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A note on differential approximation and orthogonal polynomials

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Summary. - *The problem of approximating to a given function by a sum of exponentials of the form*

$$\sum_i b_i e^{\lambda_i t},$$

where the coefficients and the exponents are both unknown, is treated in a new way in this paper.

1. Introduction.

The problem of obtaining an exponential polynomial of the form $\sum_{i=1}^{N} b_i e^{\lambda_i t}$ which closely approximates a given function $f(t$

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in an interval $a \leq t \leq b$ is a problem of some difficulty if we allow both the coefficients and the exponents to be unknowns. Either of the criteria of fit.

$$\max_{a \leq t \leq b} \left| f(t) - \sum_{i=1}^{N} b_i e^{\lambda_i t} \right| , \tag{1.1}$$

or

$$\int_a^b \left| f(t) - \sum_{i=1}^{N} b_i e^{\lambda_i t} \right|^2 dt \tag{1.2}$$

lead to difficulties; see LANCZOS [1].

In this note, we wish to consider a different way of measuring the closeness of $f(t)$ to a sum of esponentials. If $f(t)$ were a function of the form $\sum_{i=1}^{N} b_i e^{\lambda_i t}$, it would satisfy a linear differential equation of the form

$$f^{(N)} + c_1 f^{(N-1)} + \dots + c_N f = 0. \tag{1.3}$$

Hence, let us attempt to determine real coefficients c_i which minimize the integral

$$\int_{-\infty}^{\infty} (f^{(N)} + c_1 f^{(N-1)} + \dots + c_N f)^2 dt. \tag{1.4}$$

We call this *differential approximation.*

Results of the type we obtain have applications in the fields of control theory, circuit synthesis, and numerical analysis.

2. Fourier transforms.

It is clear that we must impose some conditions on the function $f(t)$ in order to pose the problem.

Assume then that $f^{(k)} \in L^2(-\infty, \infty)$ for $k = 0, 1, 2, .. , N$. Then if

$$g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt, \tag{2.1}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{-ist} ds,$$

we have

$$f^{(k)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-is)^k g(s) e^{-ist} ds. \tag{2.2}$$

Using the PLANCHEREL–PARSEVAL formula, we have

$$\int_{-\infty}^{\infty} (f^{(N)} + c_1 f^{(N-1)} + ... + c_N f)^2 dt \tag{2.3}$$

$$= \int_{-\infty}^{\infty} | g(s) |^2 \, | c_N - is c_{N-1} + (-is)^2 c_{N-2} + ... |^2 ds$$

$$= \int_{-\infty}^{\infty} | g(s) |^2 [(c_N - s^2 c_{N-2} + ...)^2 + s^2 (c_{N-1} - s^2 c_{N-3} + ...)^2] ds.$$

3. Orthogonal polynomials and minimization.

It is clear that a change of variable reduces the problem to that of finding the orthogonal polynomials associated respectively with the weight functions $|g(s)|^2$ and $s^2|g(s)|^2$. Since these orthogonal polynomials can be constructed in a systematic fashion, we have a simple way of obtaining the coefficients c_i, and further measures of the asymptotic behavior as $N \to \infty$; see SZEGO [2]. A direct treatment of the question of minimizing the quadratic form of (1.4) would not be simple computationally for large N (although a SCHMIDT orthogonalization could be used), and would not readily furnish asymptotic behavior. The tecnique presented above is most useful in connection with the treatment of various classical functions where $g(s)$ has a simple analytic form.

Similar results can be obtained for the case where the interval is finite or semi-infinite, but not of the same simplicity.

4. Stability.

The study of the precise connection between the solution of the linear differential equation

$$u^{(N)} + c_1 u^{(N-1)} + \ldots + c_N u = 0, \tag{4.1}$$

and the function $f(t)$ leads to a stability question which we shall study elsewhere.

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See also

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