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A characterization of sets of cardinal $\leq C$

by HELEN F. CULLEN (University of Massachusetts) (*)

Summary. - *First a property for sets called set-separability is defined. It is then shown with the aid of some results of Marczewski that a set T is set-separable if and only if T has cardinal $\leq C$.*

INTRODUCTION. Marczewski ([1]), in proving separability for certain product spaces, uses the following lemma: the product space N^T, where N is the set of natural numbers with the discrete topology, is separable if T is any subset of the real numbers. In the proof of this lemma, a property related to the density of the rationals in the reals is used. It is the purpose of this note first to abstract this property to obtain a set-theoretic property and then to use it to characterize sets of cardinal $\leq C$.

1. SET-SEPARABITITY. A finite partition of a set, T, will, as usual, denote a finite collection of disjoint subsets of T whose union contains T.

Definition 1. A partition, P, of a set, T, is said to separate the subset M of T if and only if distinct points in M lie in distinct sets in P,

Definition 2. A set T is said to be set-separable if and only if there exists a countable collection, $\mathcal{S}$, of finite partitions such that each finite subset of T is separated by a partition in $\mathcal{S}$.

LEMMA 1. *If a set A is similar to a set B and if A is set-separable then B is set-separable.*

The proof is straighforward and will be omitted.

LEMMA 2. *Every subset T of real numbers is set-separable.*

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Proof. For each finite subset $\{r_1, r_2, \dots, r_n\}$ of rational numbers, such that $r_1 < r_2 < \dots < r_n$, define the partition

$$A_1 = \{x \mid x < r_1\},\ A_2 = \{x \mid r_1 \leq x < r_2\},\ \dots,\ A_n = \{x \mid r_n \leq x\}$$

of T. Since the set of all finite subsets of the set of rationals is countable, the set, $\mathcal{S}$, of all such partitions is countable. From the density of the rationals in the reals, it follows immediately that each finite set of elements in T is separated by some partition in $\mathcal{S}$ and hence T is set-separable.

COROLLARY 1. *Every set of cardinal $\leq C$ is set-separable.*

Proof. From the definition of $\leq$ and from Lemma 1, the corollary follows.

2. SET-SEPARABILITY AND TOPOLOGICAL SEPARABILITY.

THEOREM 1. *If T is any set-separable set, then N^T is a separable space.*

Proof. Let $\mathcal{S}$ denote a countable collection of finite partitions which separate the finite sets of T. Let $D = \{f \text{ in } N^T \mid f \text{ is constant on each of the subsets } A_1, A_2, \dots, A_n \text{ of some partition in } \mathcal{S}\}$. Each function in D is identified by a finite partition $\{A_1, A_2, \dots, A_n\}$ and an n-tuple $(f(A_1), f(A_2), \dots, f(A_n))$ of natural numbers. Since the set of all n-tuples of natural numbers is countable and since $\mathcal{S}$ is countable, D is countable. Next, let Φ be any element in N^T and let G^* be any basic open set in N^T which contains Φ. $G^* = \prod_{\alpha \varepsilon T} G_\alpha$ where $G_\alpha = N$ except for $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k$, k a natural number. Let $P = \{A_1, A_2, \dots, A_n\}$ be a finite partition in $\mathcal{S}$ which separates $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$. Let $\alpha_1 \varepsilon A_{i_1}, \alpha_2 \varepsilon A_{i_2}, \dots, \alpha_k \varepsilon A_{i_k}$. There exists in D a function f such that $f(A_{i_1}) = \Phi(\alpha_1)$, $f(A_{i_2}) = \Phi(\alpha_2), \dots, f(A_{i_k}) = \Phi(\alpha_k)$. Hence, f is in G^* and D is dense in N^T. Thus N^T is separable.

3. SET-SEPARABILITY AND THE CARDINAL, C. In [1], Marczewski establishes the following: a product space $\prod_{\alpha \varepsilon T} X_\alpha$ where heach X_α has disjoint non-empty open sets is separable if and only if (1) each X_α is separable and (2) the cardinal of T is less than or equal to C.

THEOREM 2. *A set T has cardinal greater than C if and only if T is not set-separable.*

Proof. A. Let T have cardinal greater than C. The space N^T is not, then, separable by Marczewski's theorem. Hence, by Theorem 1, T cannot be set-separable.

B. Let T be not set-separable. By corollary 1 T does not have cardinal $\leq C$. By comparability of cardinals numbers, T has cardinal $> C$.

COROLLARY 2. *A set T has cardinal $\leq C$ if and only if T is set-separable.*

REFERENCES

[1] MARCZEWSKI, E., *Separabilité et Multiplication Cartésienne des Espace Topologiques,* Fund. Math., Vol. 34 (1947), pp. 127-143.