.

Then

$$R_{2n}^*(x) = \binom{2n+\alpha+\beta}{n}^{-1} P_n^{(\alpha,\beta)}(2x^2-1)$$

$$R_{2n+1}^*(x) = \binom{2a+\alpha+\beta+1}{n}^{-1} xP_n^{(\alpha,\beta+1)}(2x^2-1)$$

are the polynomials orthogonal with respect to the weight function

$$w^*(x) = |x|^{2\beta+1}(1-x^2)^\alpha, \qquad -1 \leqq x \leqq 1.$$

(For $\beta = -1/2$, these reduce to well-known formulas for ultraspherical polynomials while for $\alpha = 0$ they yield a special case studied by SZEGÖ [4, (4.1.5), (4.1.6)]).

We thus obtain, for arbitrary real α_0,

$$R_{2n}(x) = \binom{2n+\alpha+\beta}{n}^{-1} P_n^{(\alpha,\beta)}(2x^2-2\alpha_0^2-1)$$

$$R_{2n+1}(x) = \binom{2n+\alpha+\beta+1}{n}^{-1} (x-\alpha_0)P_n^{(\alpha,\beta+1)}(2x^2-2\alpha_0^2-1)$$

as the polynomials orthogonal with respect to the weight function

$$w(x) = \begin{cases} |x+\alpha_0| \cdot |x^2-\alpha_0^2|^\beta (1+\alpha_0^2-x^2)^\alpha, & |\alpha_0| \leq |x| \leq \sqrt{1+\alpha_0^2} \\ 0 & \text{otherwise.} \end{cases}$$

For the recurrences, (3.10) and (3.11), we obtain the coefficients

$$\gamma_{2n} = \frac{(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)},$$

$$\gamma_{2n+1} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

4. A problem of Carlitz. Addressing a question posed by CARLITZ [1], DICKINSON and WARSI [3] have shown that if $\{P_n(x)\}$ is a set of polynomials orthogonal on a subset of $(0, \infty)$ (with respect to some distribution $d\psi(x)$), then there exists a set $\{\Phi_n(x)\}$ of real orthogonal polynomials such that

$$\Phi_{2n}(x) = P_n(x^2). \tag{4.1}$$

This follows also from [2, § 2] and the particular solution constructed by DICKINSON and WARSI is seen to be the polynomial set orthogonal with respect to the distribution, $(\operatorname{sgn} x)d\psi(x^2)$.

It is also easy to show that the converse of the above is true. For if $\{\Phi_n(x)\}$ is an orthogonal set with respect to the distribution $d\varphi(x)$, and (4.1) holds. then for $m \neq n$,

$$0 = \int_{-\infty}^{\infty} \Phi_{2m}(x)\Phi_{2n}(x)d\varphi(x) = \int_{-\infty}^{\infty} P_m(x^2)P_n(x^2)d\varphi(x)$$

$$= \int_0^{\infty} P_m(x^2)P_n(x^2)d[\varphi(x)-\varphi(-x)]$$

$$= \int_0^{\infty} P_m(z)P_n(z)d\psi(z)$$

where $\psi(z) = \varphi(\sqrt{z}) - \varphi(-\sqrt{z})$ is non-decreasing on $[0, \infty)$.

Theorem 1 also shows that the orthogonal set satisfying (4.1) is not, in general, unique. Specifically suppose $\{P_n(x)\}$ is a set of monic polynomials, $P_n(x)$ of degree n, which are orthogonal with respect to a distribution, $d\psi(x)$, and that the «true» interval of orthogonality (smallest interval which contains the support of $d\psi(x)$) is (a, b). Let

$$R_{2n}(x) = P_n(z), \qquad z = x^2 + A. \tag{4.2}$$

If $A \leqq a$, then for each a_0 satisfying

$$A \leqq a_0 \leqq a,$$

we can let $\alpha = \pm \sqrt{a_0 - A}$ and use (2.3) to define $R_{2n+1}(x)$ thus obtaining an ortogonal set of the type described by theorem 1.

Moreover, every orthogonal polynomial set $\{R_n(x)\}$ satisfying (4.2) must be of this type. To see this, let (3.1) and (3.3) be the recurrences satisfied by the respective polynomials.

Write (3.3) for $n = 2m$ and use (4.2) to obtain

$$(x - f_{2m})R_{2m-1}(x) = P_m(z) + \gamma_{2m}P_{m-1}(z) \qquad (m \geqq 1) \tag{4.4}$$

Since the right side is an even function of x, we conclude that

$$R_{2m-1}(x) = (x + f_{2m})Q_{m-1}(z) \qquad (m \geqq 1) \tag{4.4}$$

where $Q_n(z)$ is a monic polynomial of degree n in z.

Next using (3.3) with $n = 2m - 1$ to eliminate $R_{2m-1}(x)$ from (4.3), we get for $m \geqq 1$

$$(x - f_{2m})(x - f_{2m-1})P_{m-1}(z) - \gamma_{2m-1}(x - f_{2m})(x + f_{2m-2})Q_{m-2}(z) =$$
$$= P_m(z) + \gamma_{2m}P_{m-1}(z).$$

Comparing coefficients of x^{2m-1}, we conclude that $f_{2m}+f_{2m-1}=0$. Then comparing coefficients of x^{2m-3} we find that

$$f_{2m} = f_{2m-2} \qquad (m \geqq 1).$$

Thus, writing $\alpha_0 = - f_{2m'}$, we obtain from (4.3) and (4.4)

$$Q_n(z) = (x^2 - \alpha_0^2)^{-1}[P_{n+1}(z) + \gamma_{2n+2}P_n(z)] \qquad (m \geqq 0).$$

Hence letting $a_0 = \alpha_0^2 + A$, we have

$$\int_a^b Q_n(z) z^k (z - a_0) d\psi(z) =$$

$$= \int_a^b [P_{n+1}(z) + \gamma_{2n+2} P_n(z)] z^k d\psi(z) = 0 \qquad \text{for } k < n.$$

Now the zeros of $P_m(z)$ are located in (a, b), hence the positive and negative zeros of $R_{2m}(x)$ are located in $(-\beta, -\alpha)$ and (α, β), respectively, where $\alpha = \sqrt{a - A}$ and $\beta = \sqrt{b - A}$. Since the zeros of $R_{2m+1}(x)$ are interlaced with those of $R_{2m}(x)$, it follows from (4.4) that $-\alpha \leqq \alpha_0 \leqq \alpha$. It follows that $a_0 \leqq a$. Hence, $\{Q_n(x)\}$ is a set of kernel polynomials and $\{R_n(x)\}$ is of the type described in Theorem 1.

Of course, a more general quadratic transformation than that used in (4.2) can be reduced to the case just considered by a linear transformation (whic does not affect orthogonality). It is also abvious that the corresponding problem for a cubic or higher order transformation would be of considerably greater complexity. Indeed, it would be interesting to find an example of an orthogonal polynomial set $\{R_n(x)\}$ such that

$$R_{3n}(x) = P_n(z)$$

where z is a cubic in x.

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