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SEZIONE SCIENTIFICA

BREVI NOTE

A note on power series with integral coefficients

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Summary. - *A criterion is obtained for the solvability in integers $a_0 = 1$ $a_2, a_3, \ldots$ of the system*

$$na_n = \sum_{r=1}^{n} b_r a_{n-r} \qquad (n = 1, 2, 3, \ldots),$$

where $b_1, b_2, b_3, \ldots$ are given integers.

Consider the system of equations

$$na_n = \sum_{r=1}^{n} b_r a_{n-r} \qquad (r = 1, 2, 3, \ldots) \tag{1}$$

with $a_0 = 1$. If the a_n are arbitrary integers it is evident the b_n are also integers. However the converse is not always true.

If we put

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{r=1}^{\infty} b_r x^r$$

it is evident that (1) is equivalent to

$$x\frac{f'(x)}{f(x)} = g(x). \tag{2}$$

Thus if $f(x)$ is a power series with integral coefficients, its logarithmic derivative also has integral coefficients but the converse is not always true.

In the second place consider the system

$$b_n = \sum_{r/n} rc_r \qquad (n = 1, 2, 3, \ldots), \tag{3}$$

If the c_n are arbitrary integers it is clear that the b_n are also integers but the converse is not always true.

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We shall now show that the systems (1), (3) are related in the following way.

THEOREM 1. - *Given the integers $b_1, b_2, b_3, \ldots$ The a_n determined by* (1) *are integral if and only if the c_n determined by* (3) *are integral.*

The proof of the theorem depends on the the following device employed by Schur [1]. If $a_0 = 1, a_1, a_2, \ldots$ are a sequence of integers then the numbers $c_1, c_2, c_3, \ldots$ determined by

$$\sum_0^\infty a_n x^n = \prod_1^\infty (1 - x^r)^{c_r} \tag{4}$$

are also integers and conversely. Since

$$a_1 = -c_1,\ a_2 = \binom{c_1}{2} - c_2,\ a_3 = -\binom{c_1}{3} + c_1c_2 - c_3,$$

and so on, the above statement follows at once.

Now by logarithmic differentation of (4) we get

$$\frac{\sum_0^\infty n a_n x^n}{\sum_0^\infty a_n x^n} = \sum_{r=1}^\infty \frac{r c_r x^r}{1 - x^r} = \sum_{n=1}^\infty x^n \sum_{r/n} r c_r .$$

If we put

$$b_n = \sum_{r/n} r c_r ,$$

this becomes

$$\sum_0^\infty n a_n x^n = \sum_0^\infty a_n x^n \sum_1^\infty b_r x^r ,$$

which is equivalent to (1).

The theorem is now immediate. For if the b_n are given integers and the a_n are also integers, then by the remark concerning (4) the c_n are integers. Conversily if the c_n are integers the a_n

A familiar criterion that the c_n in (3) are integral is furnished by

$$\sum_{rs=n} \mu(r) b_s \equiv 0 \pmod{n} \qquad (n = 1, 2, 3. \ldots), \tag{5}$$

where $\mu(n)$ is the Möbius function. Combining this with THEOREM 1 we get

THEOREM 2. – *Given the integers* $b_1, b_2, b_3, \ldots$, *the* a_n *determined by* (1) *are integral if and only if the* b_n *satisfy* (5).

REFERENCE

[1] I. SCHUR, Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung, Compositio Math., vol. 4(1937), pp. 432-444.