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On the computational solution of two-point boundary-value problems.

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Summary. - *Two-point boundary-value problems for second-order systems of linear differential equations are usually solved by a process involving the inversion of a certain matrix. If the system is too large, it may be difficult to compute this inverse to a high degree of accuracy.*

The purpose of this paper is to discuss a method of overcoming this difficulty.

1. Introduction.

Consider (as in [1]) the n-dimensional vector differential equation

$$x'' + A(t)x = 0 \tag{1.1}$$

where the solution is subject to the boundary conditions

$$x(0) = c, \qquad x(1) = d. \tag{1.2}$$

The problem is generally solved as follows. Let X_1 and X_2 denote the matrix solutions of

$$X'' + A(t)X = 0 \tag{1.3}$$

satisfying the initial conditions

$$\begin{aligned} X_1(0) &= I, \qquad X_1'(0) = 0, \\ X_2(0) &= 0, \qquad X_2'(0) = I. \end{aligned} \tag{1.4}$$

If g represents the (unknown) value of $x'(0)$, where $x(t)$ is the solution to the problem, then

$$g = X_2(1)^{-1}[d - X_1(1)c]. \tag{1.5}$$

(*) Pervenuta alla Sergeteria dell'U·M.I il 16 novembre 1963.

If $X_2(1)$ is singular, then there may be *m* any solutions, or none, and (1.5), of course, makes no sense.

If n is large, it may be difficult to compute $X_2^{-1}(1)$ to a high degree of accuracy. The purpose of this paper is to discuss a method of overcoming this difficulty.

2. An iterative technique.

Let $X_2^*(1)$ be some approximation to $X_2^{-1}(1)$. Define

$$g_1 = X_2^*(1)[d - X_2(1)c], \tag{2.1}$$

$$g_n = X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1}.$$

Then we have the following theorem:

THEOREM *If the spectral radius of* $\mathrm{I} - \mathrm{X}_2^*(1)\mathrm{X}_2(1)$ *is less than one, then the sequence* $[\mathrm{g_n}]$ *defined by* (2.1) *converges to* g, *the unique solution of* (1.5).

PROOF. - First note that if $I - X_2^*(1)X_2(1)$ has spectral radius less than one, then $X_2^*(1)X_2(1)$ must be nonsingular. Thus $X_2^*(1)$ and $X_2(1)$ are nonsingular, which means that (1.5) has a unique solution. If g is the unique solution of (1.5), then

$$\begin{aligned} g_n - g &= X_2^*(1)[d - X_1(1)c - X_2(1)\, g_{n-1}] + g_{n-1} - g = \\ &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] - \\ &- X_2^*(1)[d - X_1(1)c - X_2(1)g] + g_{n-1} - g = \\ &= (I - X_2^*(1)X_2(1))(g_{n-1} - g). \end{aligned} \tag{2.2}$$

If the spectral radius of $I - X_2^*(1)X_2(1)$ is less than one, this shows that $[g_n - g]$ goes to zero as n goes to infinity, and this concludes the proof. This theorem may be viewed as an application of a method of matrix inversion like that of BODEWIG and HOTELLING (see [3], [4] for additional references).

CORLLARY. - *If* $\mathrm{A(t)} = \mathrm{B}^2$, *a constant positive-definite matrix, then taking* $\mathrm{X}_2^*(1) = \mathrm{X}_2(1)$ *makes* $[\mathrm{g_n}]$ *converge to the solution.*

PROOF. Since $X_2(1) = B^{-1}\sin B$, it follows that the eigenvalues of $X_2(1)$ all have absolute value less than one, and thus all the eigenvalues of $X_2^2(1)$ are between zero and one.

COROLLARY. - *If each element of* $I - X_2^*(1)X_2(1)$ *is less in absolute value than* $1/n$, *then* $[g_n]$ *converges to the solution.*

COROLLARY. - *If* $A(t) = -B^2$, *where* B *is a matrix with only real eigenvalues each of which is greater than zero then taking* $X_2^*(1) = 2Be^{-B}$ *makes* $[g_n]$ *converge to the solution.*

PROOF. - $X_2(t = B^{-1}\left(\frac{e^{Bt} - e^{-Bt}}{2}\right)$, whence $X_2^*(1)X_2(1)$ equals $I - e^{-2B}$.

COROLLARY. - *If* $Y_1(t)$, $Y_2(t)$ *are solutions to* $Y'' + A(1-t)Y = 0$ *satisfying initial conditions like* (1.4), *then taking* $X_2^*(1) = Y_1'(1)$ *will make* $[g_n]$ *converge to the solution if* $Y_2'(1)X_2'(1)$ *has spectral radius less than one.*

PROOF. - $Y_2'(1)X_2'(1) = I - Y_1'(1)X_2(1)$.

COROLLARY. - *If* $X_2^*(1) = dA$, *where* A *is the transpose of* $X_2(1)$ *and* d *is a positive constant chosen to be less than twice the reciprocal of the sum of the absolute values of each row of* $AX_2(1)$, *then* $[g_n]$ *converges to the solution.*

Note that this last corollary is not apt to be computationally useful, however, since if $X_2(1)$ has some very small eigenvalues (and thus is hard to invert) under the above procedure $I - X_2^*(1)X_2(1)$ will have spectral radius very close to one, so that convergence will be slow.

REFERENCES

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