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On a paper of Banerjee.

By S. K. CHATTERJEA (a Calcutta) (*)

Summary. - *Recently D. P. Banerjee has proved Turán's inequality for the Bessel polynomials* $y_n(x)$. *Here it is pointed out that Banerjee's inequality is incorrect and that in course of demonstration he fails to notice that for* $n=2m$, $y_n(x)$ *has no real zeros. Secondly Turán's inequality for the Bessel functions is known for a long time. Lastly Banerjee's series*

$$\sum_{n=0}^{p} (2n+1) y_n^2(x)$$

was already considered by C. K. Chatterjee and W. A. Al-Salam. Further we have proved some properties of the Bessel polynomials, of which the following two are characteristic:

(i) $$\sum_{r=0}^{n} (-1)^{n-r}(2r+1) y_r^2(x) = y'_{n+1}(x) y_n(x) - y'_n(x) y_{n+1}(x),$$

(ii) $$y_n(x) y_{n+2}(x) - y^2_{n+1}(x) = x + 2x^2 \sum_{r=0}^{n} \Lambda_r,$$

where

$$\Lambda_r \equiv y'_{r+1}(x) y_r(x) - y'_r(x) y_{r+1}(x), \quad y'_r(x) \equiv \frac{d}{dx} y_r(x).$$

(*) Pervenuta allla Segreteria dell'U.M. I. l'8 febbraio 1964.

In a recent paper [1], D. P. BANERJEE has proved the theorem that for all real $x > 0$ and $n > 0$, the following TURÁN's inequality holds:

$$y^2_n(x) - y_{n+1}(x)y_{n-1}(x) \geq 0, \tag{1}$$

where $y_n(x)$ is KRALL–FRINK's BESSEL polynomials defined by

$$y_n(x) = \sum_{r=0}^{n} \frac{(n+r)!}{r!\ (n-r)!} (x/2)^r . \tag{2}$$

Here we like to remark that this theorem is incorrect. For, when $n = 1$, we have

$$y_1^2(x) - y_2(x)y_0(x) = -x(1 + 2x).$$

Again when $n = 2$, we notice

$$y_2^2(x) - y_3(x)y_1(x) = -x(1 + 6x + 12x^2 + 6x^3) .$$

Indeed, we have proved in a recent note [2] that

$$y_n(x)\ y_{n+2}(x) - y^2_{n+1}(x) = x + 2x^2 \sum_{i=0}^{[n/2]} (2n + 1 - 4i)y^2_{n-2i}(x) , \tag{3}$$

by means of the formula

$$\Delta_n = \Delta_{n-2} + 2(2n+1)x^2y^2_n(x) , \tag{4}$$

where

$$\Delta_n \equiv y_n(x)y_{n+2}(x) - y^2_{n+1}(x) .$$

When written in BURCHNALL's notation for KRALL–FRINK's BESSEL polynomials, (3) assumes the form [3]:

$$\begin{aligned} &\theta_n(x)\theta_{n+2}(x) - \theta^2_{n+1}(x) \\ &= x^{2n+1} + 2 \sum_{j=0}^{[n/2]} (2n + 1 - 4j)x^{4j}\theta^2_{n-2j}(x) , \end{aligned} \tag{5}$$

where

$$\theta_n(x) = x^n y_n(1/x) .$$

Here it is worth mentioning that TURÁN's inequality for the BESSEL polynomials is not new, for CARLITZ [4] had already proved that

$$\theta_{n-1}(x)\theta_{n+1}(x) - \theta^2_n(x) \geq 0, \quad (x \geq 0), \tag{6}$$

in a different manner. We have proved (5) with the help of the relation

$$\Delta(n, 1, 1; x) = x^4\Delta(n-2, 1, 1; x) + 2(2n+1)\theta^2_n(x), \tag{7}$$

where

$$\Delta(n, 1, 1; x) \equiv \theta_n(x)\theta_{n+2}(x) - \theta^2_{n+1}(x).$$

We also add that in the proof of (1), BANERJEE fails to notice that for even n, $y_n(x)$ has no real zeros.

Secondly, BANERJEE has proved in Theorem 2, that TURÁN's inequality holds good for the BESSEL functions $J_n(x)$ also. But LOMMEL's series [5] of squares of BESSEL functions viz.,

$$\frac{x^2}{4}\left\{ J^2_n(x) - J_{n-1}(x)J_{n+1}(x) \right\} = \sum_{k=0}^{\infty} (n+1+2k)J^2_{n+1+2k}(x), \tag{8}$$

is known to us. SZÁSZ [6] had established the inequality

$$J_n^2(x) - J_{n-1}(x)J_{n+1}(x) > \frac{1}{n+1} J_n^2(x), \ (n > 0, \ -\infty < x < +\infty). \tag{9}$$

We have proved in a previous note [7] that

$$\begin{aligned} \Delta_{n,1,2;x}(J) &< 0, \quad -\infty < x < 0 \\ &= 0, \quad x = 0 \\ &> 0, \quad 0 < x < \infty \end{aligned} \tag{10}$$

where

$$\Delta_{n,1,2;x}(J) \equiv J_{n+1}(x)J_{n+2}(x) - J_n(x)J_{n+3}(x).$$

We shall now discuss some properties of the BESSEL polynomials. We notice

$$\begin{aligned} & y_{n+1}(x)y_n(z) - y_n(x)y_{n+1}(z) \\ &= [(2n+1)xy_n(x) + y_{n-1}(x)]y_n(z) - y_n(x)[(2n+1)zy_n(z) + y_{n-1}(z)] \\ &= (2n+1)(x-z)y_n(x)y_n(z) - [y_n(x)y_{n-1}(z) - y_{n-1}(x)y_n(z)]. \end{aligned}$$

Now we write

$$(11) \qquad \sigma_n(x, z) + \sigma_{n-1}(x, z) = (2n+1)(x-z)y_n(x)y_n(z)$$

where

$$\sigma_n(x, z) \equiv y_{n+1}(x)y_n(z) - y_n(x)y_{n+1}(z).$$

Similarly we have

$$(12) \qquad \sigma_{n-1}(x, z) + \sigma_{n-2}(x, z) = (2n-1)(x-z)y_{n-1}(x)y_{n-1}(z).$$

It therefore follows from (11) and (12) that

$$(13) \qquad \frac{\sigma_n(x, z) - \sigma_{n-2}(x, z)}{x-z} = (2n+1)y_n(x)y_n(z) - (2n-1)y_{n-1}(x)y_{n-1}(z).$$

Putting $n = 2m$ and by repeated application we obtain from (13):

$$(14) \qquad \begin{aligned} \frac{\sigma_{2m}(x, z)}{x-z} &= \sum_{i=0}^{m} (4i+1)y_{2i}(x)y_{2i}(z) - \sum_{i=1}^{m} (4i-1)y_{2i-1}(x)y_{2i-1}(z) \\ &= \sum_{r=0}^{2m} (-1)^{2m-r}(2r+1)y_r(x)y_r(z). \end{aligned}$$

Again using $n = 2m+1$ and by repeated application we derive from (13)

$$(15) \qquad \begin{aligned} \frac{\sigma_{2m+1}(x, z)}{x-z} &= \sum_{i=1}^{m+1} (4i-1)y_{2i-1}(x)y_{2i-1}(z) - \sum_{i=0}^{m} (4i+1)y_{2i}(x)y_{2i}(z) \\ &= \sum_{r=0}^{2m+1} (-1)^{2m+1-r}(2r+1)y_r(x)y_r(z). \end{aligned}$$

Thus we note that (14) and (15) can be comprised into a single formula

$$(16) \qquad \frac{\sigma_n(x, z)}{x-z} = \sum_{r=0}^{n} (-1)^{n-r}(2r+1)y_r(x)y_r(z).$$

Now let $z \to x$, then (16) yields us

$$y'_{n+1}(x)y_n(x) - y'_n(x)y_{n+1}(x) = \sum_{r=0}^{n} (-1)^{n-r}(2r+1)y_r^2(x). \tag{17}$$

In this connection we mention that BANERJEE [1, p. 85] proved that

$$\sum_{n=0}^{p} (2n+1)y_n^2(x) = \frac{1}{x} \{ y_{p+1}(x)y_p(x) - y_0(x)y_{-1}(x) \}. \tag{18}$$

Here it is worth mentioning that (18) was already proved by C. K. CHATTERJEE [8] and by W. A. AL-SALAM [9]. It should be remembered that one can extend the following definition

$$x^2 \frac{d^2y}{dx^2} + 2(x+1)\frac{dy}{dx} = n(n+1)y$$

of the BESSEL polynomial $y_n(x)$ to negative subscripts by defining $y_{-n}(x)$ to be $y_{n-1}(x)$. Thus if we put $y_{-1}(x) = y_0(x)$ in (18) we actually obtain CHATTERJEE'S or AL-SALAM'S result.

Next returning to (14) we derive

$$\begin{aligned} &2x^2[y'_{2m+1}(x)y_{2m}(x) - y'_{2m}(x)y_{2m+1}(x)] \\ &= 2x^2 \sum_{i=0}^{m} (4i+1)y^2_{2i}(x) - 2x^2 \sum_{i=1}^{m} (4i-1)y^2_{2i-1}(x) \end{aligned} \tag{19}$$

Again we obtain from (3)

$$\Delta_{2m} = x + 2x^2 \sum_{i=0}^{m} (4i+1)y^2_{2i}(x) \tag{20}$$

and

$$\Delta_{2m-1} = x + 2x^2 \sum_{i=1}^{m} (4i-1)y^2_{2i-1}(x) \tag{21}$$

where

$$\Delta_n \equiv y_n(x)y_{n+2}(x) - y^2_{n+1}(x).$$

It therefore follows from (19), (20) and (21) that

$$\Delta_{2m} - \Delta_{2m-1} = 2x^2 \Lambda_{2m} \tag{22}$$

where

$$\Lambda_{2m} = y'_{2m+1}(x)y_{2m}(x) - y'_{2m}(x)y_{2m+1}(x).$$

From (14) we similarly derive

$$2x^2[y'_{2m+2}(x)y_{2m+1}(x) - y'_{2m+1}(x)y_{2m+2}(x)]$$

$$= 2x^2 \sum_{i=1}^{m+1} (4i-1)y^2_{2i-1}(x) - \sum_{i=0}^{m} (4i+1)y^2_{2i}(x).$$

Thus we obtain

(23) $$\Delta_{2m+1} - \Delta_{2m} = 2x^2 \Lambda_{2m+1}.$$

Now from (22) and (23) we derive a single formula

(24) $$\Delta_n - \Delta_{n-1} = 2x^2 \Lambda_n.$$

Lastly by repeated applicaton of (24) we get

(25) $$\Delta_n = x + 2x^2 \sum_{r=0}^{n} \Lambda_r,$$

where

$$\Delta_n = y_n(x)y_{n+2}(x) - y^2_{n+1}(x)$$

and

$$\Lambda_r = y'_{r+1}(x)y_r(x) - y'_r(x)y_{r+1}(x).$$

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