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Heat transfer of laminar forced convection in doubly connected regions

by U. A. SASTRY (India) (*)

Summary. - *In this paper the heat transfer of laminar forced-convection in a pipe whose outer cross-section is a Booths' lemniscate and inner cross section a circle has been investigated by using Schwarz's Alternating Method.*

Mathematical Formulation.

Let us consider a steady fully developed laminar flow with arbitrbry heat generation in a pipe of cross-section D bounded by a closed curve L. Suppose the axis of the pipe be in the z-direction. The basic momentum and energy equations of the constant property non-dissipative fluid in cartesian coordinates are

(1.1) $$\nabla^2 u = C_1 ,$$

(1.2) $$\nabla^2 t = (C_2 u - C_3).$$

where

$$C_1 = \frac{1}{u}\frac{\partial p}{\partial z}, \quad C_2 = \frac{\varphi}{k} c_p \frac{\partial t}{\partial z}, \quad C_3 = Q/k,$$

c_p = specific heat at constant pressure, u = viscosity.

k = thermal conductivity and φ = density.

Q = heat source intensity, ∇^2 is the LAPLACE operator in two dimensions.

Boundary conditions.

Consider the problem of forced-convection in non-circular pipes with the boundary conditions

(1.3) $$u = 0, \quad t = t_w .$$

where

u = local velecity, t = local temperature, t_w = wall temperature.

(*) Pervenuta alla Segreteria dell'U. M. I. il 17 maggio 1963.

Writing $z = x + iy$, $\bar{z} = x - iy$, $T = (t - t_w)$ we can easily deduce the expressions for the velocity and temperature (heat generation is constant) in the form

$$(1.4) \qquad u = C_1 z\bar{z}/4 + (4/C_2)[\Phi'(z) + \overline{\Phi'(z)}],$$

$$(1.5) \qquad T = \frac{C(z\bar{z})^2}{64} + \bar{z}\Phi(z) + z\overline{\Phi(z)} + \psi(z) + \overline{\psi(z)}$$

$$+ \Phi_1(z) + \overline{\Phi_1(z)} - \frac{C_3 z\bar{z}}{4}$$

where $\Phi(z)$, $\psi(z)$ and $\Phi_1(z)$ are functions holomorphic in the region D of the cross-section satisfying the given boundary conditions.

The average velocity u_m, average temperature T_m, the mixed mean temperature T_M, the heat transfer rate q, the heat transfer coefficient h and the NUSSELT number based on the mixed mean temperature are given by

$$(1.6) \qquad Au_m = \int_D u\,dA,$$

$$(1.7) \qquad AT_m = \int_D T\,dA,$$

$$(1.8) \qquad Au_m T_M = \int_D uT\,dA,$$

$$(1.9) \qquad q = (C_2 u_m - C_3)kA,$$

$$(1.10) \qquad h = -q/ST_M,$$

$$(1.11) \qquad Nu = (hD_e/k) = 4(A/S)^2(C_3 - C_2 u_m)/T_M.$$

where A is the area of the section, D_e is equivalent hydraulic diameter $= (4A/S)$, S is the circumferential length of the section.

Alternating Method of Schwarz.

The region R_{12} can be considered as the intersection of infinite region R_1 bounded by L_1 with finite region R_2 interior to L_2.

First we write the boundary condition on $(L_1 + L_2)$ as

$$L(\Phi) = f(t) \tag{2}$$

where L is an operator with the usual meaning.

To obtain the first approximation $\Phi^{(1)}$ to Φ we determine the functions in the region R_1 so that

$$L(\Phi^{(1)})\,|_{L_1} = f\,|_{L_1} \tag{2.1}$$

To obtain the second approximation we consider the solution in R_2 such that

$$L(\Phi^{(2)})\,|_{L_2} = f\,|_{L_2} - L(\Phi^{(1)})\,|_{L_2} \tag{2.2}$$

For the third approximation we determine in the region R_1 the solution satisfying the condition

$$L(\Phi^{(3)})\,|_{L_1} = f\,|_{L_1} - L(\Phi^{(2)})\,|_{L_1} \tag{2.3}$$

and so on. Then $\Phi = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \ldots$ is the required solution.

Cross-section bounded externally by a Booths' lemniscate and internally a circle.

Let us find the velocity field satisfying the boundary conditions

$$u = 0, \text{ on } L_2 \tag{2}$$

$$u = u_0. \text{ on } L_1 \tag{3}$$

u_0 is a constant to be determined.

Using (2) and (3) in (1.4) we find

$$\Phi(t) + \overline{\Phi(t)} = -Ct\bar{t}/16, \text{ on } L_2 \tag{4}$$

$$= -CR^2/16 + C_2 u_0/4, \text{ on } L_1 \tag{5}$$

$t\bar{t} = \bar{R}^2$, on L_1.

First approximation.

Let $\Phi^{(1)}$ be the first approximation to Φ which is to be determined in the region interior to L_2 satisfying

$$L(\Phi^{(1)})\,|_{L_2} = f\,|_{L_2} \tag{6}$$

The function

$$(7) \qquad z = b\zeta/(1 + m\zeta^2), \quad |m| < 1/2$$

maps the region interior to L_2 onto the unit circle in the ζ-plane. Using (4), (7) in (6) and multiplying the resulting equation by $\frac{1}{2\pi i} \frac{d\sigma}{(\sigma - \zeta)}$ where ζ is a point inside the unit circle we easily find

$$(8) \qquad \Phi_1(\zeta) = - Cb^2k/32 + Cb^2mk\zeta^2/16(1 + m\zeta^2)$$

The above equation can also be written in the form

$$(9) \qquad \Phi^{(1)}(z) = - (Ckb^2/32) \sum_0^\infty c'_n z^{2n}$$

where $c_n' = c_n\beta^n$, $\beta = (\quad 4m/b^2)$

$$(10) \qquad \zeta = (b - \sqrt{b^2 - 4mz^2})/2mz, \quad k = 1/(1 - m^2)$$

Integrating (9) we obtain

$$(11) \qquad \Phi^{(1)}(z) = - (Ckb^2/32) \sum_0^\infty c'_n z^{2n+1}/(2n + 1)$$

Second approximation.

Let $\Phi^{(2)}$ be the second approximation to Φ which is to be calculated subject to the condition

$$(12) \qquad L(\Phi^{(2)})\,|_{L_1} = f\,|_{L_1} - L(\Phi^{(1)})\,|_{L_1}$$

using (5), (9) in (12) we find

$$(13) \qquad \Phi^{(2)}(t) + \overline{\Phi^{(2)}(t)} = -CR^2/16 + C_2u_0\,4 + \\ + (Ckb^2/32) \sum_0^\infty c'_n (t^{2n} + \bar{t}^{2n})$$

since $\Phi^{(2)}$ is holomorphic outside L_1 we may take $\Phi^{(2)}(\infty) = 0$.

Multiplying (13) by $dt/2\pi i(t - z)$ where z is a point outside the circle L_1 we obtain after integration

$$(14) \qquad \Phi^{(2)}(z) = (Ckb^2/32) \sum_1^\infty c'_n R^{4n}/z^{2n}.$$

From the equation (13) we also have

$$u_0 = C_1(R^2 - kb^2)/4. \tag{15}$$

Integrating (14) we find

$$\Phi^{(2)}(z) = -(Ckb^2/32) \sum_1^\infty c'_n R^{4n}/(2n-1)z^{2n-1} \tag{16}$$

Using (9) and (14) in (1.4) we find

$$u = C_1(z\bar{z} - kb^2)/4 - \frac{Ckb^2}{4} \operatorname{Re} \sum_1^\infty c'_n(z^{2n} - R^{4n}/z^{2n}). \tag{17}$$

Let us find the temperature field subject to the boundary conditions

$$T = 0, \text{ on } L_2 \tag{18}$$

$$T = T_0 \cdot \text{on } L_1 \tag{19}$$

T_0 is a constant to be determined. From (18), (19) and (1.5) we have

$$\bar{t}\Phi(t) + t\overline{\Phi(t)} + \psi(t) + \overline{\psi(t)} + \Phi_1(t) + \overline{\Phi_1(t)} = \tag{20}$$

$$= \frac{C_2 t\bar{t}}{4} - \frac{C(t\bar{t})^2}{64} \text{ on } L_2$$

$$= \frac{C_2 R^2}{4} - \frac{CR^4}{64} + T_0 \text{ on } L_1 \tag{21}$$

First approximation.

Let $\Phi^{(1)}$, $\psi^{(1)}$, $\Phi_1^{(1)}$ be the first approximation to be determined in the region interior to L_2 satisfying

$$L(\Phi^{(1)}, \psi^{(1)}, \Phi_1^{(1)})|_{L_2} = f|_{L_2} \tag{22}$$

From (20), (22) we obtain after multiplying the resulting equation by $\frac{1}{2\pi i} \frac{d\sigma}{(\sigma - \zeta)}$ and integrating

$$\psi^{(1)}(z) + \Phi^{(1)}(z) + = \frac{H}{2}(1 + \sum_0^\infty c'_p z^{2p}) - \tag{23}$$

$$- \frac{Cb^4k^2}{256}(1 + \sum_0^\infty c'_p z^{2p})^2 - G + Ck \sum_0^\infty (c'_n b^{2n+4}/(2n+1)$$

$$\Big\} \sum_{p=0}^{\infty} E_p Z_1^{2(n+p)} + \sum_{p=0}^{\infty} E_{-p} Z_1^{2(p-n)} \Big\}$$

where

$$z_1 = -(b/2m) \sum_1^\infty c'_j z^{2j-1}$$

Second appoximation

Let $\Phi^{(2)}$, $\psi^{(2)}$, $\Phi_1^{(2)}$ be the second approximation to Φ, ψ, Φ_1 which is to be determined in the region exterior to L_1 subject to the condition

$$L(\Phi^{(2)}, \psi^{(2)}, \Phi_1^{(2)})|_{L_1} = f|_{L_1} - L(\Phi^{(1)}. \psi^{(1)}, \Phi_1^{(1)})|_{L_1} \tag{24}$$

Using (21), (23) in (24) and multiplying the resulting equation by $dt/2\pi i(t-z)$, where z is a point outside the circle L_1 and integrating we find

$$\psi^{(2)}(z) + \Phi_1^{(2)}(z) = (Ckb^2/32) \sum_1^\infty c'_n 4nR^{4n+2}/(4n^2-1)z^{2n} \tag{25}$$

$$- (H/2) \sum_1^\infty c'_p R^{4p}/z^{2p} + (Cb^4k^2/256)$$

$$\Big\} 2 \sum_1^\infty c_p R^{4p}/z^{2p} + \Big(\sum_1^\infty c'_p R^{4p}/z^{2p} \Big)^2 \Big\}$$

$$+ (Ck/32) \sum_1^\infty c'_n b^{2n+4}/(2n+1) \Big\} \sum_{p=1}^{\infty} E_p (b/2m)^{2(n+p)} z_2^{2(n+p)} +$$

$$+ \sum_{p=1}^{\infty} E_{-p} (b/2m)^{2(p-n)} Z_2^{2(p-n)} \Big\}$$

where

$$z_2 = \sum_1^\infty c'_j R^{2(2j-1)}/z^{2j-1}$$

$$H = (b^2k/8)(C_2 - Cb^2m^2k/8), \quad G = (H - Cb^4k^2/64 - Ckb^4/2 \tag{26}$$

T_0 can be determined from (6.23) using the fact that $\psi^{(2)}(\infty) = \Phi_1^{(2)}(\infty) = 0$.

The usual complex torsion function is related to the function $\Phi(z)$ by the relation

$$\Phi(z) = iF(z) \tag{27}$$

writing $C_1 = -2$, $C_2 = 8$, $C = -16$ we obtain the expression for the torsion function and the torsional rigidity is given by

$$D^* = 2\mu A u_m \tag{28}$$

where

$$(C_2 A u_m / 4) = 2\pi C \left[\frac{\{b^4k^4(1+4m^2+m^4) - R^4\}}{64}\right] - \frac{kb^2}{32}\{b^2k^2(1+m^2) - R^2\}$$

$$-(kb^3/32)\left\{\sum_1^\infty c'_n b^{2n+2}/(2n+1) \sum_{p=0}^\infty \left(\sum_{r=0}^\infty (-1)^r m^{p+r}(2r+1) \overset{-(2n+1)}{C_p}\right) + \right.$$

$$\left. + \sum_1^\infty \frac{c'_n R^{4n}}{(2n-1)G^{2n-2}} \sum_{p=0}^\infty \left(\sum_{r=0}^\infty (-1)^r (2r+1) m^{p+r} \overset{-(2n+1)}{C_p}\right)_{n+r=p}\right\} \tag{29}$$

and

$$A = \pi[k^2b^2(1+m^2) - R^2] \tag{30}$$

The stress components are given by

$$\frac{X_z}{\mu\tau} = \left[\frac{Ckb^2}{2} \sum_1^\infty nc'_n \left\{ r^{2n-1}\sin(2n-1)\theta - \frac{R^{4n}\sin(2n+1)\theta}{r^{2n+1}}\right\} - y\right] \tag{31}$$

$$\frac{Y_z}{\mu\tau} = \left[\frac{Ckb^2}{2} \sum_1^\infty nc'_n \left\{ r^{2n-1}\cos(2n-1)\theta + \frac{R^{4n}\cos(2n+1)\theta}{r^{2n+1}}\right\} + x\right] \tag{32}$$

Writing $R = 0$, in the expressions for the torsional rigidity and the torsion function we obtain known results.

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