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The inverse Laplace transform of the product $\exp[-\frac{1}{2}a^{1/n}p^{1/n}]K_{n\nu}[\frac{1}{2}a^{1/n}p^{1/n}]$, $n = 2, 3, 4, \ldots$.

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Zanichelli

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The inverse Laplace transform of the Product

$$\exp\left[-\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right]K_{n\nu}\left[\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right], \qquad n=2,\ 3,\ 4, \ldots$$

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Summary. - *The inverse Laplace transform of*

$$\exp\left[-\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right]K_{n\nu}\left[\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right]$$

where $n=2,\ 3,\ 4,\ldots$ *is obtained by formula* (11) *below.*

1. Introductory.

Nothing is known in the literature [1] of the inverse LAPLACE transform of the function

$$\exp\left[-\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right]K_{n\nu}\left[\frac{1}{2}a^{\frac{1}{n}}p^{\frac{1}{n}}\right], \tag{1}$$

where n is any positive integer greater than one and $K_\nu(z)$ is the modified BESSEL function of the second kind. In this paper

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it will be shown that the original function, whose LAPLACE transform is the function (1), is the function

$$(2)\quad 2^{-\frac{1}{2}-\frac{1}{2}n}\pi^{-\frac{1}{2}n}p\sum_{i,-i}\frac{1}{i}E\left[\Delta(n;\ n\nu),\ \Delta(n;\ -n\nu):\Delta\left(n;\ \frac{1}{2}\right):\frac{ae^{i\pi}}{n^n t}\right], \dots$$

where p, as usual, is complex, $R(p)>0$, $R(a)>0$, n is any positive integer greater than one and the symbol $\sum_{i,-i}$ means that in the expression following it i is to be reelaced by $-i$ and the two expressions are to be added. Also the symbol $\Delta(n;\ \alpha)$ represents the set of parameters $\frac{\alpha}{n}, \frac{\alpha+1}{n}, \dots, \frac{\alpha+n-1}{n}$.

The function appearing in (2) is MACROBERT's E-function whose definitions and properties are to be found in [2], pp. 348-358.

The following formulae are required in the proof of the LAPLACE transform relationship between the function (1) and (2):

([2], p. 352): $p \leq q$

$$(3)\qquad E(p;\ \alpha_r:q;\ \rho_s:z)=\frac{\Gamma(\alpha_1)\dots\Gamma(\alpha_p)}{\Gamma(\rho_1)\dots\Gamma(\rho_q)}\,{}_pF_q(p;\ \alpha_r;\ q;\ \rho_s:-1/z),$$

([2], p. 353): If $p \geq q+1$

$$(4)\qquad E(p;\alpha_r:q;\rho_s:z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}'\,\Gamma(\alpha_s-\alpha_r)\,\{\prod_{t=1}^{p}\Gamma(\rho_t-\alpha_r)\}\,z^{\alpha_r}\cdot$$
$$\cdot\,{}_{q+1}F_{p-1}\{\alpha_r,\ \alpha_r-\rho_1+1,\dots,\ \alpha_r-\rho_q+1;\ \alpha_r-\alpha_1+1,$$
$$\dots,\ *,\dots,\ \alpha_r-\alpha_p+1;\ (-1)^{p-q}z\}, \dots$$

([2], p. 394, ex 106): If $R(\alpha_{p+1})>0$

$$(5)\qquad \int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1}E(p;\ \alpha_r:q;\ \rho_s:z/\lambda)d\lambda=E(p+1;\ a_r:q;\ \rho_s:z)$$

([2], p. 267):

$$(6)\qquad K_\nu(z)=\frac{\pi}{2\sin\nu\pi}[I_\nu(z)-I_\nu(z)],$$

where $I_\nu(z)$ is given by ([2], p. 347):

$$(7)\qquad I_\nu(z)=\frac{1}{\Gamma(1+\nu)}\left(\frac{1}{2}z\right)^\nu e^{-z}\,{}_1F_1\left(\nu+\frac{1}{2};\ 2\nu+1;\ 2z\right)$$

([2], p. 346):

(8) $$ {}_1F_1(\alpha;\ \rho;\ z) = e^z {}_1F_1(\rho-\alpha;\ \rho;\ -z). $$

([3]. P; 759 and (4) p. 398): If $p > q + 1$ and n is any positive integer

(9) $$ \sum_{i,-i} \frac{1}{i} E[p;\ n\alpha_r,\ 1 : p;\ n\rho_s : ze^{i\pi}] = (2\pi)^{-\frac{1}{2}(n-1)(p-q)} \cdot $$

$$ \cdot\, n^{n(\sum_{r=1}^{p}\alpha_r - \sum_{s=1}^{q}\rho_s) - \frac{1}{2}(p-q)+1} $$

$$ \sum_{i,-i} \frac{1}{i} E\left\{\begin{matrix} p;\ \Delta(n;\ n\alpha_r),\ \Delta(n;\ 1) : (z/n^{p-q})^n e^{i\pi} \\ \frac{1}{n},\ \frac{2}{n}, \ldots, \frac{n-1}{n},\ q;\ \Delta(n;\ n\rho_s) \end{matrix}\right\} = $$

$$ = (2\pi)^{-\frac{1}{2}(n-1)(p-q)} n^{n\sum_{r=1}^{p}\alpha_r - n\sum_{s=1}^{q}\rho_s - \frac{1}{2}p + \frac{1}{2}q + 1} \cdot $$

$$ \cdot \sum_{i,-i} \frac{1}{i} E[p;\ \Delta(n;\ n\alpha_r),\ 1 : q,\ \Delta(n : n\rho_s) : (z/n^{p-q})^n e^{i\pi}], \ldots $$

where the symbols Δ, Σ have the same meaning as in (2).

Also the following relations will be utilized:

(10) $$ \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z. $$

2. The Laplace transform relationship.

We shall now establish the following LAPLACE transfsrm theorem in which the original function is the function of t is the LAPLACE transform parameter whose real part is greater than 0:

(11) $$ \int_0^\infty e^{-pt} \sum_{i,-i} \frac{1}{i} E\left[\Delta(n;\ n\nu),\ \Delta(n;\ -n\nu) : \Delta\left(n;\ \frac{1}{2}\right) : \frac{ae^{i\pi}}{n^n t}\right] dt = $$

$$ = 2^{\frac{1}{2}n+\frac{1}{2}} \pi^{\frac{1}{2}n} p^{-1} \exp\left[-\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right] K_{n\nu}\left[\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right], $$

where n is any positive integer greater than one, $R(a) > 0$ and the symbols Δ, $\sum_{i,-i}$ have the same meanings in (2).

To prove (11) substitute in the left side $t = \lambda/p$ and apply (5) in which the value of α_{p+1} is set equal to one. This yields

$$\int_0^\infty e^{-pt} \sum_{i,-i} \frac{1}{i} E\left[\Delta(n;\ n\nu),\ \Delta(n;\ -n\nu) : \Delta\left(n;\ \frac{1}{2}\right) : \frac{ae^{i\pi}}{n^n t}\right] dt =$$

$$= \frac{1}{p} \sum_{i,-i} \frac{1}{i} E\left[1,\ \Delta(n;\ n\nu),\ \Delta(n;\ -n\nu) : \Delta\left(n;\ \frac{1}{2}\right) : \frac{ape^{i\pi}}{n^n t}\right] =$$

$$= (2\pi)^{\frac{1}{2}(n-1)} p^{-1} \sum_{i,-i} \frac{1}{i} E\left[1,\ n\nu,\ -n\nu : \frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}} e^{i\pi}\right],$$

by (9) with $p = 2$ and $q = 1$ there.

We now express each of the last two E-functions in terms of the value given by (4) and combine the two resulting expressions by factoring out common terms. The terms which result from z^{α_r} in (4) are transformed by means of (10). We also use the definition of the generalized hypergeometric function to cancel factors in the numerator and denominator.

Thus we have

$$\int_0^\infty e^{-pt} \sum_{i,-i} \frac{1}{i} E\left[\Delta(n;\ n\nu),\ \Delta(n;\ -n\nu) : \Delta\left(n;\ \frac{1}{2}\right) : \frac{ae^{i\pi}}{n^n t}\right] dt =$$

$$= (2\pi)^{\frac{1}{2}(n+1)} p^{-1} \sum_{\nu,-\nu} \frac{\Gamma(-2n\nu)}{\Gamma\left(\frac{1}{2} - n\nu\right)} (ap)^{\nu}\, {}_1F_1\left[\frac{1}{2} + n\nu;\ 1 + 2n\nu;\ -a^{\frac{1}{n}} p^{\frac{1}{n}}\right] =$$

$$= (2\pi)^{\frac{1}{2}(n+1)} p^{-1} \cdot \frac{1}{2} \exp\left[-\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right] \cdot$$

$$\cdot \sum_{\nu,-\nu} \Gamma(-n\nu) \left[\frac{a^{\frac{1}{n}} p^{\frac{1}{n}}}{4}\right]^{n\nu} \exp\left[-\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right] {}_1F_1\left[\frac{1}{2} + n\nu;\ 1 + 2n\nu;\ a^{\frac{1}{n}} p^{\frac{1}{n}}\right],$$

by (8)

$$= (2\pi)^{\frac{1}{2}(n+1)} \pi^{-\frac{1}{2}} p^{-1} \exp\left[-\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right] \sum_{\nu,-\nu} \frac{\pi}{2 \sin(-n\nu\pi)} I_{n\nu}\left[\frac{1}{2} a^{\frac{1}{n}} p^{\frac{1}{n}}\right],$$

by (7). The result now follows from (6) and (11) is proved.

As particular cases, we take $n=2$ in (1) and (2). Then the original function whose LAPLACE transform is the function

$$\exp\left[-\frac{1}{2}a^{\frac{1}{2}}p^{\frac{1}{2}}\right]K_{2\nu}\left[\frac{1}{2}a^{\frac{1}{2}}p^{\frac{1}{2}}\right], \tag{12}$$

is the function

$$2^{-\frac{3}{2}}\pi^{-1}p\sum_{i,-i}\frac{1}{i}E\left(\nu,\ \nu+\frac{1}{2},\ -\nu,\ -\nu+\frac{1}{2}:\frac{1}{4},\ \frac{3}{4}\frac{ae^{i\pi}}{4t}\right). \tag{13}$$

Also when $n=3$, then the original function whose LAPLACE transform is

$$\exp\left[-\frac{1}{2}a^{\frac{1}{3}}p^{\frac{1}{3}}\right]K_{3\nu}\left[\frac{1}{2}a^{\frac{1}{3}}p^{\frac{1}{3}}\right], \dots \tag{14}$$

is the function

$$\frac{p}{4}\pi^{-\frac{3}{2}}\sum_{,-i}\frac{1}{i}E\left(\nu,\ \nu+\frac{1}{3},\ \nu+\frac{2}{3},\ -\nu,\ -\nu+\frac{1}{3},\right. \tag{15}$$

$$\left.-\nu+\frac{2}{3}:\frac{1}{6},\ \frac{1}{2},\ \frac{5}{6}:\frac{ae^{i\pi}}{27t}\right).$$

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