Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **19** (1964), n.2, p. 98–105.

Zanichelli

<http://www.bdim.eu/item?id=BUMI_1964_3_19_2_98_0>

Characterization of certain q-identities

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Summary. - *Certain well known* q*-identities are characterized; see Theorem* 1, 2, 3 *below.*

1. Consider a set of polynomials $\{\Phi_n(x)\}$ such that

$$\Phi_0(x) = 1 \tag{1}$$

and

$$\Phi_n(x) = (1 + \lambda_n x)\Phi_{n-1}(x) \qquad (n = 1, 2, 3, \ldots). \tag{2}$$

Moreover assume that

$$\Phi_n(x) = \sum_{r=0}^{n} C(n, r)x^r, \tag{3}$$

where

$$C(n, r) = \frac{f(n)f(n-1)\ldots f(n-r+1)}{g(1)g(2)\ldots g(r)} \tag{4}$$

and of course

$$g(r) \neq 0 \qquad (r = 1, 2, 3, \ldots).$$

Combining (2) and (3) we get

$$C(n, r) = C(n-1, r) + \lambda_n C(n-1, r-1) \qquad (1 \leq r < n) \tag{5}$$

and

$$C(n, n) = \lambda_n C(n-1, n-1). \tag{6}$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 3 ottobre 1963.
Supported in part by NSF grant GP - 1593.

In view of (4), (5) and (6) become

(7) $$f(n) = f(n-r) + \lambda_n g(r) \qquad (1 \leq r < n)$$

and

(8) $$f(n) = \lambda_n g(n).$$

Eliminating λ_n we get

(9) $$1 = \frac{f(n-r)}{f(n)} + \frac{g(r)}{g(n)} \qquad (1 \leq r < n).$$

It is also evident from (7) that

$$\lambda_n g(1) + \lambda_{n-1} g(1) = \lambda_n g(2),$$

so that

(10) $$\lambda_n (g(2) - g(1)) = \lambda_{n-1} g(1) \qquad (n > 2).$$

If $g(2) = g(1)$ it follows that

$$\lambda_n = 0 \qquad (n \geq 0),$$

a case of little interest. Excluding this case, (10) gives

(11) $$\lambda_n = A\alpha^n,$$

where A and α are independent of n. Substituting from (8) and (11) in (9) we get

$$g(n) = \alpha^{-r} g(n-r) + g(r) = \alpha^{-n+r} g(r) + g(n-r).$$

Thus

$$\frac{g(n-r)}{1-\alpha^{-n+r}} = \frac{g(r)}{1-\alpha^{-r}}.$$

If we take $r = 1$ and replace n by $n+1$ we get

(12) $$g(n) = B(1-\alpha^{-n}),$$

where B is a constant. Then by (8), (11) and (12), it is clear that

$$f(n) = AB(\alpha^n - 1). \tag{13}$$

It follows from (4), (12) and (13) that

$$C(n, r) = A^r \alpha^{\frac{1}{2}r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix},$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-\alpha^n)(1-\alpha^{n-1}) \ldots (1-\alpha^{n-r+1})}{(1-\alpha)(1-\alpha^2) \ldots (1-\alpha^r)}. \tag{14}$$

Hence (2) and (3) reduce to the familiar identity

$$\prod_{r=1}^{n} (1 - \alpha^r Ax) = \sum_{r=0}^{n} \alpha^{\frac{1}{2}r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix} A^r x^r.$$

We have therefore proved the following

THEOREM 1. – *Let the set of polynomials* $\{\Phi_n(x)\}$ *satisfy* (1), (2), (3), (4) *and*

$$\deg \Phi_n(x) = n \qquad (n = 0, 1, 2, \ldots).$$

Then

$$\Phi_n(x) = \sum_{r=0}^{n} \alpha^{\frac{1}{2}r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix} A^r x^r,$$

where A *and* α *are arbitrary constants,* $A \neq 0$.

2. In the next place we consider a set of power series $\{\Psi_n(x)\}$ such that

$$\Psi_0(x) = 1, \tag{15}$$

$$(1 - \lambda_n x)\Psi_n(x) = \Psi_{n-1}(x) \qquad (n = 1, 2, 3, \ldots) \tag{16}$$

and

$$\Psi_n(x) = \sum_{r=0}^{n} C(n + r - 1, r) x^r, \tag{17}$$

where $C(n, r)$ is defined by (4). Then exactly as above, we get

$$C(n+r-1, r) - \lambda_n C(n+r-2, r-1) = C(n+r-2, r) \qquad (n=1, 2, 3, \ldots)$$

This reduces to

$$f(n + r - 1) - f(n - 1) + \lambda_n g(r). \tag{18}$$

It follows from (18) that

$$\lambda_n g(1) + \lambda_{n+1} g(1) = \lambda_n g(2),$$

so that

$$\lambda_n = A\alpha^{n-1}, \tag{19}$$

where A and α are constants. Substituting from (19) in (18) we get

$$f(n + r - 1) - f(n - 1) = A\alpha^{n-1} g(r). \tag{20}$$

In particular, when $r = 1$, this reduces to

$$f(n) - f(n - 1) = A\alpha^{n-1} g(1).$$

This evidently implies

$$f(n) = Ag(1) \frac{1 - \alpha^n}{1 - \alpha} + B, \tag{21}$$

where B is a constant. Thus (20) becomes

$$Ag(1) \frac{\alpha^{n-1} - \alpha^{n+r-1}}{1 - \alpha} = A\alpha^{n-1} g(r),$$

so that

$$g(r) = g(1)\frac{1-\alpha^r}{1-\alpha}. \tag{22}$$

Now (16) gives

$$\Psi_1(x) = 1 + \lambda_1 x + \lambda_1^2 x^2 + \dots;$$

moreover

$$C(1,\ 1) = \frac{f(1)}{g(1)}$$

so that

$$f(1) = \lambda_1 g(1) = Ag(1).$$

Comparing this with (21), it follows that $B = 0$. We therefore get

$$C(n + r - 1,\ r) = A^r \begin{bmatrix} n + r - 1 \\ r \end{bmatrix}.$$

This proves

THEOREM 2. – *Let the set of power series* $\{\Psi_n(x)\}$ satisfy (15), (16), (17) *and*

$$\lambda_n \neq 0 \qquad (n = 1,\ 2,\ 3, \ldots).$$

Then

$$\Psi_n(x) = \sum_{r=0}^{\infty} \begin{bmatrix} n + r - 1 \\ r \end{bmatrix} A^r x^r = \prod_{s=0}^{n-1} (1 - \alpha^s A x)^{-1},$$

where A *and* α *are arbitrary constants*, $A \neq 0$.

3. The identity [1, p. 66]

$$\prod_{1}^{\infty} \frac{1 - \alpha^{n-1}\lambda x}{1 - \alpha^{n-1} x} = \sum_{0}^{\infty} \frac{(\lambda)_n}{(\alpha)_n} x^n, \tag{23}$$

where

(24) $$(\lambda)_n = (1-\lambda)(1-\alpha\lambda)\dots(1-\alpha^{n-1}\lambda),$$

$$(\alpha)_n = (1-\alpha)(1-\alpha^2)\dots(1-\alpha^n),$$

suggests consideration of the following functional equation:

(25) $$\frac{F(\lambda x)}{F(x)} = \sum_0^\infty \frac{(\lambda,\ 1)(\lambda,\ 2)\dots(\lambda,\ n)}{g(1)g(2)\dots g(n)} x^n,$$

where

$$F(x) = \sum_0^\infty c_n x^n \qquad (c_0 = 1).$$

It follows from (25) that $(\lambda,\ 1)(\lambda,\ 2)\dots(\lambda,\ n)$ is a polynomial in λ of degree n; hence $(\lambda,\ n)$ is a polynomial of degree 1. We shall assume that

(26) $$(0,\ n) = 1 \qquad (n = 1,\ 2,\ 3, \dots).$$

We have therefore

(27) $$(\lambda,\ n) = 1 - \beta_n\lambda.$$

It also follows from (25) that

(28) $$(\lambda\mu,\ 1)\dots(\lambda\mu,\ n) =$$

$$= \sum_{r=0}^{n} \frac{g(n)\dots g(n-r+1)}{g(1)\dots g(r)} \lambda^{n-r}(\lambda,\ 1)\dots(\lambda,\ r)(\mu,\ 1)\dots(\mu,\ n-r).$$

In particular, for $n = 1$,

$$(\lambda\mu,\ 1) = (\lambda,\ 1) + \lambda(\mu,\ 1) = (\mu,\ 1) + \mu(\lambda,\ 1)$$

so that

$$(1-\mu)(1-\beta_1\lambda) = (1-\lambda)(1-\beta_1\mu)$$

and $\beta_1 = 1$. Similarly, when $n = 2$, we find that

$$(1+\mu)\left(1-\beta_2\lambda-\frac{g(2)}{g(1)}\right) = (1+\lambda)\left(1-\beta_2\mu-\frac{g(2)}{g(1)}\right)$$

which implies

$$1-\frac{g(2)}{g(1)} = -\beta_2.$$

We put

$$\alpha = \beta_2, \qquad g(1) = A^{-1}(1-\alpha), \tag{29}$$

where A is a constant. We shall assume that

$$\beta_r = \alpha^{r-1}, \qquad g(r) = A^{-1}(1-\alpha^r) \qquad (1 \leq r < n). \tag{30}$$

Then (28) becomes after a little reduction

$$\left\{\frac{Ag(n)}{1-\alpha^n} - \frac{1-\alpha^{n-1}\lambda\mu}{(\lambda\mu,\ u)}\right\}(\lambda\mu)_n + (\lambda)_n\left\{\frac{(\lambda,\ n)}{1-\alpha^{n-1}\lambda} - 1\right\} + \tag{31}$$

$$+\lambda^n(\mu)_n\left\{\frac{(\mu,\ n)}{1-\alpha^{n-1}\mu} - 1\right\} = 0.$$

Interchanging λ and μ. we get

$$(1-\mu^n)(\lambda)_n\left\{\frac{(\lambda,\ n)}{1-\alpha^{n-1}\lambda} - 1\right\} =$$

$$= (1-\lambda^n)(\mu)_n\left\{\frac{(\mu,\ n)}{1-\alpha^{n-1}\mu} - 1\right\},$$

which reduces to

$$(1-\mu^n)(\lambda)_n\lambda(\alpha^{n-1}-\beta_n) = (1-\lambda^n)(\mu)_n\mu(\alpha^{n-1}-\beta_n).$$

Therefore $\beta_n = \alpha^{n-1}$ and by (31)

$$Ag(n) = 1 - \alpha^n.$$

Substituting from (30) in (25) we get

$$\frac{F(\lambda x)}{F(x)} = \sum_0^\infty \frac{(\lambda)_n}{(\alpha)_n} A^n x^n.$$

This completes the proof of the following

THEOREM 3. - *Let*

$$F(x) = \sum_0^\infty c_n x^n \qquad (c_0 = 1)$$

satisfy the functional equation

$$\frac{F(\lambda x)}{F(x)} = \sum_0^\infty \frac{(\lambda, 1)(\lambda, 2) \dots (\lambda, n)}{g(1)g(2) \dots g(n)} x^n$$

where

$$(0, n) = 1 \qquad (n = 1, 2, 3, \dots).$$

Then

$$F(x) = \prod_0^\infty (1 - \alpha^n Ax),$$

where A, α *are arbitrary constants.*

REFERENCE

[1] W. N. BAILEY, *Generalized hypergeometric series*, Cambridge 1955.