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On the uniqueness of limit cycles

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Sunto. - *Si generalizza un noto criterio dell'unicità dei cicli dovuto a Massera e Hudaî-Veronov.*

By a variety of methods and under different hypotheses LIÉNARD [3], LEVINSON and SMITH [2], SANSONE [5], MASSERA [4] and HUDAÎ-VERONOV [1] have proved that the equation

$$x'' + f(x)x' + x = 0 \tag{1}$$

has at most one limit cycle. The object of the present note is to extend the method of HUDAÎ-VERONOV to the system

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y). \end{aligned} \tag{2}$$

Moreover we replace by rigorous proof the appeal which this author makes to geometric intuition.

The proof of our uniqueness criterion is based on the following lemma, which is perhaps of independent interest.

LEMMA. - *Let $f(x, y)$ be a continuous real-valued function such that a unique solution of the differential equation*

$$dy/dx = f(x, y) \tag{3}$$

passes through any point of the rectangle $\alpha < x < \beta$, $\gamma < y < \delta$. Moreover let there exist a continuous function $x = \varphi(y)$, defined for $\gamma < y < \delta$, such that $f(x, y) \gtrless 0$ according as $x \gtrless \varphi(y)$.

Then the derivative of any solution $y(x)$ of the differential equation (3) *vanishes at most once in the interval $\alpha < x < \beta$. Moreover if $y'(\xi) = 0$ then $y'(x) \gtrless 0$ according as $x \gtrless \xi$.*

(*) Pervenuta alla Segreteria dell'U. M. I. il 10 agosto 1964.

PROOF. - A solution of the differential equation (3) cannot have its derivative equal to zero throughout an interval $x_1 < < x < x_2$. For this would imply the existence of a constant c such that $f(x, c) = 0$ for $x_1 < x < x_2$, whereas $x = \varphi(c)$ is the only value of x for which $f(x, c) = 0$.

Suppose that $y'(\xi_1) = 0$ and $y'(x) \neq 0$ for $\xi_1 < x \leq \xi_2$. Then either $y'(x) > 0$ or $y'(x) < 0$ for $\xi_1 < x \leq \xi_2$. We will show that the second alternative is impossible. In fact it implies that $y = y(x)$ has a continuous, strictly decreasing inverse $x = \psi(y)$ for $\eta_2 \leq y \leq \eta_1$, where $\eta_1 = y(\xi_1)$ and $\eta_2 = y(\xi_2)$ Moreover $\psi(y) < \varphi(y)$ for $\eta_2 \leq y \geq \eta_1$, since $y'(x) = f[x, y(x)] < 0$.

Define a new function $\bar{f}(x, y)$ throughout the rectangle $\xi_1 \leq \leq x \leq \xi_2$, $\eta_2 \leq y \leq \eta_1$, by setting

$$\bar{f}(x, y) = f(x, y) \text{ if } x \leq \varphi(y), \ = 0 \text{ otherwise.}$$

Also put

$$\bar{f}(x, y) = \bar{f}(x, \eta_1) \quad \text{for} \quad y > \eta_1, \ = \bar{f}(x, \eta_2) \quad \text{for} \quad y < \eta_2.$$

Then $\bar{f}(x, y)$ is continuous, bounded and non-positive in the entire strip $\xi_1 \leq x \leq \xi_2$, $-\infty < y < \infty$. Choose any value η_0 between η_2 and η_1 and take ξ_0 greater than $\psi(\eta_0)$ and less than both ξ_2 and $\varphi(\eta_0)$. The differential equation

$$dy/dx = \bar{f}(x, y)$$

has a solution $y = w(x)$ which passes through the point (ξ_0, η_0) and is defined for $\xi_1 \leq x \leq \xi_0$. Moreover $w(x)$ is a non-increasing function of x.

The graph of $y = w(x)$ is contained in the region R: $x \leq \varphi(y)$, $\eta_0 \leq y \leq \eta_1$. For suppose the point $(x_1, w(x_1))$ lay outside R. Since (ξ_0, η_0) belongs to R there must exist a value $x_2 > x_1$ such that $(x_2, w(x_2))$ is situated on the boundary of R and $(x, w(x))$ lies outside R for $x_1 \leq x < x_2$. It follows that $w'(x) = 0$ for $x_1 \leq x \leq x_2$ and hence $w(x_1) = w(x_2)$. Moreover $w(x_2) > \eta_0$, because $w'(\xi_0) = = f(\xi_0, \eta_0) < 0$, and $w(x_2) < \xi_1$, because $x_2 > \eta_1$. Thus

$$\eta_0 < w(x_1) = w(x_2) < \eta_1.$$

Hence, by the definition of the points $(x_1, w(x_1))$ and $(x_2, w(x_2))$,

$$x_2 = \varphi[w(x_2)],\ x_1 > \varphi[w(x_1)].$$

Since $x_1 < x_2$ this is a contradiction.

It follows that $w(x)$ is a solution of the original differential equation (3). Therefore the graphs of $w(x)$ have no common point and $w(x)$ is always greater than $y(x)$. Since $y(\xi_1) = \eta_1$ this implies $w(\xi_1) > \eta_1$, contrary to what we have just proved.

Similarly it may be shown that if $y'(\xi_2) = 0$ and $y'(x) \neq 0$ for $\xi_1 \leq x < \xi_2$ then $y'(x) < 0$ for $\xi_1 \leq x < \xi_2$.

Suppose now that $y'(x)$ vanished at least twice. At some point x_0 between the two zeros $y'(x)$ must be different from 0. Let x_1 and x_2 be the nearest zeros of $y'(x)$ on either side of $x_0 (x_1 < x_0 < x_2)$. Then by what has been shown $y'(x)$ is positive to the right of x_1 and negative to the left of x_2. Therefore it vanishes between x_1 and x_2, which is a contradiction. This completes the proof.

After these preparations we can prove without difficulty our main result:

THEOREM. – *Let $P(x, y)$, $Q(x, y)$ be continuous functions such that the solutions of the system* (2) *are uniquely determined by their initial values. Snppose also*

(i) *the system* (2) *has no critical points, except possibly the origin,*

(ii) *for every $\lambda > 1$ and every point (x, y)*

$$\Delta \equiv P(\lambda x, \lambda y) Q(x, y) - P(x, y) Q(\lambda x, \lambda y) \geq 0, \tag{4}$$

(iii) *strict inequality holds in* (4) *at all points $(x, y) \neq (0, 0)$ for which $xQ(x, y) = yP(x, y)$ and at all points of a curve extending from the origin to infinity.*

Then the system (2) *has at most one closed path.*

We can suppose the origin is a critical point, since otherwise there are certainly no closed paths. Changing to polar coordinates $x = r \cos \Theta$, $y = r \sin \Theta$ we get

$$r' = P \cos \Theta + Q \sin \Theta$$

$$r\Theta' = Q \cos \Theta - P \sin \Theta.$$

If Θ' vanishes for $t = t_0$ then $r' \neq 0$ for $t = t_0$ by (i). Thus in the neighbourhood of (r_0, Θ_0) we can write

$$\frac{d\Theta}{dr} = \frac{1}{r} \frac{Q \cos \Theta - P \sin \Theta}{P \cos \Theta + Q \sin \Theta} \equiv \varphi(r, \Theta).$$

By (iii) we have strict inequality in (4) near the point $(x_0, y_0) = (r_0 \cos \Theta_0, r_0 \sin \Theta_0)$. If $P(x_0, y_0) \neq 0$ then $\cos \Theta_0 \neq 0$ and (4) tells us that $Q(r \cos \Theta, r \sin \Theta)/P(r \cos \Theta, r \sin \Theta)$ is a decreasing function of r near (r_0, Θ_0). Hence, by the most elementary form of the implicit function theorem, for each Θ near Θ_0 there is a unique value $\rho(\Theta)$ of r near r_0 such that

$$Q(r \cos \Theta, r \sin \Theta)/P(r \cos \Theta, r \sin \Theta) = \tan \Theta .$$

Moreover $\rho(\Theta)$ is a continuous function of Θ and $\varphi(r, \Theta) \lesseqgtr 0$ according as $r \gtreqless \rho(\Theta)$. The same holds if $P(x_0, y_0)=0$ and $Q(x_0, y_0) \neq 0$. By the lemma, with y replaced by $-y$, it follows that at any zero of Θ' $d\Theta/dr$ changes sign from $+$ to $-$ as r increases. Consequently Θ' changes sign from $+$ to $-$ as t increases. Therefore Θ' vanishes at most once on any path and does not vanish at all on a closed path.

Thus any closed path is defined by an equation $r = r(\Theta)$, where $r(\Theta)$ is a solution of the equation

$$\frac{1}{r} \frac{dr}{d\Theta} = \frac{P \cos \Theta + Q \sin \Theta}{Q \cos \Theta - P \sin \Theta}$$

such that $r(2\pi) = r(0)$. Integrating with respect to Θ we get

$$0 = \int_0^{2\pi} \frac{P \cos \Theta + Q \sin \Theta}{Q \cos \Theta - P \sin \Theta} d\Theta.$$

If there were two closed paths, defined by equations $r = r_1(\Theta$ and $r = r_2(\Theta)$, where $r_1(\Theta) < r_2(\Theta)$, then by subtraction we would get

$$0 = \int_0^{2\pi} \frac{P_2 Q_1 - P_1 Q_2}{(Q_1 \cos \Theta - P_1 \sin \Theta)(Q_2 \cos \Theta - P_2 \sin \Theta)} d\Theta.$$

The denominator of the integrand has constant sign by what we have already proved. The numerator is non-negative by (ii), and actually positive for at least one value of Θ by (iii). Thus we have a contradiction.

The equation (1) is equivalent to the system

$$x' = y - F(x)$$

$$y' = -x,$$

where $F(x) = \int_0^x f(\xi)d\xi$. It follows from the theorem that the equation (1), where $f(x)$ is continuous, has at most one non-constant periodic solution if $F(x)/x$ is an increasing function for $x > 0$ and a decreasing function for $x < 0$. This is more general than the requirement of MASSERA and HUDAĬ-VERENOV that $f(x)$ be an increasing function for $x > 0$ and a decreasing function for $x<0$, since

$$[F(x)/x]' = x^{-2}[xf(x) - F(x)]$$

$$= x^{-2}\int_0^x [f(x) - f(\xi)]d\xi.$$

Moreover in most practical applications it is the function $F(x)$ which is given directly, rather than its derivative $f(x)$.

REFERENCES

[1] M. G. HUDAĬ-VERONOV, *Some theorems on limit cycles for the equation of Liénard* (Russian), « Uspehi Mat. Naux (N. S.) 12 (1957) no. 3(75), 389-396.

[2] N. LEVINSON and O. K. SMITH, *A general equation for relaxation oscillations*, « Duke Math. J. », 9(1942) 382-403.

[3] A. LIÉNARD, *Étude des oscillations entretenues*, « Rev. Gén. Élec. », 23 (1928) 901-912; 946-954.

[4] J. L. MASSERA, *Sur un théoreme de G. Sansone sur l'équation de Liénard*, « Boll. Un. Mat. Ital. », (3) 9 (1954) 367-369.

[5] G. SANSONE, *Equazioni differenziali nel campo reale*, II, 2 ed., Zanichelli, Bologna, 1949, pp. 396-402.