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U. A. Sastry

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Torsion of hollow prismatic cylinders.

By U. A. SASTRY (India) (*)

Summary. - *In this paper (a) the torsion of a hollow cylinder whose outer cross-section is a circle and inner cross-section a square with rounded corners and (b) the torsion of a hollow prismatic been bounded externally by a quartic curve and internally by a circle have benn studied by using Schwarz's Alternating Method.*

Introduction.

D. I. SHERMAN has developed complex variable method to obtain the solution of some doubly connected regions in the plane theory of elasticity. D. I. SHERMAN and M. Z. NARODETSKI have obtained the solution of torsion of a hollow cylinder bounded externally by a square with rounded corners and internally by a circle by reducing the problem to the solution of the integral equation of Fredholm. The have also investigated a number of important torsion problems.

Mathematical Formulation of the Problem.

To solve the torsion problem for a beam of doubly connected cross-section we have to find a function $\psi(x, y)$ which is harmonic in the region occupied by the material and satyisfing the condition

$$\psi = (x^2 + y^2) / 2 + c_i\,, \qquad (i = 1,\ 2) \tag{1.1}$$

on the boundaries L_1 and L_2.

One constant can be choosen arbitrarily and the other is to be determined so that Re $F(z)$ is single valued throughout the region. Let

$$F(z) = \Phi + i\psi\,, \tag{1.2}$$

$$F(z) = i\Phi(z)\,, \tag{1.3}$$

where $F(z)$ is the usual complex torsion function. Now the boundary conditions (1.1) can be written with the aid of (1 3) in

(*) Pervenuta alla Segreteria dell'U.M.I. il 19 ottobre 1963.

the from

$$\Phi(t) + \overline{\Phi(t)} = t\,\bar{t} + c_i \qquad (i = 1,\ 2) \tag{1.4}$$

where t is a point in the z – plane.

The stress components and the torsional rigidity are given by

$$T_x - iT_y = i\mu\tau[\Phi'(z) - \bar{z}], \tag{1.5}$$

$$D^* = \mu(I + D_0), \tag{1.6}$$

where

$$4iI = \int z\,\bar{z}^2 dz, \tag{1.7}$$

$$4iD_0 = \int [\Phi(z) - \overline{\Phi(z)}] d(z\bar{z}), \tag{1.8}$$

$z = x + iy$, $\bar{z} = x - iy$ and integrals are to be evaluated over the doubly connected region.

Schwarz's Alternating Method.

The region R_{12} occupied by the material can be considered as the intersection of the infinite region R_1 bounded by the inner cross-section L_1 with finite region R_2 interior to the outer cross-section L_2.

With the usual notation we write the boundary condition in the form

$$L(\Phi) = \Phi(t) + \overline{\Phi(t)} = f(t). \tag{1.9}$$

To obtain the first approximation $\Phi^{(1)}$ we have to determine a function in the infinite region exterior to L_1 such that

$$L(\Phi^{(1)})\,|_{L_1} = f\,|. \tag{1.10}$$

Tho find the second approximatio $\Phi^{(2)}$ we consider the solution in the finite region interior to L_2 subject to the condition

$$L(\Phi^{(2)})\,|_{L_2} = f\,|_{L_2} - L(\Phi^{(1)})\,|_{L_2}. \tag{1.11}$$

For the third approximation we have to determine the function $\Phi^{(3)}$ satisfying

$$L(\Phi^{(3)}\,|_{L_1} = f\,|_{L_1} - L(\Phi^{(2)})\,|_{L_1}, \tag{1.12}$$

and so on.

(a) – *Cross-section bounded externally by a circle and internally by a square with rounded corners.*

The boundary conditions for the torsion problem reduces to

$$\Phi(t) + \overline{\Phi(t)} = R^2 + c_2, \text{ on the circle } L_2 \tag{2.1}$$

$$= t\,\bar{t} \mp c_1 \text{ on the square } L_1 \tag{2.2}$$

we can take $c_2 = -R^2$ and the constant c_1 is to be determined.

Conformal Transformation.

The function

$$z = a(\zeta + m/\zeta^3) \quad a > 0, \quad |m| < \frac{1}{3} \tag{2.3}$$

maps conformally the region exterior to the square L_1 on to the region exterior to the unit circle in the ζ — plane. To find ζ in terms of z let us assume

$$\left\{ \begin{array}{l} \zeta = dz + \sum\limits_1^{\infty} c_n / z^n, \ 1/\zeta = \sum\limits_1^{\infty} b_n / z^n \\ 1/\zeta^3 = \sum\limits_1^{\infty} E_n / z^n, \ 1/\zeta^4 = \sum\limits_1^{\infty} H_{4n} / z^{4n} \end{array} \right. \tag{2.4}$$

where

$$E_n = \sum_{\gamma=1}^{\infty} G_{n-r} b_r, \quad G_n \sum_{\gamma=1}^{\infty} b_{n-r} b_r, \quad H_{4n} = \sum_{\gamma=2}^{\infty} G_{4n-r} G_r .$$

Using (1.4) in (1.3) and comparing the like powers of z we find

$$\left\{ \begin{array}{l} d = 1/a, \ c_3 = -ma^3, \ c_7 = -3m^2a^7, \ c_{11} = -15m^3a^{11}, \\ c_{15} = -91m^4a^{15}, \ c_{19} = -712m^5a^{19}, \ldots \\ b^1 = a, \ b_5 = ma^5, \ b_9 = 4m^2a^9, \ b_{13} = 22m^3a^{13}, \ b_{17} = 140m^4a^{17}, \ldots \\ H_4 = a^4, \ H_8 = 4ma^8, \ H_{12} = 22m^2a^{12}, \ H_{16} = 140m^3a^{16}, \ldots \end{array} \right. \tag{2.5}$$

First approximation.

From (2.2) and (1,10) we find

$$\Phi^{(1)}(t) + \overline{\Phi^{(1)}(t)} = t\,\bar{t} + c_1 \tag{2.6}$$

(2.7) since $t = a(\sigma + m/\sigma^3)$, $\bar{t} = a(1/\sigma + m\sigma^3)$

σ is a point on the unit circle in the ζ — plane the above equation becomes

$$\Phi_1^{(1)}(\sigma) + \overline{\Phi_1^{(1)}(\sigma)} = a^2(1 + m^2 + m/\sigma^4 + m\sigma^4) + c_1 . \tag{2.8}$$

Multiplying (2.8) by $d\sigma/2\pi i\ (\sigma - \zeta)$ where ζ is a point outside the unit circle and integrating we find

$$\Phi_1^{(1)}(\zeta) = a^2 m/\zeta^4 . \tag{2.9}$$

From (2.4) and (2.9) we have

$$\Phi^{(1)}(z) = a^2 m \sum_1^{\infty} H_{4n}/z^{4n} . \tag{2.10}$$

From (2.8) and (2.9) it follows that

$$c_1 = - a^2(1 + m^2) . \tag{2.11}$$

Second approximation.

Le $\Phi^{(2)}$ be the second approximation to be determined interior to the circle L_2 subject to the condition

$$L(\Phi^{(2)})\,|_{L_2} = f\,|_{L_2} - L(\Phi^{(1)})\,|_{L_2} . \tag{2.12}$$

The above equation becomes

$$\Phi^{(2)}(t) + \overline{\Phi^{(2)}}(t) = - a^2 m \sum_1^{\infty} H_{4n}(1/t^{4n} + t^{4n}/R^{8n}) , \tag{2.13}$$

since $t\,\bar{t} = R^2$.

Multiplying the above equation by $dt/2\pi i(t - z)$ where z is a point inside the circle L_2 we find after integration

$$\Phi^{(2)}(z) = - a^2 m \sum_1^{\infty} H_{4n} z^{4n}/R^{8n} . \tag{2.14}$$

Third approximation.

Let $\Phi^{(3)}$ be the third approximation to be determined in the infinite region exterior to the square L_1 subject to

$$L(\Phi^{(3)})\,|_{L_1} = f\,|_{L_1} - L(\Phi^{(2)})\,|_{L_1} . \tag{2.15}$$

Using (2.14), (2.7) in the above equation and then multiplying the resulting equation by $d\sigma / 2\pi i(\sigma - \zeta)$ where ζ is a point outside the unit circle and integrating we find

$$(2.16) \quad \begin{cases} \Phi_1^{(3)}(\xi) = a^2 m / \zeta^4 + a^2 m \sum_1^{\infty} H_{4n} a^{4n} R^{-8n} \cdot \\ \left\{ \sum_{p=n+1}^{4n} \binom{4n}{p} m^p \zeta^{4(n-p)} + \sum_{p\ 0}^{n-1} \binom{4n}{p} m^p \zeta^{4(p-n)} \right\}. \end{cases}$$

For the first two approximations we find from (210) and (2.14)

$$(2.17) \qquad \Phi(z) = a^2 m \sum_1^{\infty} H_{4n} (1 / z^{4n} - z^{4n} / R^{8n}).$$

It is easy to prove that

$$(2.18) \qquad I = \pi a^4 (1 - 4m^2 - 3m^4) / 2,$$

$$(2.19) \; D_0 = 4\pi a^4 m^2 \left[- H \sum_1^{\infty} H_{4n} a^{4n} R^{-8n} \left\{ \binom{4n}{n+1} m^{n+1} - \binom{4n}{n-1} m^{n-1} \right\} \right]$$

(b) - *Cross-section bounded externally by a Quartic curve and internally by a circle.*

The boundary conditions for the torsion problem are

$$(3.1) \qquad \Phi(t) + \overline{\Phi(t)} = t\,\bar{t} \quad \text{on the quartic curve } L_2$$

$$(3.2) \qquad = R^2 + c_1 \quad \text{on te circle } L_1 .$$

The constant on the outer boundary is taken to be zero. c_1 is a constant to be determined.

First approximation.

Let $\Phi^{(1)}$ be the first approximation to be determined in the finite region interior to L_2 subject to the condition

$$(3.3) \qquad L(\Phi^{(1)}) |_{L_2} = f |_{L_2} .$$

The above equation becomes

$$(3.4) \qquad \Phi^{(1)}(t) + \overline{\Phi^{(1)}(t)} = t\,\bar{t} .$$

Conformal Transformation.

The function

$$(3.5)\qquad z = b\zeta/(1+p\zeta+m\zeta^2),\ b>0,\ |p|<2,\ |m|<|,$$

maps conformally the region interior to the quartic curve on to the unit circle in the ζ – plane. Since we have to express ζ in terms of z let us take

$$(3.6)\qquad \zeta = \sum_0^\infty a_n z^n,\quad \zeta^2 = \sum_0^\infty A_n z^n,\quad A_n = \sum_{\gamma=0}^\infty a_{n-r} a_r .$$

Using (3.6) in (3.5) and comparing like powers of z we can find the constants a_n.

$$(3.7)\begin{cases} a_0 = 0,\ a_1 = 1/b,\ a_2 = p/b^2,\ a_3 = (p^2+m)/b_3 \\ a_4 = (p^3+3pm)/b^4,\ a_5 = (p^4+6p^2m+2m^2)/b^5 \\ a_6 = (p^5+10p^3m+10pm^2)/b^6,\ a^7 = (p^6+15p^4m+30p^2m^2+5m^3)/b^7 \text{ etc.}\end{cases}$$

Using (3.5) in (3.4) we find

$$(3.8)\qquad \Phi_1^{(1)}(\sigma) + \overline{\Phi_1^{(1)}(\sigma)} = b^2\sigma^2/(1+p\sigma+m\sigma^2)(\sigma^2 \mp p\sigma + m)$$

where σ is point on the unit circle and $t = b\sigma/(1+p\sigma+m\sigma^2)$ s

Multiplying (3 8) by $d\sigma/2\pi i(\sigma-\zeta)$ where ζ is a point inside the unit circle and integrating we find

$$(3.9)\qquad \Phi_1^{(1)}(\zeta) = b^2k\left[\frac{(1+m)-(1-m)p\zeta - m(1+m)\zeta^2}{1+p\zeta+m\zeta^2}\right].$$

Using (3.5), (3.6) in (3.9) we find

$$(3.10)\qquad \Phi^{(1)}(z) = bk[(1+m)b - 2pz - 2m(1+)\sum_0^\infty a_n z^{n+1}].$$

Second approximation.

Let $\Phi^{(2)}$ be the second approximation to be determined in the infinite region exterior to the circle L_1 subject to

$$(3.11)\qquad L(\Phi^{(2)})\,|_{L_1} = f|_{L_1} - L(\Phi^{(1)})\,|_{L_1}.$$

Fron (3.10) and (3.11) we find

$$\Phi^{(2)}(t) + \Phi^{(2)}(t) = (R^2 + c_1) - \tag{3.12}$$
$$-bk[2(1+m)b - 2p(t + R^2/t) - 2m(1+m)\sum_0^\infty a_n(t^{n+1} + R^{2n+2}/t^{n+1})$$

Multiplying (3.12) by $dt/2\pi i(t - z)$ where z is a point outside the circle L_1, we find after integration

$$\Phi^{(2}_{(}(z) = bk[2pR^2/z + 2m(1 + m)\sum_0^\infty a_n R^{2n+2}/z^{n+1}]\,. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$R^2 + c_1 = 2(1 + m)kb^2\,. \tag{3.14}$$

Also

$$k = 1/2\{(1 + m)^2 - p^2\}\,. \qquad (1 - m) \tag{3.15}$$

From (3.10) and (3.13) we find

$$\Phi(z) = bk[(1+m)b - 2p(z - R^2/z) - 2m(1+m)\sum_0^\infty a_n(z^{n+1} - R^{2n+2}z^{n+1})]. \tag{3.16}$$

Special case.

By writing $p = 0$ we get the result for a cross-section bounded externally by a BOOTHS lemniscate section and internally by a circle. The formula (3.5) becomes

$$z = b\zeta/(1 + m\zeta^2), \quad b > 0, \quad m < 1 \tag{3.17}$$

from which we have

$$2mz\zeta = b - (b^2 - 4mz^2)^{\frac{1}{2}}\,. \tag{3.19}$$

We can easily calculate the torsional rigidity.

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