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An integral representation for the product of two generalized Bessel polynomials

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Summary. - *The product of two generalized Bessel polynomials is represented by a double integral.*

1. Introduction.

A few years ago WATSON [1] gave an integral representation for the product $L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)$, where $L_n^{(\alpha)}(x)$ denotes the general LAGUERRE polynomial of degree n. CARLITZ [2] has recently proved the following formula

$$
\begin{aligned}
(1.1)\qquad L_m^{(\alpha)}(x)L_n^{(\beta)}(y) = {} & \frac{2^{\alpha+\beta+m+n}}{\pi^2}\frac{\Gamma(\alpha+m+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+m+n+1)}\cdot \\
& \int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} e^{(m-n)\varphi i+(\alpha-\beta)\theta i}\cos^{m+n}\varphi\cos^{\alpha+\beta}\theta\cdot \\
& \cdot L_{m+n}^{(\alpha+\beta)}\left(\frac{xe^{(\theta-\varphi)i}+ye^{-(\theta-\varphi)i}}{\cos\varphi}\cdot\cos\theta\right)d\varphi d\theta.
\end{aligned}
$$

This formula is therefore a generalization of WATSON's consideration. CARLITZ [2, p. 28] has also proved that

$$
\begin{aligned}
& \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{\Gamma(c+d+1)}{\Gamma(c+1)\Gamma(d+1)}\Phi(a;\,c+1;\,x)\Phi(b;\,d+1;\,y) = \\
(1.2)\qquad & = \frac{2^{c+d}}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} t^{a-1}(1-t)^{b-1}e^{(c-d)\theta i}\cos^{c+d}\theta\cdot \\
& \cdot\Phi(a+b;\,c+d+1;\,2\cos\theta\,(xte^{\theta i}+y(1-t)e^{-\theta i}))d\theta dt
\end{aligned}
$$

(*) Pervenuta alla Segreteria dell'U. M. I. il 17 giugno 1963.

where $\Phi(a; c; x)$ is defined by

$$\Phi(a;\, c;\, x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_r} \frac{x^r}{r!}$$

Here we propose to give an integral representation for the product of two generalized BESSEL polynomials. In the terminology of KRALL and FRINK [3] the generalized BESSEL polynomial is

$$(1.3) \qquad y_n(x,\, a,\, b) = {}_2F_0(-n,\, a-1+n;\, -;\, -x/b).$$

In this connection we mention AL-SALAM'S result [4]:

$$(1.4) \qquad Y_n^{(\alpha)}(u)\, Y_n^{(\beta)}(v) =$$

$$= \frac{n!}{\Gamma(1+\alpha+n)} \int_0^{\infty} e^{-t} t^{\alpha} P_n^{(\alpha,\, 0)}\left(1 + t\left(u + v + uv\frac{t}{2}\right)\right) dt$$

where $Y_n^{(\alpha)}(x) = y_n(x,\, \alpha+2,\, 2)$ in the notation of KRALL-FRINK.

But the results of CARLITZ suggest that the product of two generalized BESSEL polynomials can be represented by a double integral involving another polynomial of the same kind.

2. Polynomials related to the Bessel polynomials.

We first consider the polynomials $\Phi_m(c, x)$ related to the BESSEL polynomials $y_n(x, a, b)$, where $\Phi_m(c, x)$ is defined by [5]:

$$(2.1) \qquad \Phi_n(c,\, x) = \frac{(c)_n}{n!}\, {}_2F_0(-n,\, c+n;\, -;\, x)$$

To obtain the KRALL-FRINK generalized BESSEL polynomial, we introduce the redundant parameter b by replacing x by $(-x/b)$, put $c = a - 1$ and multiply $\Phi_m(a-1,\, -x/b)$ by $m!/(a-1)_m$.

Now it follow from (2.1) that

$$(2.2) \qquad \Phi_m(c,\, x) = \sum_{r=0}^{m} \frac{(-1)^r (c)_{m+r}}{r!\,(m-r)!} x^r; \qquad \left((c)_r = \frac{\Gamma(c+r)}{\Gamma(c)}\right).$$

Then we derive

$$\Phi_m(c, x,\Phi_n(d, y) =$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-1)^{r+s} x^r y^s}{r!\, s!} \frac{\Gamma(c+m+r)\Gamma(d+n+s)}{\Gamma(c)\Gamma(d)} \cdot \frac{1}{(m-r)!\,(n-s)!}.$$

In other words,

$$\Gamma(c)\Gamma(d)\Phi_m(c, x)\Phi_n(d, y) = \tag{2.3}$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-1)^{r+s} x^r y^s}{r!\, s!} \frac{\Gamma(c+m+r)\Gamma(d+n+s)}{\Gamma(c+d+m+n+r+s)} \frac{(m+n-r-s)!}{(m-r)!(n-s)!} \frac{\Gamma(c+d+m+n+r+s)}{(m+n-r-s)!}$$

Now we notice the results [6]:

$$\frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} = \frac{2^{\mu+\nu}}{\pi} \int_{-\pi/2}^{\pi/2} e^{(\mu-\nu)\theta i} \cos^{\mu+\nu} d\theta, \qquad (\mu+\nu > -1) \tag{2.4}$$

and

$$\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt, \qquad (\mu > 0,\ \nu > 0). \tag{2.5}$$

It follows therefore from (2.3) that

$$\Gamma(c)\Gamma(d)\Phi_m(c, x)\Phi_n(d, y) =$$

$$\begin{aligned} &= \frac{2^{m+n}}{\pi} \int_0^1 \int_{-\pi/2}^{\pi/2} t^{c+m-1}(1-t)^{d+n-1} e^{(m-n)\theta i} \cos^{m+n}\theta \,\cdot \\ &\cdot \sum_{k=0}^{m+n} (-1)^k \frac{\Gamma(c+d+m+n+k)}{k!\,(m+n-k)!} (2\cos\theta)^{-k} \,\cdot \\ &\sum_{r+s=k} \binom{k}{r} (xt)^r\, [y(1-t)]^s\, e^{-(r-s)\theta i} d\theta\, dt, \qquad (c > 0,\ d > 0). \end{aligned} \tag{2.6}$$

In other words,

$$\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\Phi_m(c,\ x)\Phi_n(d,\ y)=$$

$$(2.7)\qquad =\frac{2^{m+n}}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} t^{c+m-1}(1-t)^{d+n-1}e^{(m-n)\theta i}\cos^{m+n}\theta\ \cdot$$

$$\cdot\sum_{k=0}^{m+n}\frac{(-1)^k(c+d)_{m+n+k}}{k!\,(m+n-k)!}\left(\frac{xte^{-\theta i}+y(1-t)e^{\theta i}}{2\cos\theta}\right)^k d\theta dt.$$

Finally using (2 2) we derive

$$\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\Phi_m(c,\ x)\Phi_n(d,\ y)=$$

$$(2.8)\qquad =\frac{2^{m+n}}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} t^{c+m-1}(1-t)^{d+n-1}e^{(m-n)\theta i}\cos^{m+n}\theta\ \cdot$$

$$\cdot\,\Phi_{m+n}\left(c+d,\quad \frac{xte^{-\theta i}+y(1-t)e^{\theta i}}{2\cos\theta}\right)d\theta dt. \qquad (c>0,\ d>0).$$

3. Formula for the product $y_m(x,\ a,\ b)\cdot y_n(y,\ a',\ b')$.

In this section we express the formula (2.8), which is our main result, in terms of the generalized BESSEL polynomials. Indeed we derive from (2 8)

$$\frac{\Gamma(a-1)\Gamma(a'-1)}{\Gamma(a+a'-2)}\frac{m!}{(a-1)_m}\Phi_m(a-1,\ -x/b)\frac{n!}{(a'-1)_n}\Phi_n(a'-1,\ -y/b')=$$

$$=\frac{m!\,n!}{(a-1)_m(a'-1)_n}\frac{2^{m+n}}{\pi}\int_0^1\int_{-\pi/2}^{\pi/2} t^{a+m-2}(1-t)^{a'+n-2}e^{(m-n)\theta i}\cos^{m+n}\theta\ \cdot$$

$$\cdot\,\Phi_{m+n}\left(a+a'-2,\ -\left\{\frac{x}{b}te^{-\theta i}+\frac{y}{b'}(1-t)e^{\theta i}\right\}\Big/2\cos\theta\right)d\theta dt.$$

Now using (1.3) and (2.1) we easily obtain from the result just now deduced from (2 8):

$$\frac{\Gamma(a+m-1)\Gamma(a'+n-1)}{\Gamma(a+a'+m+n-2)} y_m(x, a, b) y_n(y, a', b') =$$

$$(3.1) \qquad = \frac{2^{m+n}}{\pi} \frac{m!\,n!}{(m+n)!} \int_0^1 \int_{-\pi/2}^{\pi/2} t^{a+m-2}(1-t)^{a'+n-2} e^{(m-n)\theta i} \cos^{m+n}\theta \cdot$$

$$\cdot\, y_{m+n}\left(\frac{b'txe^{-\theta i} + by(1-t)e^{\theta i}}{2\cos\theta},\ a+a'-1,\ bb'\right) d\theta\, dt.$$

Next we know that KRALL–FRINK simple BESSEL polynomial $y_n(x)$ is obtained by putting $a = b = 2$. Thus we notice that

$$y_n(x, 2, 2) = y_n(x).$$

So using $a = b = a' = b' = 2$, we derive from (3.1)·

$$(3.2) \qquad y_m(x) y_n(y) = \frac{2^{m+n}}{\pi} (m+n+1) \cdot$$

$$\cdot \int_0^1 \int_{-\pi/2}^{\pi/2} t^m (1-t)^n e^{(m-n)\theta i} \cos^{m+n}\theta\, y_{m+n}\left(\frac{xte^{-\theta i} + y(1-t)e^{\theta i}}{2\cos\theta},\ 3,\ 2\right) d\theta\, dt.$$

From (3.2) we thus observe that the product of two simple BESSEL polynomials is represented by a double integral involving a generalized BESSEL polynomial whose first parameter « a » is 3, while the redundant parameter « b » remains the same.

REFERENCES

[1] G. N WATSON, *A note on the polynomials of Hermite and Laguerre*, « J. Lond. Math. Soc. », Vol. 13 (1938), pp. 204-209.

[2] L. CARLITZ, *An integral for the product of two Laguerre polynomials*, « Boll. Un. Mat. Ital. », (3) Vol. 17 (1962), pp. 25-28.

[3] H. L. KRALL and O. FRINK, *A new class of orthogonal polynomials · The Bessel polynomials*, «Trans. Amer. Math. Soc.», Vol. 65 (1949), pp. 100-115.

[4] W. A. AL-SALAM, *The Bessel polynomials*, « Duke Math. Jour. » Vol. 24 (1957), pp. 529 546.

[5] E. D. RAINVILLE, *Special Functions*, New York (1960), p. 294.

[6] E. T. WHITTAKER and G. WATSON, *A course of modern analysis*, 4th. edition, Cambridge (1952), pp. 263, 253.