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On a note of E. Stipanić

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Summary. - *We extend a theorem of E Stipanic cocncerning positive term series.*

If $\sum_{p=0}^{\infty} c_p$ is a convergent series, let $r_{n-1} = \sum_{p=n}^{\infty} c_p$, $n = 1, 2, 3, \ldots$. STIPANIĆ [2] proved the following statement.

If $\{c_p\}_{p=0}^{\infty}$ is a positive term sequence such that

(1) $$\frac{c_n}{c_{n-1}} \longrightarrow \mu < 1$$

and

(2) $$c_n^{v+1} \leqq c_{n+v}(c_{n-1})^v, \qquad n \geqq 1, \ v \geqq 0,$$

then the series

(3) $$\sum_{p=1}^{\infty} \left(\frac{c_{p-1} - c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} \right)$$

is convergent.

We note that (2) holds for positive c_p's if and only if

(4) $$\frac{c_1}{c_0} \leqq \frac{c_2}{c_1} \leqq \frac{c_3}{c_2} \leqq \ldots .$$

Thus STIPANIC's result is essentially that (1) and (4) imply convergence of (3).

In this note we prove the following theorem which includes STIPANIC's result.

(*) Pervenuta alla segreteria dell'U. M. I. il 6 luglio 1964.

THEOREM 1. – If $\{c_p\}_{p=0}^{\infty}$ is a complex sequence such that $\{c_{p+1}/c_p\}$ converges absolutely to μ, $|\mu| < 1$, then

$$\sum_{p=N}^{\infty} \left| \frac{c_{p-1} - c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} \right| \tag{5}$$

converges for some N. (« $\{c_{p+1}/c_p\}$ converges absolutely » means $\sum_{p=k}^{\infty} |(c_{p+1}/c_p) - (c_p/c_{p-1})|$ converges for some k).

The following lemma may be of independent interest.

LEMMA. – If $\{a_p\}_{p=1}^{\infty}$ is a complex sequence which converges absolutely to δ, $|\delta| < 1$, and $b_p = 1 + a_p + a_p a_{p+1} + a_p a_{p+1} a_{p+2} + \cdots$, $p = 1, 2, 3, \ldots$, then $\{b_p\}$ converges absolutely to a nonzero limit.

Proof. of Lemma. – We note that $\{b_p\}$ is bounded; let L be a number such that $|b_p| < L$, $p = 1, 2, 3, \ldots$. If p is a positive integer, then $b_p = 1 + a_p b_{p+1}$, and so

$$\begin{aligned} b_p - b_{p+1} &= a_p b_{p+1} - a_{p+1} b_{p+2} \\ &= a_p(b_{p+1} - b_{p+2}) + b_{p+2}(a_p - a_{p+1}). \end{aligned}$$

Let N be a positive integer such that $|a_p| < t < 1$ if $p < N$. Then if i is a positive integer,

$$|b_{N+i} - b_{N+i+1}| \leqq t\,|b_{N+i+1} - b_{N+i+2}| + L\,|a_{N+i} - a_{N+i+1}| .$$

Hence if $m > 1$,

$$\begin{aligned} &|b_{N+1} - b_{N+2}| + (1 - t) \sum_{p=2}^{m} |b_{N+p} - b_{N+p+1}| \\ &\leqq t\,|b_{N+m+1} - b_{N+m+2}| + L \sum_{p=1}^{m} |a_{N+p} - a_{N+p+1}| \\ &\leq 2tL + L \sum_{p=1}^{\infty} |a_p - a_{p+1}|. \end{aligned}$$

Thus $\{b_p\}$ converges absolutely. Since $b_p = 1 + a_p b_{p+1}$, $p = 1, 2, 3, \ldots$, and $a_n \to \delta$, we see that $b_n \not\to 0$. This completes the proof of the lemma.

We now prove Theorem 1. Suppose $\{c_{p+1}/c_p\}$ converges absolutely to μ, $|\mu| < 1$. Let N_1 be a positive integer such that $c_p \neq 0$ if $p > N_1$. Then if $p > N_1$, we have

$$\frac{r_{p-1}}{c_p} = \frac{c_p + c_{p+1} + c_{p+2} + \dots}{c_p}$$

$$= 1 + \frac{c_{p+1}}{c_p} + \frac{c_{p+1}}{c_p}\frac{c_{p+2}}{c_{p+1}} + \frac{c_{p+1}}{c_p}\frac{c_{p+2}}{c_{p+1}}\frac{c_{p+3}}{c_{p+2}} + \dots .$$

Thus from the lemma, $\{r_{p-1}/c_p\}$ converges absolutely to a nonzero limit. Let N_2 be a positive integer such that $r_p \neq 0$ if $p > N_2$. Let $N = N_1 + N_2 + 2$. Then if $p \geqq N$, we have

$$(6) \qquad \frac{c_{p-1} - c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} = 1 - \frac{c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} = 1 - c_p\left[\frac{r_{p-1} + c_{p-1}}{c_{p-1}r_{p-1}}\right]$$

$$= 1 - \frac{c_p r_{p-2}}{c_{p-1} r_{p-1}} = 1 - \frac{c_p/r_{p-1}}{c_{p-1}/r_{p-2}} .$$

Hence (5) converges, and Theorem 1 is established.

As a partial converse of Theorem 1 we have the following theorem.

THEOREM 2. - If Σc_p is a convergent complex series and (5) converges for some N, then $\{c_{p+1}/c_p\}$ converges absolutely to a complex number μ such that $|\mu| \leqq 1$ and $\mu \neq 1$.

Proof. - Suppose (5) converges for some N. Then from (6) and Theorem 2.1 of [1], we see that $\{c_p/r_{p-1}\}$ converges absolutely to a nonzero limit. If $p \geqq N$, we have

$$(7) \qquad \frac{c_{p-1} - c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} = \frac{r_{p-1} - c_p}{r_{p-1}} - \frac{c_p}{c_{p-1}} = \frac{r_p}{r_{p-1}} - \frac{c_p}{c_{p-1}},$$

and

$$(8) \qquad \frac{r_p}{r_{p-1}} - \frac{r_{p+1}}{r_p} = \frac{r_{p-1} - c_p}{r_{p-1}} - \frac{r_p - c_{p+1}}{r_p} = \frac{c_{p+1}}{r_p} - \frac{c_p}{r_{p-1}} .$$

Thus $\{c_{p+1}/c_p\}$ converges absolutely, say to μ. Clearly $|\mu| \leq 1$, since Σc_p converges. Suppose $\mu = 1$. Then $c_p/r_{p-1} \to 0$, since

$$\frac{c_{p-1} - c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} = 1 - \frac{c_p}{c_{p-1}} - \frac{c_p}{r_{p-1}} \to 0$$

because of the convergence of (5). But this is a contradiction.

REMARK. - From (7), (8), and the proof of Theorem 1 we see that if $\{c_{p+1}/c_p\}$ converges absolutely to μ, $|\mu| < 1$, then $\{r_{p+1}/r_p\}$ converges absolutely to μ, and so for each positive integer n, $\sum_{p=n}^{\infty} r_p$ is defined. Now, if $\{c_{p+1}/c_p\}$ converges absolutely to μ, $|\mu| < 1$, and j is a nonnegative integer, let

$$r_j^{(n)} = \sum_{p=j+1}^{\infty} r_p^{(n-1)} \qquad n = 1, 2, 3, \ldots,$$

where $r_j^{(0)} = c_j$. Then we have the following corollary to Theorem 1.

COROLLARY. - Suppose $\{c_p\}_{p=0}^{\infty}$ is a complex sequence and $\{c_{p+1}/c_p\}$ converges absolutely to μ, $|\mu| < 1$. Then if n is a positive integer, there exists a positive integer N such that

$$\sum_{p=N}^{\infty} \left| \frac{r_{p-1}^{(n-1)} - r_p^{(n-\ )}}{r_{p-1}^{(n-1)}} - \frac{r_p^{(n-1)}}{r_{p-1}^{(n)}} \right|$$

converges.

The proof is by induction and is obvious in light of Theorem 1 and the remark.

REFERENCES

[1] D. F. DAWSON, *Concerning a theorem of Hadamard.* « Americam Mathematical Monthly », Vol. 69 (1962), pp. 981-983.

[2] E. STIPANIĆ, *Due teoremi su alcune serie divergenti di Dini a termini positivi*, « Boll. Un. Mat. Ital. », (3), Vol. 14 (1959), pp. 516-524.