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BREVI NOTE

Mosaic Spaces, P_1 - Mappings, and Property K

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Summary. - *A characterization of those mosaic spaces which have property K is given, and a necessary and sufficient condition for an M-space which is the image of an hereditary mosaic space under a quasi-compact mapping to be an hereditary mosaic space is established.*

Davison [2, def. 1.1, p. 526][1] introduced the following definition of a mosaic space. A collection $\{(X_a, \mathfrak{F}_a): a \in A\}$ is said to be a *mosaic of topological spaces* on a set X if and only if (i) each $X_a, \mathfrak{F}_a)$ is a topological space; (ii) $X = \bigcup \{X_a : a \in A\}$; and (iii) for all subsets E of X and all $a, b \in A$, if $E \subseteq X_a$ and E is $\mathfrak{F}_a$-closed then $E \cap X_b$ is $\mathfrak{F}_b$-closed. For a mosaic of topological spaces on X the *mosaic topology* $\mathfrak{F}$ is defined as follows: for all $E \subseteq X$, E is $\mathfrak{F}$-closed if and only if $E \cap X_a$ is $\mathfrak{F}_a$-closed for all $a \in A$. If each $(X_a, \mathfrak{F}_a)$ is a compact metric space, then the topological space $(X, \mathfrak{F})$ determined by the collection $\{(X_a, \mathfrak{F}_a): a \in A\}$ is called a *mosaic space*. Every mosaic space is T_1 but not, in general, HAUSDORFF. In fact, DAVISON [2, ex. 6.1, p. 541] has given an examble of a compact, hereditary mosaic space which is not HAUSDORFF, where by an *hereditary mosaic space* we mean a mosaic space with the property that each subspace is a mosaic space.

Let f be a function from a topological space $(X, \mathfrak{F})$ onto a

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(1) Numbers in square brackets refer to the bibliography.

topological space $(Y, \mathfrak{S})$. We shall call f a *quasi-compact mapping* provided that for every subset E of Y, E is $\mathfrak{S}$-closed if and only if $f^{-1}(E)$ is $\mathfrak{F}$-closed. Davison [2, theorem 3.7, p. 534] has shown that if f is a quasi-compact mapping from a mosaic space $(X, \mathfrak{F})$ onto an *M-space* $(Y, \mathfrak{S})$ i. e., a space $(Y, \mathfrak{S})$ for which limits of $\mathfrak{S}$-convergent sequences are unique [5, p. 474], then $(Y, \mathfrak{S})$ is a mosaic space.

In this note we shall establish a necessary and sufficient condition for an M-space which is the image of an hereditary mosaic space under a quasi-compact mapping to be an hereditary mosaic space and we shall characterize those mosaic spaces $(X, \mathfrak{F})$ which have *property* K (for each point x and each subset E of X having x as a $\mathfrak{F}$-limit point, there exists a $\mathfrak{F}$-compact subset of $E \cup \{x\}$ which has x as a $\mathfrak{F}$-limit point [3, def. 2, p. 689]).

Bae [1, p. 39] proved the following theorem.

THEOREM 1. – Let f be a quasi-compact mapping of a HAUSDORFF space $(X, \mathfrak{F})$ having property K onto a HAUSDORFF space $(Y, \mathfrak{S})$. Then $(Y, \mathfrak{S})$ has property K if and only if f is a P_1-mapping (if $y \in Y$ and U is a $\mathfrak{F}$-open set containing $f^{-1}(y)$ then $y \in \operatorname{int} f(U)$ [5, p. 474]).

THEOREM 2. – Let f be a quasi-compact mapping of a mosaic space $(X, \mathfrak{F})$ having property K onto an M-space $(Y, \mathfrak{S})$. Then $(y, \mathfrak{S})$ has property K if and only if f is a P_1-mapping.

PROOF. Bae's argument for theorem 1 can be seen to hold for any T_1-space $(X, \mathfrak{F})$ having property K and for any topological space $(Y, \mathfrak{S})$ in which every compact set is closed. Thus the result follows directly from Bae's argument and Davison's results that every mosaic space is a T_1-space [2, p. 527], that $(Y, \mathfrak{S})$ is a mosaic space [2, theorem 3.7, p. 534]and that every compact set in a mosaic space is closed [2, cor. 1·9, p. 527].

THEOREM 3. – Let $(X, \mathfrak{F})$ be a mosaic space. $(X, \mathfrak{F})$ has property K if and only if $(X, \mathfrak{F})$ is hereditary.

PROOF. – Let $(X, \mathfrak{F})$ be a mosaic space with property K. To show that $(X, \mathfrak{F})$ is hereditary it is sufficient to show that every $\mathfrak{F}$-limit point of a subset of X is also a $\mathfrak{F}$-sequential limit point of the subset [2, theorem 4.3, p. 536]. So let x be a $\mathfrak{F}$-limit point of a subset E of X. Since $(X, \mathfrak{F})$ has property K, there exists a

$\mathfrak{F}$-compact subset F of $E \cup \{x\}$ which has x as a $\mathfrak{F}$-limit point. Since $(X, \mathfrak{F})$ is a mosaic space, F is $\mathfrak{F}$-closed and x is therefore in F. It follows that $F - \{x\}$ is not $\mathfrak{F}$-closed. Thus there exist a sequence S and a point y in X such that S $\mathfrak{F}$-converges to y, $y \varepsilon F - \{x\}$, and $S_J \subseteq F - \{x\}$, where S_J is the point set associated with the sequence S [2, theorem 1.3, p. 526]. Now y is a $\mathfrak{F}$-limit point of S_J and hence of $F - \{x\}$ and, since $y \varepsilon F - \{x\}$, then $y = x$. Therefore x is a $\mathfrak{F}$-sequential limit point of $F - \{x\}$ and hence of E and it follows that $(X, \mathfrak{F})$ is hereditary.

Now let $(X, \mathfrak{F})$ be an hereditary mosaic space. To show that $(X, \mathfrak{F})$ has property K, let x be a $\mathfrak{F}$-limit point of a subset E of X. Since $(X, \mathfrak{F})$ is hereditary, there exists a sequence S such that S $\mathfrak{F}$-converges to x and $S_J \subset E$. If the mosaic space $(X, \mathfrak{F})$ is determined by the mosaic $\{(X_\alpha, \mathfrak{F}_\alpha): \alpha \varepsilon A\}$ of compact metric spaces on the set X with the mosaic topology $\mathfrak{F}$. then there exist a subsequence S' of S and an $\alpha \varepsilon A$ such that $S'_J \cup \{x\} \subseteq X_\alpha$ [2, lemma 1.5, p. 526]. $S'_J \cup \{x\}$ is $\mathfrak{F}_\alpha$-closed and therefore $\mathfrak{F}_\alpha$-compact. Moreover, since $\mathfrak{F}_\alpha$ is the topology $\mathfrak{F}$ relativized to X_α, $S'_J \cup \{x\}$ is $\mathfrak{F}$-compact. Since $S'_J \cup \{x\}$ is a $\mathfrak{F}$-compact subset of $E \cup \{x\}$ which has x as a $\mathfrak{F}$-limit point then $(X, \mathfrak{F})$ has property K.

The next result follows immediately from theorems 2 and 3.

THEOREM 4. - Let f be a quasi-compact mapping of an hereditary mosaic space $(X, \mathfrak{F})$ onto an M-space $(Y, \mathfrak{S})$. Then $(Y, \mathfrak{S})$ is an hereditary mosaic space if and only if f is a P_1-mapping.

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