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On a result of D. Gallarati concerning semigroups

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Summary. - *We extend the result of D. Gallarati given in* [1].

An element λ of a semigroup Σ is called a zeroid element of Σ in case it is true that for every $a \varepsilon \Sigma$, the equations $xa = \lambda$ and $ay = \lambda$ have solutions $x, y \varepsilon \Sigma$ [2].

GALLARATI [1] showed that if Σ is a semigroup with zeroid element λ such that for each $a \varepsilon \Sigma$, the solutions x and y in Σ of the equations $xa = \lambda$ and $ay = \lambda$ are unique, then Σ is a group.

In this note we prove the following extension of GALLARATI's result: If Σ is a semigroup with zeroid element λ such that for each $a \varepsilon \Sigma$, the solution $x \varepsilon \Sigma$ of the equation $xa = \lambda$ is unique, then Σ is a group (Theorem 2).

THEOREM 1. – Suppose Σ is a semigroup with zeroid element λ. If the solution $e \varepsilon \Sigma$ of the equation $e\lambda = \lambda$ is unique, then

$$\Sigma' = \{ ea \mid a \varepsilon \Sigma \}$$

is a subgroup of Σ.

PROOF. – Suppose $a \varepsilon \Sigma$. We wish to show that a has a left inverse with respect to e. Determine c such that $ac = \lambda$ and then determine t such that $ct = \lambda$. Now determine b such that $b(\lambda t) = \lambda$. Suppose $ba = k$. Then $kc = (ba)c = b(ac) = b\lambda$. Thus

$$(kc)t = (b\lambda)t$$

$$k(ct) = b(\lambda t)$$

$$k\lambda = \lambda.$$

(*) Pervenuta alla Segreteria dell' U. M. I. il 10 agosto 1964.

Hence $k = e$, since $k\lambda = e\lambda = \lambda$. Therefore $ba = e$ and b is a left inverse of a with respect to e.

We next note that $ee = e$, since $(ee)\lambda = e(e\lambda) = e\lambda = \lambda$.

Thus if $ea \in \Sigma'$, we have $e(ea) = (ee)a = ea$, and so $e \in \Sigma'$ is a left identity for Σ'. If $a \in \Sigma'$ and b is a left inverse of a with respect to e ($b \in \Sigma$), we have $(eb)a = e(ba) = ee = e$, $eb \in \Sigma'$. Then a has a left inverse $eb \in \Sigma'$ with respect to e. Clearly Σ' is a subsemigroup of Σ.

Thus Σ' is a group.

THEOREM 2. – *Suppose Σ is a semigroup with zeroid element λ. If for each $a \in \Sigma$, the solution $x \in \Sigma$ of $xa = \lambda$ is unique, then $\Sigma = \Sigma'$ and Σ is a group.*

PROOF. Suppose $a \in \Sigma$. Determine c such that $ac = \lambda$. Let $ea = b$. Then $bc = (ea)c = e(ac) = e\lambda = \lambda$. Thus $b = a$, since $bc = ac = \lambda$. Therefore $a = ea$ and $a \in \Sigma'$. Thus $\Sigma = \Sigma'$, and therefore by Theorem 1, Σ is a group.

We note in closing that if in Theorem 1, the solution $f \in \Sigma$ of $\lambda f = \lambda$ is unique, then $e = f$. This is shown as follows. We can show by a proof analogous to one given in the proof of Theorem 1 that each element of Σ has a right inverse with respect to f. Thus there exist s, $u \in \Sigma$ such that $s\lambda = e$ and $\lambda u = f$. Hence $e = f$, since

$$f = \lambda u = (e\lambda)u = e(\lambda u) = ef$$

and

$$e = s\lambda = s(\lambda f) = (s\lambda)f = ef.$$

REFERENCES

[1] DIONISIO GALLARATI, *Un'osservazione sopra i semigruppi*, Boll. Un. Mat. Ital., (3), 18 (1963), pp. 279-280

[2] D. D. MILLER and A. H. CLIFFORD, *Semigroups having zeroid elements*, Amer. J. Math., 70 (1948), pp. 117-125.