Bollettino
Unione Matematica Italiana

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Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. **19** (1964), n.2, p. 124–126.

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<http://www.bdim.eu/item?id=BUMI_1964_3_19_2_124_0>

Transformations in a three - space.

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Summary. - *The set of 3 x 3 non-singular matrices over the scalar field of real numbers is a group with respect to matrix multiplication. A number of its subgroups have interesting geometric significance.*

1. – *Purposes of the problem:* The purposes of this paper are to analyze certain eigenvectors and their corresponding eigenvalues of 3 x 3 matrices, which transform in a rectangular coordinatized E_3 the vector $v = (a_1 . a_2, a_3)$ leaving its length invariant, and to survey the group properties of these matrices.

2. – *Transformations:* Consider in E_3 the vector v.

Let π be the set of transformations which permute the components of v allowing for change in sign. Then π leaves the origin 0 as a fixed point.

If we define an *augmented permutation matrix* as a monomial matrix [1] having non-zero entries 1 and -1, then π will be the set of corresponding augmented permutation transformations. The $2^3 \cdot 3! = 48$ transformations of π are:

Identity	I	I
Inversion w.r.t. 0	$-I$	$(-I)^2 = I$
Reflections w.r.t. lines through 0	$A_i \quad i = 1, ..., 9$	$A_i^2 = I$
Reflections w.r.t. planes through 0	$B_i \quad i = 1, ..., 9$	$B_i^2 = I$
90° rotations about axes	$C_i \quad i = 1, ..., 6$	$C_i^4 = I$
Reflection-rotations w.r.t. planes $X_j = 0$ and X_j axes	$D_i = B_j C_i = 1, ..., 6$ $j \equiv i \pmod 3$	$D_i^4 = I$
$(a_j)T_i = [\pm a_{\Phi(j)}]$, Φ permutes subscripts $j = 1, 2, 3$	$T_i \quad i = 1, ..., 8$ $i = 9, ..., 16$	$T_i^3 = I$ $T_i^6 = I$

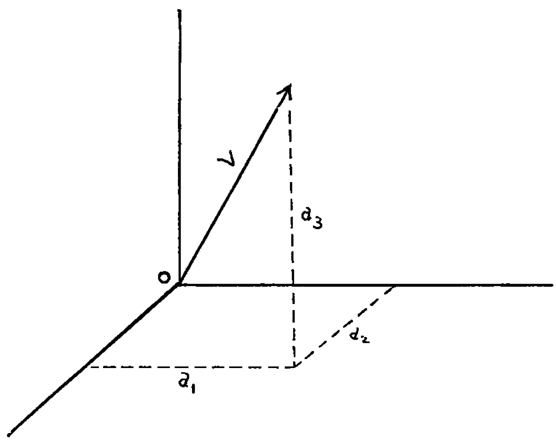
(*) Pervenuta alla Segreteria dell' U. M. I. il 7 dicembre 1963.

3. - *Eigenrays:* A given point (x_i) lies on an eigenray of some augmented permutation matrix A iff $\exists$ a complex scalar $\lambda \ni (x_i') = (\lambda x_i)\, i = 1, 2, 3$ and $(x_i)A = (x_i')$. This is tantamount to solving the secular equation $|A - \lambda I| = 0$ for λ.

Suppose:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$. represent the 90° counterclockwise rotation about the x_1 axis. The eigenvalues of C are found to be 1 and $\pm i$. The corresponding eigenvectors are $(c_1, 0, 0)$ and $(0, c_2, \mp c_2 i)$ where c_1 and c_2 are real scalars. It can then be concluded that the only real eigenray for C is the x_1 axis. Similarly, if

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$ represents the mapping $(x_1, x_2, x_3) \rightarrow (x_2,$



$- x_3, x_1)$, then the eigenrays corresponding to the eigenvalues -1, $\lambda_1 = (1/2) + (\sqrt{3}/2)\, i$, and $\bar{\lambda}_1$ are $(c_1, -c_1, -c_1)$, $(c_1, \lambda_1 c_1\ \bar{\lambda}_1 c_1)$ and $(c_1, \bar{\lambda}_1 c_1 . \lambda_1 c_1)$ respectively: the first of these is the only real eigenray for T.

The secular equation resulting from an augmented permutation matrix yields three eigenvalues. The corresponding real eigenvectors generate a subspace of E_3. In general, a line of reflection has eigenvalue 1 and all orthogonal lines have eigenvalue -1; rays lying in a plane of reflection have eigenvalue 1 and orthogonal rays eigenvalue -1; the axis of a rotation has an eigenvalue 1, all other eigenrays being imaginary; a reflection-rotation mapping has its line of rotation as an eigenray with eigenvalue -1; a tranformation of the form $(a_i) \to (\pm a_{\Phi(i)})$ where one or three of the signs are $+$ has eigenvalue 1 and where none or two of the signs are $+$ has eigenvalue -1.

4. - *Group Structure*: The set M of $48 = 2^4 \cdot 3$ augmented permutation matrices is a group generated by C and T above with defining relations.

$C^4 = T^6 = I$	$C^3 T^4 = T^2 C$	$CT^2 = T^4 C^2$
$CT^5 = TC^3$	$T^3 = -I$	$C^2 T = T^5 C$

Let $\| m_{ij} \| \in M$. Denote the k^{th} Sylow p-subgroup of M by $S_k(p)$. Then $M \supset S_k(2)$ and $S_l(3)$, $k = 1, 2, 3$ and $l = 1, \dots, 4$. Each of the 16 $m_{ij} \in S_k(2)$ for fixed k_0 has an entry $m_{ii} \neq 0$. $\bigcap_{k=1}^{3} S_k(2)$ is a self-conjugate subgroup of order $8 \ni \forall i$, $m_{ii} \neq 0$. $\forall l$, $S_l(3) \subset Z_l(3)$ where $Z(n)$ is a cyctic group of order n. $\bigcup_{=1}^{4} S_l(3) = [T_i : i = 1, \dots, 8]$. $\bigcup_{l=1}^{4} Z_l(6) = [T_i : i = 9, \dots, 16]$. $\exists$ 6 $Z_l(4)s$.

Any two matrices of order 3 and 4 in M generate a subgroup of order 24. An interesting exploration would be the extension of this topic to E_n where the number of augmented permutation matrices is $2^n \cdot n!$

REFERENCES

[1] BIRKHOFF and MACLANE *A Survey of Modern Algebra,* New York, 1957.

[2] V. ROJANSKY, *Introductory Quantum Mechanics*' New York, 1946.