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The inverse Laplace transform of the product $\exp\left[-\frac{1}{2}a^{1/n}p^{1/n}\right]K_{\nu\nu}\left[\frac{1}{2}a^{1/n}p^{1/n}\right]$, $n = 2, 3, 4, \ldots$


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<http://www.bdime.eu/item?id=BUMI_1964_3_19_1_26_0>
The inverse Laplace transform of the Product
\[ \exp \left[ -\frac{1}{2} a^n p^n \right] K_{\nu} \left[ \frac{1}{2} a^n p^n \right], \quad n = 2, 3, 4, \ldots \]

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Summary. - The inverse Laplace transform of
\[ \exp \left[ -\frac{1}{2} a^n p^n \right] K_{\nu} \left[ \frac{1}{2} a^n p^n \right] \]
where \( n = 2, 3, 4, \ldots \) is obtained by formula (11) below.

1. Introductory.

Nothing is known in the literature [1] of the inverse Laplace transform of the function

\[ \exp \left[ -\frac{1}{2} a^n p^n \right] K_{\nu} \left[ \frac{1}{2} a^n p^n \right], \]

where \( n \) is any positive integer greater than one and \( K_{\nu}(z) \) is the modified Bessel function of the second kind. In this paper

it will be shown that the original function, whose Laplace transform is the function (1), is the function

\[ 2^{-\frac{1}{2}} \pi^{-\frac{n}{2}} \exp \left[ \frac{1}{2} \right] \left[ \Delta(n; -nv), \Delta(n; -nv) \right] \left( \frac{n}{2} \right)^{\frac{\alpha-\pi}{2\pi n}} \]

where \( p, \) as usual, is complex, \( R(p) > 0, R(\alpha) > 0, \) \( n \) is any positive integer greater than one and the symbol \( \Sigma \) means that in the expression following it \( i \) is to be replaced by \( -i \) and the two expressions are to be added. Also the symbol \( \Delta(n; \alpha) \) represents the set of parameters \( \frac{\alpha}{n}, \frac{\alpha+1}{n}, \ldots, \frac{\alpha+n-1}{n} \).

The function appearing in (2) is MacRobert's \( E \)-function whose definitions and properties are to be found in [2], pp. 348-358.

The following formulae are required in the proof of the Laplace transform relationship between the function (1) and (2):

([2], p. 352): \( p \leq q \)

(3) \[ E(p; \alpha_r; q; \rho_r; z) = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_p)}{\Gamma(\rho_1) \cdots \Gamma(\rho_q)} E_q(p; \alpha_r; q; \rho_r; -1/z), \]

([2], p. 353): If \( p \geq q + 1 \)

(4) \[ E(p; \alpha_r; q; \rho_r; z) = \sum_{r=1}^{q+1} \left( \prod_{s=1}^{p} \Gamma(\alpha_s - \alpha_r) \right) \left( \prod_{t=1}^{p} \Gamma(\rho_t - \alpha_r) \right) s^\alpha_r \cdot \]

\[ q+1 \text{F}_{p-1}(\alpha_r, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1; \alpha_r - \alpha_1 + 1, \ldots, \alpha_r - \alpha_p + 1; (-1)^{p-q} z), \ldots \]

([2], p. 394, ex 106): If \( R(\alpha_{p+1}) > 0 \)

(5) \[ \int_0^\infty \exp^{-\lambda \alpha+1-1} E(p; \alpha_r; q; \rho_r; z/\lambda) d\lambda = E(p+1; \alpha_r; q; \rho_r; z) \]

([2], p. 267):

(6) \[ K_v(z) = \frac{\pi}{2 \sin \pi v} [I_v(z) - I_v(z)], \]

where \( I_v(z) \) is given by ([2], p. 347):

(7) \[ I_v(z) = \frac{1}{\Gamma(1 + v)} \left( \frac{1}{2} \right)^v e^{-z} F_1 \left( v + \frac{1}{2}; 2v + 1; 2z \right) \]
\((2\text{. p. } 346)\):

\[ F_1(x; \rho; z) = e^{z}F_1(\rho - x; \rho; -z). \]

\([3\text{. P. } 759 \text{ and } (4) \text{ p. } 398]\): If \(p > q + 1\) and \(n\) is any positive integer

\[
\sum_{i=-1}^{1} \frac{1}{i} E\left[p; n\alpha_r, 1 : p; n\rho_s: z e^{in}\right] = (2\pi)^{-1} \left(1 = \frac{2}{3}(n - 1)(p - q) + 1\right)
\]

\[
\sum_{i=-1}^{1} \frac{1}{i} E\left\{p; \Delta(n; n\alpha_r), \Delta(n; 1) : (\alpha/n^{p-q})^n e^{in}\right\} =
\]

\[
\frac{1}{2}(n - 1)(p - q) + 1 = (2\pi)^{-1} \left(1 = \frac{2}{3}(n - 1)(p - q) + 1\right)
\]

where the symbols \(\Delta, \Sigma\) have the same meaning as in (2).

Also the following relations will be utilized:

\[(10) \quad \Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z.\]

2. The Laplace transform relationship.

We shall now establish the following Laplace transform theorem in which the original function is the function of \(t\) is the Laplace transform parameter whose real part is greater than 0:

\[
\int_{0}^{\infty} e^{-zt} \sum_{i=-1}^{1} \frac{1}{i} E\left[\Delta(n; n\nu), \Delta(n; -n\nu) : \Delta \left(n; \frac{1}{2} : \frac{ae^{in}}{n^t}\right)\right] dt =
\]

\[
= 2^{\frac{1}{2}n + \frac{1}{2}} \pi^{\frac{1}{2}} p^{-1} \exp \left[-\frac{1}{2} \frac{1}{2} a^{n} p^{n}\right] K_{nv} \left[\frac{1}{2} a^{n} p^{n}\right],
\]

where \(n\) is any positive integer greater than one, \(R(a) > 0\) and the symbols \(\Delta, \Sigma\) have the same meanings in (2).
To prove (11) substitute in the left side $t = \lambda/p$ and apply (5) in which the value of $x_{p+1}$ is set equal to one. This yields

$$\int_0^\infty e^{-st} \sum_{i, -i} \frac{1}{i} E \left[ \Delta(n; nv), \Delta(n; -nv) : \Delta \left( n; \frac{1}{2} \right) : ae^{i\pi} \frac{1}{n^t} \right] dt =$$

$$= \frac{1}{p} \sum_{i, -i} \frac{1}{i} E \left[ 1, \Delta(n; nv), \Delta(n; -nv) : \Delta \left( n; \frac{1}{2} \right) : ap^{i\pi} \frac{1}{n^t} \right] =$$

$$= (2\pi)^\frac{1}{2} (n+1) p^{-1} \sum_{i, -i} \frac{1}{i} E \left[ 1, nv, -nv : \frac{1}{2} a^n p^n e^{i\pi} \right],$$

by (9) with $p = 2$ and $q = 1$ there.

We now express each of the last two $E$-functions in terms of the value given by (4) and combine the two resulting expressions by factoring out common terms. The terms which result from $x_r$ in (4) are transformed by means of (10). We also use the definition of the generalized hypergeometric function to cancel factors in the numerator and denominator.

Thus we have

$$\int_0^\infty e^{-st} \sum_{i, -i} \frac{1}{i} E \left[ \Delta(n; nv), \Delta(n; -nv) : \Delta \left( n; \frac{1}{2} \right) : ae^{i\pi} \frac{1}{n^t} \right] dt =$$

$$= (2\pi)^\frac{1}{2} (n+1) p^{-1} \sum_{v, -v} \Gamma(-2nv) \frac{(ap)^\nu F_1 \left[ 1 + nv; 1 + 2nv; - \frac{1}{2} a^n p^n \right]}{\Gamma \left( \frac{1}{2} - nv \right)} =$$

$$= (2\pi)^\frac{1}{2} (n+1) p^{-1} \cdot \frac{1}{2} \exp \left[ - \frac{1}{2} a^n p^n \right].$$

$$\cdot \sum_{v, -v} \Gamma(-nv) \left( \frac{a^n p^n}{2} \right)^a \exp \left[ - \frac{1}{2} a^n p^n \right] F_1 \left[ \frac{1}{2} + nv; 1 + 2nv; \frac{1}{2} a^n p^n \right],$$

by (8)

$$= (2\pi)^\frac{1}{2} (n+1) \pi^{-\frac{1}{4}} p^{-1} \exp \left[ - \frac{1}{2} a^n p^n \right] \sum_{v, -v} \frac{\pi}{2 \sin (-nv\pi)} I_{nv} \left[ \frac{1}{2} a^n p^n \right].$$
by (7). The result now follows from (6) and (11). is proved.

As particular cases, we take $n = 2$ in (1) and (2). Then the original function whose Laplace transform is the function

$$\exp\left[-\frac{1}{2} a^\frac{1}{2} p^\frac{1}{2}\right] K_{2v}\left[\frac{1}{2} a^\frac{1}{2} p^\frac{1}{2}\right],$$

is the function

$$2^{-\frac{3}{2}} \pi^{-1} p \sum_{i, -4} \frac{1}{2} E\left(v, v + \frac{1}{2}, -v, -v + \frac{1}{4}, \frac{3}{4}, \frac{4}{4}, \frac{4}{4}, \frac{4}{4}\right).$$

Also when $n = 3$, then the original function whose Laplace transform is

$$\exp\left[-\frac{1}{2} a^\frac{1}{2} p^\frac{1}{2}\right] K_{3v}\left[\frac{1}{2} a^\frac{1}{2} p^\frac{1}{2}\right], ...$$

is the function

$$\frac{3}{4} \pi^{-\frac{3}{2}} \sum_{i, -4} \frac{1}{2} E\left(v, v + \frac{1}{3}, v + \frac{2}{3}, -v, -v + \frac{1}{3}, \right.$$

$$\left. -v + \frac{2}{3}, \frac{1}{3}, \frac{1}{5}, a e^{\frac{3\pi}{2}}\right).$$

REFERENCES


